# FINITE CORRESPONDANCES AND TRANSFERS OVER A REGULAR BASE

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Abstract. In this article, we establish the base of the theory of motivic complexes of V. Voevodsky, more precisely the theory of sheaves with transfers which is exposed in [FSV00], chap. 2 and chap. 4. We intended to give a full treatment of the theory, that is full detailed proofs with the less possible references.

The purpose is twofold. First, we establish the theory over a general regular noetherian base by using the Tor formula for intersection multiplicities of [Ser58]. Secondly we give all the proofs of [FSV00] in the case of a perfect field. Though this relies on the ideas of [FSV00], the exposition differs notably as we consider solely the Nisnevich topology (instead of Zariski) and we use directly correspondances up to homotopy.

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## General notations and conventions

All schemes in this paper are implicitly assumed to be noetherian.

We simply say smooth (resp. étale) where one should say smooth (resp. étale) of finite type for a scheme over a base or a morphism. For any scheme $S$, we will denote by $\mathscr{S}m_S$ the category of smooth $S$-schemes. We will also consider the category $\mathscr{S}m_S^{\mathrm{cor}}$ of smooth $S$-schemes equipped with finite $S$-correspondences with its canonical graph functor$\gamma : \mathscr{S}m_S \rightarrow \mathscr{S}m_S^{\mathrm{cor}}$ (see def. 1.19).

Every presheaf (or sheaf) considered in this paper is assumed to be a presheaf of abelian groups, unless explicitly stated otherwise.

Let $S$ be a scheme and $F$ be a presheaf over $\mathscr{S}m_S$. We will extend the presheaf $F$ to the category of pro-smooth $S$-schemes in the obvious way : if $X_\bullet = (X_i)_{i\in\mathcal{I}}$ is such a pro-object we put

$$F(X_\bullet) = \varinjlim_{i\in\mathcal{I}^{op}} F(X_i),$$

the colimit being computed in the category of abelian groups.

We will identify isomorphic pro-smooth $S$-schemes. This implies that a pro-object $X_\bullet$ which admits a limit in the category of $S$-schemes is determined uniquely by this limit (because the terms of the pro-object are of finite presentation over $S$). If for example this limit is affine over $S$, we can consider $X_\bullet$ as a pro-object of affine $S$-schemes. We often adopt the suggestive notation

$$X_\bullet = \tilde{\varprojlim_{i\in\mathcal{I}}} X_i$$

to denote a pro-smooth scheme $(X_i)_{i\in\mathcal{I}}$ where $\mathcal{I}$ is a right filtering essentially small category. The symbol $\tilde{\varprojlim}$ indeed denotes the projective limit taken tautologically in the category of pro-smooth schemes.

The topology used to compute associated sheaves, cohomology or Čech cohomology, is the Nisnevich topology, unless explicitly stated otherwise.

We will denote by $\mathscr{N}_S^{\mathrm{tr}}$ the category of sheaves with transfers over $S$ (see def. 2.3) and by $\mathrm{H}\mathscr{N}_S^{\mathrm{tr}}$ the subcategory of sheaves with transfers which are homotopy invariant. Such sheaves will simply be called "homotopy sheaves". This terminology is inspired by the theory of perverse sheaves. Indeed, at least in the case of a perfect base field $k$, the category $\mathrm{H}\mathscr{N}_k^{\mathrm{tr}}$ is the heart of the category $DM_-^{eff}(k)$ for the homotopy t-structure[1].

[1]Note that the homotopy t-structure is also defined by Morel on the stable homotopy category of schemes over $k$. Its heart is indeed the category of homotopy invariant sheaves but these do not have transfers in general, in the sense given here. Thus the correct

## Introduction

The generalisation of the theory of sheaves with transfers over a regular base is presented in the first two sections.

In the first section we prove all the basic facts concerning finite correspondences over a regular base, using only Serre's Tor formula for intersection multiplicities. The original part here is the study of the functoriality of finite correspondences, the base change and "forget the base" functor, in subsection 1.5.

