Workshop
F-isocristaux et cohomologie rigide

PROPER COHOMOLOGICAL DESCENT IN RIGID COHOMOLOGY

FRÉDÉRIC DÉGLISE

Introduction

We fix a complete discrete valuation ring $\mathcal{V}$ of mixed characteristic $(0, p)$ with residue field k and fraction field K.

The purpose of the talk is to prove the theorem of proper descent in rigid cohomology due to Tsuzuki and to expose some of its consequences.

To do so, we will have to recall a piece of the general descent theory which rely deeply on simplicial methods. We will do that as exposed by Chiarellotto and Tsuzuki to make technicalities less heavy.

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Contents

1. Geometric setting

We will adopt the definitions and notations of [3].

In general, k-schemes will be assumed to be locally of finite type.

We will simply call pair any pair of k-schemes $(X, \bar{X})$ such that X is an open subscheme of $\bar{X}$. A morphism of pairs is a commutative diagram

$$\begin{array}{ccc} Y & \to & \bar{Y} \\ {\scriptstyle \mathring{w}}\downarrow & & \downarrow{\scriptstyle \bar{w}} \\ X & \to & \bar{X}. \end{array}$$

We say the morphism is strict if the square is cartesian.

We will overall consider triples $\mathbb{X}$ which are triples of schemes $(X, \bar{X}, \mathcal{X})$ such that

(1) $(X, \bar{X})$ is a pair,
(2) $\mathcal{X}$ is a formal π-adic $\mathcal{V}$-scheme locally of finite type
(3) $\bar{X}$ is a closed subscheme of $\mathcal{X} \times_{\mathcal{V}} k$.

We will always use the same typography for the three schemes which compose the triple $\mathbb{X}$ which will ease notation.

A morphism $w : \mathbb{Y} \to \mathbb{X}$ of triples is a commutative diagram

$$\begin{array}{ccccc} Y & \to & \bar{Y} & \to & \mathcal{Y} \\ {\scriptstyle \mathring{w}}\downarrow & & {\scriptstyle \bar{w}}\downarrow & & \downarrow{\scriptstyle \hat{w}} \\ X & \to & \bar{X} & \to & \mathcal{X}. \end{array}$$

We say the morphism is strict (resp. semi-strict) if the two squares are cartesian (resp. first square is cartesian). The category of triples is denoted by $\mathscr{T}r_{\mathcal{V}}$.

The same rule for the typesetting of morphisms will apply.

Let's introduce the concepts of rigid cohomology in this setting.

For a triple $\mathbb{X}$, we will denote by $]\bar{X}[$ the tube of $\bar{X}$ in the generic fiber $\mathcal{X}_K$ "à la Raynaud". It is functorial. For $w : \mathbb{Y} \to \mathbb{X}$ a morphism of triple, we denote by $\tilde{w} :]\bar{Y}[\to]\bar{X}[$ the induced morphism.

We will denote conventionally by $j_{\mathbb{X}} : X \to \bar{X}$ the canonical immersion and by $j^{\dagger}_{\mathbb{X}}$ the dagger construction on sheaves over the rigid analytic space $\mathbb{X}_K$. Finally, we put $\mathcal{O}^{\dagger}_{\mathbb{X}} = j^{\dagger}_{\mathbb{X}}(\mathcal{O}_{]\bar{X}[})$ and denote by $\mathcal{O}^{\dagger}_{\mathbb{X}}-\mathscr{M}od^c$ the category of coherent $\mathcal{O}^{\dagger}_{\mathbb{X}}$-module.

These categories are functorial. Let $w : \mathbb{Y} \to \mathbb{X}$ be a morphism of triples. We associate to $\tilde{w}$ a couple of adjoint functors $(\tilde{w}^{-1}, \tilde{w}_*)$. There exist a canonical morphism

$$\tilde{w}^{-1}\mathcal{O}^{\dagger}_{\mathbb{X}} \to \mathcal{O}^{\dagger}_{\mathbb{Y}}.$$

Thus we deduce a couple of adjoints functor

$$(\tilde{w}^{\dagger}, \tilde{w}_*) : \mathcal{O}^{\dagger}_{\mathbb{Y}}-\mathscr{M}od^c \to \mathcal{O}^{\dagger}_{\mathbb{X}}-\mathscr{M}od^c$$

with the formula

$$\tilde{w}^{\dagger}E = j^{\dagger}\big(\mathcal{O}^{\dagger}_{\mathbb{Y}} \otimes_{\tilde{w}^{-1}\mathcal{O}^{\dagger}_{\mathbb{X}}} \tilde{w}^{-1}E\big).$$

We will use the following fact :

Proposition 1.1. *Let $w : \mathbb{Y} \to \mathbb{X}$ be a morphism of triples such that $\hat{W} : \mathcal{Y} \to \mathcal{X}$ is flat around Y.*

Then the morphism $\tilde{w}^\dagger$ is exact.

Proof. If w is the obvious morphism $(X, X, \mathcal{X}) \to (X, \bar{X}, \mathcal{X})$, $\tilde{w}^\dagger$ is the natural inclusion

$$\mathcal{O}^\dagger_{j^\dagger \mathcal{O}_{]\bar{X}[}} - \mathscr{M}od^c \to \mathcal{O}^\dagger_{\mathcal{O}_{]X[}} - \mathscr{M}od^c$$

which is exact and faithful according to [2], 2.1.11.

Thus an easy computation showsz we can reduce to the case $X = \bar{X}$ and $Y = \bar{Y}$. Let $\mathcal{Y}'$ be an open subscheme of $\mathcal{Y}$ flat over $\mathcal{X}$. As $]Y[_{\mathcal{Y}} =]Y[_{\mathcal{Y}'}$, we can suppose $\hat{w}$ is flat.

Then the morphism $\tilde{w}$ is flat which imply the result for coherent modules. $\square$

2. Hypercoverings

2.1. **Simplicial objects.** For any integer $n \geq 0$ we denote by

(1) Δ^n the ordered set of integer $[0, n]$
(2) (*face operators*) $\delta^{n,i} : \Delta^n \to \Delta^{n+1}$ the increasing injection such that $i \notin \mathrm{Im}(\delta^{n,i})$ for $0 \leq i \leq n+1$
(3) (*degeneracy operators*) $\sigma^{n,i} : \Delta^{n+1} \to \Delta^n$ the increasing surjection such that $\sigma^{n,i}(i+1) = \sigma^{n,i}(i)$ for $0 \leq i \leq n$.

We denote as usual by Δ the category with objects Δ^n, $n \geq 0$, and with increasing map as morphisms. Any such map is a composite of maps $\delta^{n,i}$ or $\sigma^{n,i}$. Note that the definition of these latter maps impose naturally some relations on their compositions.

Let $\mathscr{C}$ be an arbitrary category. A simplicial (resp. cosimplicial) object of $\mathscr{C}$ is a contravariant (resp. covariant) functor $X_\bullet : \Delta^{op} \to \mathscr{C}$. This corresponds to a sequence of objects and maps of $\mathscr{C}$ as follows

$$\begin{tikzcd}
X_0 \arrow[r, "s_{0,0}"] & X_1 \arrow[l, shift right=2, "d_{0,1}"'] \arrow[l, shift left=2, "d_{0,0}"] \arrow[r, shift left=1, "s_{1,0}"] \arrow[r, shift right=3, "s_{1,1}"] & X_2 \arrow[l, shift right=4, "d_{1,2}"'] \arrow[l, shift right=1, "d_{1,1}"] \arrow[l, shift left=4, "d_{1,0}"] \; \ldots
\end{tikzcd}$$

We denote by $\Delta^{op}\mathscr{C}$ the category of simplicial objects with natural transformations as morphisms.

We will constantly use in this talk the case where $\mathscr{C} = \mathscr{T}r_{\mathcal{V}}$. Simplicial objects of $\mathscr{T}r_{\mathcal{V}}$ will be called simplicial triples.

Example 2.1. Let $w : \mathbb{Y} \to \mathbb{X}$ be a morphism of triples.

We can define the Čech complexe of w as the following simplicial triple

$$\begin{tikzcd}
\mathbb{Y} \arrow[r, "s_{0,0}"] & \mathbb{Y} \times_{\mathbb{X}} \mathbb{Y} \arrow[l, shift right=2, "d_{0,1}"'] \arrow[l, shift left=2, "d_{0,0}"] & \cdots
\end{tikzcd}$$

where $s_{0,0}$ is the diagonal immersion, $d_{0,0}$ and $d_{0,1}$ the two canonical projection. We denote this simplicial triple by $\mathrm{cosk}_0^{\mathbb{X}}(\mathbb{Y})$.

2.1.1. In the case of an abelian category $\mathscr{A}$, we associate to simplicial object $A_\bullet$ of $\mathscr{A}$ a complexe concentrated in non negative degrees :

$$A_0 \xleftarrow{d_0} A_1 \xleftarrow{d_1} A_2 \ ...$$

with $d_n = \sum_{i=0}^{n+1}(-1)^i d_{n,i}$. It is classically called the complexe of alternating faces.

Note that the same definition can be apply dually to cosimplicial objects of $\mathscr{A}$.

2.2. **Coskeleton.** Let fix a triple $\mathbb{X}$. We will work with the category $\mathscr{T}r_{\mathcal{V}}/\mathbb{X}$ of triples over $\mathbb{X}$.

For an integer $n \geq 0$, we let $\Delta^{\leq n}$ be the full subcategory of Δ with objects Δ^i for $0 \leq i \leq n$. A n-truncated simplicial triple over $\mathbb{X}$ will be a contravariant functor $\Delta^{\leq n} \to \mathscr{T}r_{\mathcal{V}}/\mathbb{X}$.