In the second section, we develop the theory of sheaves with transfers over a regular base. The arguments are very close to those of [FSV00], chap. 4, apart from being more precise. However, the study of the functoriality is new.

In the rest of the paper, we prove the fundamental facts about homotopy invariant sheaves with transfers over a perfect field. This proof follows the main ideas of [FSV00], chap. 2, but it seems to us more accurate for two reasons.

First, we interpret the constructions about pretheories in the framework of correspondences up to homotopy, in section 3. This shows in particular how the arguments of Voevodsky are related to cycles and even consist of constructing cycles. See especially Proposition 3.21 and Theorem 3.23. Proposition 3.21 is a corrected version of a result of [FSV00], chap. 2 (which is valid only in the case of an infinite field). Theorem 3.23 is a generalisation of a result in *loc. cit.* which allows us to use the Nisnevich topology in what follows. In this part, we use the functoriality of finite correspondences.

Secondly, we only use the Nisnevich topology. In the study of homotopy invariance and the category of sheaves with transfers, this allows us to give another proof of a fundamental result: the associated sheaf functor preserves homotopy invariance for sheaves with transfers (cf Corollary 4.14). The strategy of the proof is to use the Nisnevich Čech cohomology functor together with the computation done in Proposition 4.10 (this computation is a slightly more precise version of Proposition 5.4 of *loc. cit.*).

Another fundamental theorem states that a homotopy invariant sheaf with transfers over a perfect field has homotopy invariant cohomology. We prove this important theorem of Voevodsky in the last section. The argument of the proof is completely due to Voevodsky. Our work consisted in establishing the preliminary facts clearly (in full generality), specifically the construction of the localisation long exact sequence for homotopy sheaves (see section 5.3). These facts are slightly different from the analogue in *loc. cit.* as we use the Nisnevich topology. In particular, we have to use our generalisation of the theory established in Theorem 3.23. We hope that the reader will be able to appreciate the beautiful argument of Voevodsky in this proof, which consists in proving both localisation and homotopy invariance for cohomology (see 5.4 for more details).

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terminology for the object of $\mathrm{H}\mathscr{N}_S^{\mathrm{tr}}$ should be "homotopy sheaves with transfers" or "homotopy oriented sheaves", but there is no risk of confusion here.

## Future development

The intention of the author in establishing the theory in that generality is to construct the category of motivic complexes over a general regular base and, more importantly, to give full functoriality for this category. This will be done in a future work of D.-C. Cisinski and the author (cf [CD]).

For the full functoriality, we intend to use the recent work of J.Ayoub on cross functors to get the six functors formalism for motivic complexes. First of all, we have to construct a non effective version of motivic complexes, which can be done by considering symetric spectra of complexes. Still, there is a technical problem as the theory developped here allows only regular base schemes. However, though it is not obvious, one can do the work of Ayoub with only those regular bases. Nonetheless, the choice of the author is to generalize motivic complexes to arbitrary base by using the theory of relative cycles of [FSV00] - which was surely the original intention of Voevodsky.

## 1. Finite correspondences

### 1.1. **Relative cycles.**

In this section, we use a particular case of the notion of a relative cycle from [FSV00], chap. 2, by restricting to the case of a regular base. We set up the foundations of the theory in this case using [Ser58] and [Ful98].

1.1.1. *Definition.* We begin by recalling a few facts about equidimensionality :

**Definition 1.1.** Let $f : X \to S$ be a morphism. One says that $f$ is equidimensional if :

(1) $f$ is of finite type.
(2) the relative dimension of $f$ is constant.
(3) Every irreducible component of $X$ dominates an irreducible component of $S$.

If $S$ is regular, normal or more generaly geometrically unibranch, one obtains an equivalent set of conditions by replacing the third property with the stronger one of being universally open (cf [GD66, 14.4.4]).