There is an obvious restriction functor

$$\mathrm{sk}_n : \Delta^{op}\mathscr{T}r_{\mathcal{V}}/\mathbb{X} \to (\Delta^{\leq n})^{op}\mathscr{T}r_{\mathcal{V}}/\mathbb{X}, \mathbb{X}_\bullet \mapsto (\mathbb{X}_0, ..., \mathbb{X}_n).$$

As the category $\mathscr{T}r_{\mathcal{V}}/\mathbb{X}$ has arbitrary finite limits, this functor has ar right adjoint denoted by $c_n^{\mathbb{X}}$. Following Verdier, we will consider the coskeleton functor to be the composite $\mathrm{cosk}_n^{\mathbb{X}} = c_n^{\mathbb{X}} \circ \mathrm{sk}_n$. For any simplicial triple $\mathbb{Y}_\bullet$ over $\mathbb{X}$, we have an adjoint morphism $\mathbb{Y}_\bullet \to \mathrm{cosk}_n^{\mathbb{X}}\mathbb{Y}_\bullet$. [1]

Example 2.2. In the case $n = 0$, $\mathrm{cosk}_0^{\mathbb{X}}(\mathbb{Y})$ is simply the Čech complex introduced in the previous example.

Note that this definition keeps sense for $n = -1$. In this case, $\mathrm{cosk}_{-1}^{\mathbb{X}}$ is the functor with value the constant simplicial tripel $\mathbb{X}$ ($\mathbb{X}$ in each degree, identity as transition morphisms).

2.3. **Definition.** Let t be a class of morphisms of triples which is stable under composition and base change.

Definition 2.3. Let $\mathbb{X}$ be a triple.

A t-hypercovering of $\mathbb{X}$ is a simplicial triple $w_\bullet : \mathbb{Y}_\bullet \to \mathbb{X}$ over $\mathbb{X}$ such that for any integer $n \leq 0$, the canonical functor $\mathbb{Y}_n \to [\mathrm{cosk}_{n-1}^{\mathbb{X}}(\mathbb{Y}_\bullet)]_n$ is a covering.

Example 2.4.
(1) Let $\mathbb{Y} \xrightarrow{a} \mathbb{X}$ be a morphism in t. Then $\mathrm{cosk}_0^{\mathbb{X}}(\mathbb{Y}) \to \mathbb{X}$ is a t-hypercovering of $\mathbb{X}$ because for $n > 0$, the morphism

$$(\mathrm{cosk}_0^{\mathbb{X}}\mathbb{Y})_n \to [\mathrm{cosk}_{n-1}^{\mathbb{X}}(\mathrm{cosk}_0^{\mathbb{X}}\mathbb{Y}))]_n = (\mathrm{cosk}_0^{\mathbb{X}}\mathbb{Y})_n$$

is the identity.
(2) Based on the previous example, one can construct recursively hypercoverings using the following method :

Suppose we are given a morphism $N_1 \xrightarrow{b} \mathbb{Y} \times_{\mathbb{X}} \mathbb{Y}$ in t. Then we can define a 1-truncated simplicial object

$$\begin{tikzcd} & \mathbb{Y}\times_{\mathbb{X}}\mathbb{Y} \arrow[dl, "p_1"'] & \\ \mathbb{Y} \arrow[rr, "1_{\mathbb{Y}}"] & & N_1 \sqcup \mathbb{Y}. \arrow[ul, "b\sqcup s"'] \arrow[dl, "b\sqcup s"] \\ & \mathbb{Y}\times_{\mathbb{X}}\mathbb{Y} \arrow[ul, "p_2"] & \end{tikzcd}$$

[1]This functor is an analog of the truncation functor for complexes. This can be made precise using the Dold-Kan equivalence for simplicial abelian groups.

Then $\mathrm{cosk}_1^{\mathbb{X}}(\mathbb{Y}...N_1 \sqcup \mathbb{Y})$ is again a t-hypercovering.

We can iterate this process to produce more and more complicated hypercoverings.

Remark 2.5. Recall that in any topos, the Čech cohomology of a sheaf F does not coincide with the cohomology of F. This is the reason why hypercovering were invented. Indeed, Čech cohomology is a limit of cohomology of F calculated through t-hypercovering of example 1, for t being the class of coverings.. Now if we take a similar limit over all t-hypercoverings, then the resulting cohomology agree with the cohomology of F. This fact is the basis of cohomological descent theory.

We will use this definition particularly when t is the class of morphism $w : \mathbb{Y} \to X$ such that

1. w is semi-strict,
2. $\mathring{w}$ is proper surjective,
3. $\bar{w}$ is proper,
4. $\hat{w}$ is smooth around Y.

In this case a t-hypercovering will be called simply a proper hypercovering.

3. Čech complex

3.1. **Sheaves over a simplicial triple.** In order to define a sheaf over a simplicial triple, we introduce the category $\mathscr{F}$ with

1. objects are pairs $(F, \mathbb{X})$ such that F is a sheaf over the rigid analytic space $]\bar{X}[$.
2. morphisms are $(\phi, f) : (G, \mathbb{Y}) \to (F, \mathbb{X})$ with $f : \mathbb{Y} \to \mathbb{X}$ a morphism of triples and $\phi : f^{-1}F \to G$.

Thus there is a natural projection functor $\mathscr{F} \to \mathscr{T}r_{\mathcal{V}}$. [2]

If $\mathbb{X}_\bullet$ is a simplicial triple, a sheaf over $\mathbb{X}_\bullet$ is a simplicial object of $\mathscr{F}$ whose projection on $\mathscr{T}r_{\mathcal{V}}$ is $\mathbb{X}_\bullet$. In particular, it is given by

1. for any integer $n \geq 0$, a sheaf F^n over $]\bar{X}_n[$
2. for any transition map $\eta : \mathbb{X}_n \to \mathbb{X}_m$ of $\mathbb{X}_\bullet$, a morphism of sheaves $F^m \to \tilde{\eta}_* F^n$.

We will simply denote by $F^\bullet$ such a simplicial sheaf. The sheaf F^n will be called the fiber over n of $F^\bullet$.

Example 3.1. Let $\mathbb{X}_\bullet$ be a simplicial triple. Using the classical functoriality of $j^\dagger$, we readily see that the sequence of sheaves $\mathcal{O}^\dagger_{\mathbb{X}_n}$ with the canical compatibility morphisms as transition morphism define a sheaf over $\mathbb{X}_\bullet$ simply denoted by $\mathcal{O}^\dagger_{\mathbb{X}_\bullet}$.

We denote by $\widetilde{\mathbb{X}_\bullet}$ the category of sheaves over $\mathbb{X}_\bullet$. Following [1], exp. 6, this is in fact a topos. The sheaf $\mathcal{O}^\dagger_{\mathbb{X}_\bullet}$ is a ring in this topos which is moreover coherent. The category of modules over $\mathcal{O}^\dagger_{\mathbb{X}_\bullet}$ is given by sheaves $M_\bullet$ such that M_n is a $\mathcal{O}^\dagger_{\mathbb{X}_n}$-module over $]\bar{X}_n[$. As in any topos, it is an abelian category with enough injectives.

[2] In the terminology of Grothendieck, $\mathscr{F}$ is a fibered topos but we won't enter into these details here.

We say a $\mathcal{O}^\dagger_{\mathbb{X}_\bullet}$-module is coherent if is it coherent fiberwise and we denote by $\mathcal{O}^\dagger_{\mathbb{X}_\bullet}-\mathscr{M}od^c$ the corresponding category.

3.2. **Functoriality.** Let $\mathbb{X}$ be a triple. A simplicial triple over $\mathbb{X}$ is a simplicial object of the category $\mathscr{T}r_{\mathcal{V}}/\mathbb{X}$ of triples over $\mathbb{X}$.

Example 3.2. If $\mathbb{Y} \to \mathbb{X}$ is a morphism of triples, the Čech complex $\operatorname{cosk}_0^{\mathbb{X}}(\mathbb{Y})$ is naturally an $\mathbb{X}$-simplicial triple.

Let $w_\bullet : \mathbb{Y}_\bullet \to \mathbb{X}$ be a general simplicial triple over $\mathbb{X}$.

3.2.1. For any $\mathcal{O}^\dagger_{\mathbb{X}}$-module E, the sequence $\tilde{w}_n^\dagger E$ has a structure of $\mathcal{O}^\dagger_{\mathbb{Y}_\bullet}$-module :

Indeed, for any transition morphism $\eta : \mathbb{Y}_n \to \mathbb{Y}_m$, we can consider the canonical morphism

$$\tilde{w}_m^\dagger E \to \tilde{\eta}_* \tilde{w}_n^\dagger E$$

because $\tilde{\eta}^\dagger \tilde{w}_m^\dagger = \tilde{w}_n^\dagger$.

We thus have defined a functor

$$w_\bullet^\dagger : \mathcal{O}^\dagger_{\mathbb{X}}-\mathscr{M}od^c \to \mathcal{O}^\dagger_{\mathbb{Y}_\bullet}-\mathscr{M}od^c.$$

Example 3.3. Let $w : \mathbb{Y} \to \mathbb{X}$ be a morphism of triples such that $\hat{w} : \mathcal{Y} \to \mathcal{X}$ is flat around Y. Consider the morphism $w_\bullet : \operatorname{cosk}_0^{\mathbb{X}}(\mathbb{Y}) \to \mathbb{X}$. Then, from proposition 1.1, the functor $w_\bullet^\dagger$ is exact as for all n, $w_n^\dagger$ is exact.

More generally, if $w_\bullet : \mathbb{Y}_\bullet \to \mathbb{X}$ is a simplicial triple over $\mathbb{X}$ such that for any n, $\hat{w}_n$ is flat around Y_n, then $w_n^\dagger$ is exact.