The following lemma allows us to simplify this notion in the particular case which we are interested in :

**Lemma 1.2.** *Let $X$, $S$ be irreducible schemes, $S$ geometrically unibranch. The following conditions on a morphism $f : X \to S$ are equivalent :*

(1) *$f$ is finite equidimensional.*
(2) *$f$ is finite onto.*
(3) *$f$ is proper, equidimensional of dimension* 0.

*Proof.* Conditions (1) and (2) are equivalent because a finite morphism is of constant relative dimension 0 and universally closed. The equivalence between (1) and (3) follows from the Stein factorisation. □

We will use the following notion from the general theory of [SV00], which is all what we need for our constructions :

**Definition 1.3.** Let $S$ be a regular scheme and $X$ an $S$-scheme. We define the abelian group $\mathrm{c}_0(X/S)$ as the subgroup of the group of cycles on $X$ generated by points $x$ whose closure in $X$ is finite equidimensional over $S$. The elements of this group are called finite relative cycles on $X/S$.

*Remark* 1.4. In *loc. cit.*, the cycles defined above are denoted by $c_{equi}(X/S,0)$, and are called equidimensional relative cycles of relative dimension 0 on $X/S$.

A cycle $\alpha$ on $X$ is a finite relative cycle on $X/S$ if its support is finite equidimensional over $S$ (Lemma 1.2).

Remark also that if $S = S_1 \sqcup S_2$, $\mathrm{c}_0(X/S) = \mathrm{c}_0(X/S_1) \oplus \mathrm{c}_0(X/S_2)$. This allows us to reduce to the case $S$ irreducible.

**1.5.** Let $S$ be a regular scheme and $X$ an $S$-scheme. Consider a closed subscheme $Z$ of $X$ which is finite equidimensional over $S$. The irreducible components of $Z$ which dominate an irreducible component of $S$ are finite equidimensional over $S$.

Let $(z_i)_{\{i=1,\dots,n\}}$ be the generic points of $Z$ which dominate an irreducible component of $S$. One associates to $Z$ a finite relative cycle on $X/S$ :

$$[Z]_{X/S} = \sum_i \lg(\mathcal{O}_{Z,z_i}).z_i.$$

1.1.2. *Pullback.* Let $S$ and $T$ be regular schemes and consider a cartesian square

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "p"] \\ T \arrow[r, "g"'] & S \end{tikzcd}$$

with $p$ smooth.

Let $\alpha$ be a finite relative cycle on $X/S$ and $U$ be its support. We show that the pullback cycle $f^*(\alpha)$ is well defined in the sense of [Ser58], that is, the hypothesis of V.C.7 (b) is satisfied[2].

As $U$ is finite equidimensional over $S$, $V = f^{-1}(U)$ is again finite equidimensional over $T$. Suppose that $U$ is irreducible. Considering the irreducible component of $X$ containing $U$ and its image on $S$, we can suppose that $X$ and $S$ are irreducible. As the morphism $p$ is smooth, it is equidimensional of dimension $n$ and the codimension of $U$ in $X$ is $n$. Moreover, the morphism $q$ is equidimensional of dimension $n$ and the codimension of $V$ in $Y$ is $n$. This proves that $f^*(\alpha)$ is well defined. Moreover, it is a finite relative cycle on $Y/T$.

**Definition 1.6.** With the preceding notations, we will put $\Delta^*(\alpha) = f^*(\alpha)$ as a cycle in $\mathrm{c}_0(Y/T)$.

Using [Ser58], V.C.7 exercice 1, we obtain functoriality of this pullback with respect to composition of cartesian squares.

[2]More precisely, we consider the extension of the theory presented in [Ser58] to the case of arbitrary noetherian regular schemes, as described in V.C.8. Moreover, we don't need Theorem 1 of V.B.3 as the correct equality of dimension is already true in our case. Besides, the positivity of intersection multiplicities for arbitrary regular noetherian local rings has been proved recently by O.Gabber.