If $E^\bullet$ is a complex of $\mathcal{O}^\dagger_{\mathbb{X}}$-modules, applying $w_\bullet^\dagger$ to each term of $\mathcal{M}^\bullet$ gives a complex of $\mathcal{O}^\dagger_{\mathbb{Y}_\bullet}$-modules which we denote again by $w_\bullet^\dagger E^\bullet$.

3.2.2. Consider a $\mathcal{O}^\dagger_{\mathbb{Y}_\bullet}$-module $F^\bullet$. Then the sequence $(\tilde{w}_n)_* F^n$ for $n \in \mathbb{N}$ defines a cosimplicial $\mathcal{O}^\dagger_{\mathbb{X}}$-module :

For any map $\Delta^m \to \Delta^n$ corresponding to the transition morphism $\eta : \mathbb{Y}_n \to \mathbb{Y}_m$, we apply $\tilde{w}_{m*}$ to the structural morphism $F_m \to \tilde{\eta}_* F^n$ which gives a morphism

$$\tilde{w}_{m*} F^m \to \tilde{w}_{m*} \tilde{\eta}_* F^n = \tilde{w}_{n*} F^n$$

as needed.

We denote by $\mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; F^\bullet)$ the associated (cohomological) complex of alternating faces.

If we are given a bounded below complex $F^{\bullet,\bullet}$ of sheaves over $\mathbb{X}_\bullet$, then applying $\mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; .)$ to each term of this complex gives a bicomplex. We again denote by $\mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; F^{\bullet,\bullet})$ the total complex of this bi-complex. Its term in degree n is

$$\prod_{r+s=n} (\tilde{w}_r)_* F^{r,s}.$$

It is an easy exercice to see the functors

$$\left(w_\bullet^\dagger, \mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; .)\right) : Comp(\mathcal{O}^\dagger_{\mathbb{Y}_\bullet}-\mathscr{M}od^c) \to Comp(\mathcal{O}^\dagger_{\mathbb{X}}-\mathscr{M}od^c)$$

are adjoints. Indeed, in the terminology of Deligne and Saint-Donas, $\mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; .)$ would have been denoted by $w_{\bullet *}$.

3.3. **Derived Čech complex.** Consider again the situation of an augmented $\mathbb{X}$-simplicial triple $w_\bullet : \mathbb{Y}_\bullet \to \mathbb{X}$.

The functor $\mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; .)$ on complexes is left exact. As usual, it is possible to derive it on the right by choosing injective resolution in $\mathcal{O}^\dagger_{\mathbb{Y}_\bullet}-\mathscr{M}od^c$.

Any complex $F^{\bullet,\bullet}$ of $\mathcal{O}^\dagger_{\mathbb{Y}_\bullet}$-modules has an injective resolution $F^{\bullet,\bullet} \to \mathcal{I}^{\bullet,\bullet}$ and we define the derived Čech complex of $F^{\bullet,\bullet}$ as

$$\mathbb{R}\mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; F^{\bullet,\bullet}) = \mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; \mathcal{I}^{\bullet,\bullet}).$$

As in any abelian category with enough injectives, this complex is well-defined up to quasi-isomorphism and induces a triangulated functor

$$\mathbb{R}\mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; .) : D^+(\mathcal{O}^\dagger_{\mathbb{Y}_\bullet}-\mathscr{M}od^c) \to D^+(\mathcal{O}^\dagger_{\mathbb{X}}-\mathscr{M}od^c).$$

When $w_\bullet^\dagger$ is exact, we then have a couple of adjoint functors

$$\big(w_\bullet^\dagger, \mathbb{R}\mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; .)\big) : D^+(\mathcal{O}^\dagger_{\mathbb{Y}_\bullet}-\mathscr{M}od^c) \to D^+(\mathcal{O}^\dagger_{\mathbb{X}}-\mathscr{M}od^c).$$

4. The setting of cohomological descent in rigid cohomology

4.1. **Definitions.** We are ready to introduce our main definition in the theory of cohomological descent for rigid cohomology :

Definition 4.1. Let $w_\bullet : \mathbb{Y}_\bullet \to \mathbb{X}$ be an augmented simplicial triples.

We say $w_\bullet$ is cohomologically descendable if

(1) $w_\bullet^\dagger$ is exact.

(2) For any coherent $\mathcal{O}^\dagger_{\mathbb{X}}$-module E, the canonical adjunction morphism

$$E \to \mathbb{R}w_{\bullet *}w_\bullet^\dagger(E)$$

is a quasi-isomorphism.

We say $w_\bullet$ is universally cohomologically descendable if it is cohomologically descendable after any base change.

We have to introduce a variant of this definition in order to grasp rigid cohomology.

Let $w : \mathbb{Y} \to \mathbb{X}$ be a triple satisfying condition
(*) $\hat{w}$ is smooth around Y.

Then $j^\dagger\Omega^1_{]\bar{Y}[/]\bar{X}[}$ is a locally free $\mathcal{O}^\dagger_{\mathbb{Y}}$-module. For any coherent $\mathcal{O}^\dagger_{\mathbb{Y}}$-module F, we define the De Rham complex of F (considered with the null connection), by

$$\mathrm{DR}(\mathbb{Y}/\mathbb{X}; F) = F \xrightarrow{1\otimes d^1} F \otimes_{\mathcal{O}^\dagger_{\mathbb{Y}}} j^\dagger\Omega^1_{]\bar{Y}[/]\bar{X}[} \to ...$$

This definition is functorial in F and $\mathbb{Y}$. Thus we can extend it in an obvious way to the case of simplicial triples $\mathbb{Y}_\bullet$ over $\mathbb{X}$ which satisfy condition (*) fibrewise.

Definition 4.2. Let $w_\bullet : \mathbb{Y}_\bullet \to \mathbb{X}$ be a simplicial triples over $\mathbb{X}$ satisfying condition (*) fibrewise.

We say $w_\bullet$ is De Rham cohomologically descendable if for any coherent $\mathcal{O}^\dagger_{\mathbb{X}}$-module E, the canonical adjunction morphism

$$\mathrm{DR}(\mathbb{Y}_\bullet/\mathbb{X}; w_\bullet^\dagger E) \to \mathbb{R}w_{\bullet *}w_\bullet^\dagger\mathrm{DR}(\mathbb{Y}_\bullet/\mathbb{X}; w_\bullet^\dagger E)$$

is a quasi-isomorphism.

4.2. **Fundamental properties.**

Theorem 1 (Chiarellotto-Tsuzuki). *Consider a morphism of simplicial triples over $\mathbb{X}$*

$$\begin{tikzcd} \mathbb{Y}' \arrow[rr, "f"] \arrow[dr, "w'"'] & & \mathbb{Y} \arrow[dl, "w"] \\ & \mathbb{X} & \end{tikzcd}$$

Suppose the morphism f is u.c.d. (resp. De Rham u.c.d.). Then the following conditions are equivalent :

(i) *w is c.d. (resp. De Rham c.d.).*
(ii) *w' is c.d. (resp. De Rham c.d.).*

(idea). We denote by $f_\bullet$ (resp. $w_\bullet$, $w'_\bullet$) the projection morphism of the Čech simplicial triples associated to f (resp. w, w').

The idea of the proof is to extend the canonical natural transformation

$$Id \to \mathbb{R}f_{\bullet *}f_\bullet^\dagger$$

to the category of sheaves over $\operatorname{cosk}_0^{\mathbb{X}}(\mathbb{Y})$. Applying this extension to $w_\bullet^\dagger E$ for a coherent $\mathcal{O}_{\mathbb{X}}^\dagger$-module, we obtain a natural morphism

$$\mathbb{R}w_{\bullet *}w_\bullet^\dagger E \to \mathbb{R}w'_{\bullet *}(w'_\bullet)^\dagger E$$

and we proove it is still an isomorphism. □

We can always reduce the case of hypercoverings to the case of Čech simplicial triple. We adopt the following definition.

Definition 4.3. Let $w : \mathbb{Y} \to \mathbb{X}$ be a morphism of triples.

We say w is c.d. (resp. u.c.d., De Rham c.d.,De Rham u.c.d) if the induced morphism $w_\bullet : \operatorname{cosk}_0^{\mathbb{X}}(\mathbb{Y}) \to \mathbb{X}$ is so.

Example 4.4. Let $w : \mathbb{Y} \to \mathbb{X}$ ba morphism of triples. Then w is u.c.d. and De Rham u.c.d. in the following cases :

(1) if w is strict and $\hat{w}$ is an open covering
(2) if $\overset{\circ}{w}$ is an isomorphism, $\bar{w}$ is proper, $\hat{w}$ is smooth around Y.

Then the following proposition is a basic step argument in the proper descent theorem.

Proposition 4.5. *Let t be a class of morphisms of triples stable under composition and base change.*

Then the following properties are equivalent :

(1) *Any t-hypercovering $\mathbb{Y}_\bullet \to \mathbb{X}$ is u.c.d. (resp. De Rham u.c.d.)*
(2) *Any morphism $\mathbb{Y} \to \mathbb{X}$ in t is u.c.d. (resp. De Rham u.c.d.)*

(idea). We have to prove $2 \Rightarrow 1$.

Take a t-hypercover $w_\bullet : \mathbb{Y}_\bullet \to \mathbb{X}$. By hypothesis, $\operatorname{cosk}_0^{\mathbb{X}}(\mathbb{Y}_\bullet) \to \mathbb{X}$ is u.c.d.