We note also the following lemma :

**Lemma 1.7.** *Let $S$, $T$ be regular schemes. Consider a cartesian square*

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "p"] \\ T \arrow[r, "g"'] & S \end{tikzcd}$$

*such that $p$ is smooth and $g$ is flat. Then, for any closed subscheme $Z$ in $X$ which is finite equidimensional over $S$,*

$$\Delta^*([Z]_{X/S}) = [Z \times_S T]_{Y/T}.$$

*Proof.* Put $\alpha = [Z]_{X/S}$ and let $\Gamma_f$ be the graph of $f$ as a closed subscheme of $Y \times X$. As $Y$ is regular, using [Ser58], V.C.8, $\Delta^*(\alpha) = [\Gamma_f].[Y \times \alpha]$. But $\Gamma_f$ is isomorphic to $Y$, hence it is a regular scheme. Its local rings are all Cohen-Macaulay local rings and the result now follows from [Ful98], Prop. 7.1. (See also [Ser58], V.C.1 Th. 1 for the identification between Serre's and Samuel's intersection multiplicities.) □

1.1.3. *Compactness of relative cycles.* Consider $(T_i)_{i\in\mathcal{I}}$, a pro-object of affine regular noetherian $S$-schemes. It admits a limit $\mathcal{T}$ in the category of affine $S$-schemes. Indeed, $T_i$ is the spectrum over $S$ of a coherent $\mathcal{O}_S$-algebra $\mathcal{A}_i$. The family $(\mathcal{A}_i)_{i\in I^{op}}$, together with its natural transition morphisms, forms an ind-$\mathcal{O}_S$-algebra. It admits a limit, and we put $\mathcal{A} = \varinjlim_{i\in\mathcal{I}^{op}} \mathcal{A}_i$. Then $\mathcal{T} = \mathrm{Spec}_S(\mathcal{A})$.

Note that $\mathcal{T}$ need not be noetherian nor regular[3].

**Proposition 1.8.** *We adopt the notations above and suppose $\mathcal{T}$ is regular noetherian. Let $X$ be a smooth $S$-scheme. We put $X_i = X \times_S T_i$, $X_\mathcal{T} = X \times_S \mathcal{T}$ and consider the cartesian square*

$$\begin{tikzcd} X_\mathcal{T} \arrow[r, "f_i"] \arrow[d] \arrow[dr, phantom, "\Delta_i"] & X_i \arrow[d] \\ \mathcal{T} \arrow[r, "g_i"'] & T_i. \end{tikzcd}$$

*Then the morphism $\delta = \varinjlim_{i\in\mathcal{I}^{op}} \Delta_i^* : \varinjlim_{i\in\mathcal{I}^{op}} \mathrm{c}_0(X_i/T_i) \to \mathrm{c}_0(X_\mathcal{T}/\mathcal{T})$ is an isomorphism.*

*Proof.* Let us first prove that $\delta$ is surjective. Let $Z$ be a closed integral subscheme of $X_\mathcal{T}$, finite equidimensional over $\mathcal{T}$. Then $Z$ is defined by a quasi-coherent ideal of $\mathcal{O}_{X_\mathcal{T}}$. As $X_\mathcal{T}$ is noetherian this ideal is coherent, generated by a finite number of sections $f_1, ..., f_n$. The sheaf $\mathcal{O}_{X_\mathcal{T}}$ is the inductive limit of the $\mathcal{O}_{X_i}$, and we can assume that there exists $i \in I$ such that $f_1, ..., f_n$ lift to $\mathcal{O}_{X_i}$. Let $Z_i$ be the closed subscheme of $X_i$ defined by

[3] If $\mathcal{T}$ is noetherian, the author is aware essentially of two hypotheses that imply that $\mathcal{T}$ is regular: (1) the transition morphisms of $(T_i)_{i\in I}$ are flat, or (2) $S$ is of equal characteristic.