Now we use the following diagram

$$\begin{tikzcd} \operatorname{cosk}_n^{\mathbb{X}}(\mathbb{Y}) \arrow[rr, "\phi_n"] \arrow[dr, "w^{(n)}"'] & & \operatorname{cosk}_{n-1}^{\mathbb{X}}(\mathbb{Y}) \arrow[dl, "w^{(n-1)}"] \\ & \mathbb{X} & \end{tikzcd}$$

We can generalise the preceding theorem to the case of the morphisms of type ϕ_n. Thus we obtain recursively that

$$\operatorname{cosk}_n^{\mathbb{X}}(\mathbb{Y}) \to \mathbb{X}$$

is u.c.d. Now we use the fact the complex

$$\mathbb{R}w_\bullet * w_\bullet^\dagger E$$

can be all reconstructed using the complexes

$$\mathbb{R}\varprojlim_{n\in\mathbb{N}} w_*^{(n)} w^{(n)\dagger} E.$$

□

5. The proof of the proper descent theorem

We now prove the theorem of Tsuzuki :

Theorem 2. *Let $\mathbb{X}$ be a triple.*
Any proper hypercovering $w_\bullet : \mathbb{Y}_\bullet \to \mathbb{X}$ is u.c.d. and De Rham u.c.d.

We won't bother about the exactitude of $\tilde{w}_\bullet$ according to proposition 1.1.
We can reduce this theorem to the following :

Theorem 3. *Let $w : \mathbb{Y} \to \mathbb{X}$ be a proper covering such that $\bar{w}$ is surjective. Then w is c.d.*

Proof. Using proposition 4.5, we reduce to the case of a proper covering $w : \mathbb{Y} \to \mathbb{X}$.

Using that strict Zariski covering are c.d. and the theorem 1, we may assume $\bar{w}$ is of finite type.

A general argument using the Hoge filtration of $\mathrm{DR}(\mathbb{Y}/\mathbb{X};(E,0))$ shows that u.c.d. for w imply De Rham u.c.d. for w.

Finally using the second example of 4.4, we can assume $\bar{w}$ is surjective. □

Using the fundamental properties of the previous subsections, we can reduce to the following cases :

(1) $\bar{w}$ is a closed covering.
(2) $\bar{Y}/\bar{X}$ is a projective space and $\bar{w}$ is the canonical projection.
(3) $\bar{X} = \mathrm{Spec}(A)$, $\bar{Y} = \mathrm{Spec}(A[x]/(f))$ for a monic polynomial $f \in A[x]$ and $\bar{w}$ is the natural finite morphism.
(4) $\bar{X}$ and $\bar{Y}$ are integral and $\bar{w}$ is birational.

In all these cases we put $\mathbb{Y}_\bullet = \operatorname{cosk}_0^{\mathbb{X}}(\mathbb{Y})$.
We give just rough ideas :

1. The idea is to reduce to a case where $\tilde{w} :]\bar{Y}[\to]\bar{X}[$ is quasi-Stein. In this case, we have $\mathbb{R}^r \tilde{w}_{q*} w_q^\dagger E = 0$ for $r > 1$.
   Then we consider the factorisation
   $$E \to \mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; w_\bullet^\dagger E) \to \mathbb{R}\mathcal{C}^\dagger(\mathbb{X}, \mathbb{Y}_\bullet; w_\bullet^\dagger E).$$
   The second arrow is an isomorphism using the vanishing of its second line by Kiel's theorem B. We may also assume w is strict and $\hat{w}$ is affine which imply the first arrow is an isomorphism.

2. We prove more generally if w is a strict triple such that $\hat{w}$ is flat and $\bar{w}$ is surjective, the w is c.d. This in turn follows from Tate's acyclicity theorem.
3. Follows from an induction on the degree of f and the bvoe flat descent result.
4. We can reduce to the case where $\bar{w}$ is a blow-up with respect to an ideal generated by two elements. Then, it follows from the explicit formula of blow-up.

Then we make an induction on the dimension d of $\bar{Y}$ which allow to reduce to these cases. The case $d = 0$ again follows from the flat descent result.

6. Applications

6.1. **Descent spectral sequence.** We state the optimal form of the descent spectral sequence :

Theorem 4. *Let X be a separated finite type k-scheme and $g_\bullet : Y_\bullet \to X$ be a proper hypercovering of X. Then for any overconvergent isocristal E on X/K, there exists a spectral sequence*

$$E_1^{p,q} = H^q_{rig}(Y_q/K; g_{q*}E) \Rightarrow H^{p+q}_{rig}(X/K; E).$$

The proof of this theorem rely on the proper descent theorem for triples. Ideally we have to find a triple $\mathbb{X}$ over X and a simplicial triple $\mathbb{Y}_\bullet \to \mathbb{X}$ over $Y_\bullet \to X$.

Such an ideal situation cannot be achieved. Instead, we have first to try to refine the covering $Y_\bullet \to X$ such that the refinement can be covered by a proper hypercovering of pairs of $(X, \bar{X})$ for a completion $\bar{X}$ of X. This is possible only in a particular case where the hypercovering iq split and we did it inductively on the coskeleton of $Y_\bullet$.

Then for a proper hypercovering $(Y_\bullet, \bar{Y}_\bullet) \to (X, \bar{X})$ of pairs, we do the same thing which works even in the non split case.

Suppose we are now in the ideal situation stated above. We have a proper hypercovering $\mathbb{Y}_\bullet \to \mathbb{X}$. Let $\mathcal{S} = (k, k, \mathcal{V})$.

Using the proper descent theorem, we obtain a canonical isomorphism

$$\mathbb{R}^n\mathcal{C}^\dagger(\mathcal{S}, \mathbb{Y}_\bullet; \mathrm{DR}(\mathbb{Y}_\bullet/\mathcal{S}; w_\bullet^\dagger(E, \nabla))) = H^n_{rig}(X/K; E).$$

Now recall $\mathbb{R}\mathcal{C}^\dagger(\mathcal{S}, \mathbb{Y}_\bullet; \mathrm{DR}(\mathbb{Y}_\bullet/\mathcal{S}; (E, \nabla)))$ is the total complex of a bicomplex. Thus we can filtrate this complex diagonally. The spectral sequence obtained is the one of the theorem.

6.2. **Finiteness of rigid cohomology.** Let X be a separated k-scheme of finite type.

Using De Jong's lateration theorem,
(**) there exist a proper hypercover $Y_\bullet \to X$ such that every Y_n is smooth over a purely inseparable finite extension k_n of k.

Proposition 6.1. *Let X be a separated k-scheme of finite type.*
Then for all $r \in \mathbb{N}$, $H^r_{rig}(X/K)$ is of finite type over K.

Proof. Place ourselves ine the situation of (**). Thus, $Y_n = (Y_n \times_k k_n)_red$ and we have

$$H^r_{rig}(Y_n/K_n) = H^r_{rig}(Y_n \times_k k_n/K_n) = H^r_{rig}(Y_n/K) \otimes_K K_n$$

by base change. The first vector space is of finite type over K_n as Y_n/k_n is smooth using the result of Berthelot. Thus $H^r_{rig}(Y_n/K)$ is of finite type over K and the result follows from the descent spectral sequence Thus the finiteness follows from the descent spectral sequence for $Y_\bullet/X$ and the finiteness theorem of $H^r(Y_s/K)$ of Berthelot. □

Proposition 6.2. *Let X be a separated k-scheme of finite type.*

Let σ be a Frobenius endomorphism of K. Then the Frobenius morphism in rigid cohomology

$$\Phi : \sigma^* H^r_{rig}(X/K) \to H^r_{rig}(X/K)$$

is an isomorphism for all $r \in \mathbb{N}$.

Proof. Suppose k is a perfect field. Let consider the situation (**). Then $k_n = k$. THe spectral sequence associated with $Y_\bullet/X$ is compatible with the Frobenius. Thus we are reduced to the case where X/k is smooth. Thne the assertion follows from Poicar duality.

We reduce to the case of a perfect field by considering an extension K'/K of complete discrete valuation field with perfect residue field k' and using once again the base change. □

Remark 6.3. These two propositions are equally true for rigid cohomology with compact support using the existence of compactifiations and the localisation long exact sequence.

Fix an isomorphism $\iota : \bar{\mathbb{Q}}_p \to \mathbb{C}$. Suppose K is a subfield of $\bar{\mathbb{Q}}_p$ with finite residue field k with p^a elements. Let $\sigma : K \to K$ be a Frobenius endomorphism such that $\sigma^a = 1$. For any X/k seperated of finite type, we define a Frobenius endomophism

$$\Phi_a : H^r_{rig}(X/K) \to H^r_{rig}(X/K)$$

which is an isomorphism using the preceding proposition.

Recallthat we say $H^r_{rig}(X/K)$ is ι-mixed (resp. ι-pure) of weight less than n (resp. weight n) if for any eignevalue α of Φ_a, $\iota(\alpha)$ is an algebraic number such that $|\iota(\alpha)| = p^{at/2}$ for $t \leq n$ (resp. $t = n$).

Theorem 5. *Let X be a proper k-scheme of finite type.*

Then $(H^r_{rig}(X/K), \Phi_a)$ is ι-mixed of weight less than r.

If X is smooth over k, it is ι-pure of weight r.

Proof. We can apply the theorem of Katz-Messing to rigid cohomology as it satisfies Poincar duality (Berthelot), weak Lefschetz (Chiarellotto) and the Lefschetz fixed point formula (Etesse, Le Stum). Thus the theorem follows in the case where X/k is projective smooth.

There exist a proper hypercovering $Y_\bullet \to X$ such that $Y_n/$ is projective surjective using De Jong and Chow lemma. The conclusion follows from the descent spectral sequence. □

As a consequence, we obtain that the descent spectral sequence of a proper hypercovering $Y_\bullet \to X$ with Y_n smooth over k is degenerated at the E_2-term.

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