the equations $f_1 = 0, ..., f_n = 0$. Note that the square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X_{\mathcal{T}} \arrow[d, "p_i \times_S 1_X"] \\ Z_i \arrow[r] & X_i \end{tikzcd}$$

is cartesian. Denote by $\mathcal{I}/i$ the category whose objects are the arrows $j \to i$. Considering such an arrow, we let $Z_j$ be the pullback of $Z_i$ along the corresponding morphism. This defines a pro-object $(Z_j)_{j\in\mathcal{I}/i}$ such that the canonical morphism

$$Z \to \varprojlim_{j\in\mathcal{I}/i} Z_j$$

is an isomorphism. Using [GD66] there exists an arrow $j \to i$ such that :

1. (1) $Z_j$ is integral, using Cor. 8.4.3 of *loc.cit.*, because the transition morphism of the pro-object $(Z_j)_{j\geq i}$ are dominant.
2. (2) $Z_j$ is finite surjective over a component of $T_j$, using 8.10.5 of *loc.cit.*.

Thus the cycle $[Z_j]$ of $X_j$ associated to $Z_j$ is a finite relative cycle on $X_j/T_j$. As $f_j^{-1}(Z_j) = Z$ is an integral scheme, we obtain $f_j^*([Z_j]) = [Z]$, that is, $\Delta_j^*([Z_j]) = [Z]$.

We finally show that $\delta$ is injective. Let $i \in \mathcal{I}$ and $\alpha_i \in \mathrm{c}_0(X_i/T_i)$ such that $\delta(\alpha_i) = 0$. For any $j \to i$ in $\mathcal{I}$ with corresponding transition morphism $f_{ji} : X_j \to X_i$, let $Z_j$ be the support of $f_{ji}^*(\alpha)$.

Then $(Z_j)_{j\in\mathcal{I}/i}$ is a pro-object. As $f_i^*(\alpha) = 0$ this pro-object has the empty scheme as limit. This means that the canonical morphism

$$\emptyset \to \varprojlim_{j\in\mathcal{I}/i} Z_j$$

is an isomorphism. From the first point 8.10.5 of *loc. cit.* there exists $j \to i$ such that $Z_j = \emptyset$. Thus $f_{ji}^*(\alpha) = 0$, which shows $\alpha_i$ is 0 in the colimit $\varinjlim_{i\in\mathcal{I}^{op}} \mathrm{c}_0(X_i/T_i)$. □

1.1.4. *General pushout.* One of the advantages of relative cycles is that pushout by any morphism is always defined and functorial.

**Lemma 1.9.** *Let $S$ be an irreducible scheme and $f : X \to Y$ be a morphism of finite type $S$-schemes.*

*Let $Z$ be a closed integral subscheme of $X$. If $Z$ is finite and surjective over $S$ then $f(Z)$, equipped with its reduced structure of subscheme in $Y$, is closed and finite surjective over $S$. The morphism $Z \to f(Z)$ is finite surjective.*

*Proof.* Indeed, as $Z/S$ is proper, $f(Z)$ is closed in $Y$. With its induced structure of reduced subscheme of $Y$ it is proper over $S$, as can be seen for example from the valuative criterion of properness (cf [Har77]). Moreover using [GD63] 4.4.2, $f(Z)$ is finite over $S$ because its fibers are finite. Thus the induced morphism $Z \to f(Z)$ is finite. □

**Definition 1.10.** Let $S$ be a scheme, $X$ and $Y$ be finite type $S$-schemes and $f : Y \to X$ be an $S$-morphism.

For $Z$ a closed integral subscheme of $X$ which is finite and equidimensional over $S$, we set according to the preceding lemma

$$f_*([Z]) = d.[f(Z)] \in \mathrm{c}_0(X/S)$$

where $d$ is degree of the extension of function fields induced by $f$. By linearity, this defines a morphism $f_* : \mathrm{c}_0(Y/S) \to \mathrm{c}_0(X/S)$.

**1.11.** This pushout coincides with the one of [Ser58, V-27.6]. As it is always reduced to a pushout by a proper morphism according to the preceding lemma, it is functorial in $f$. This is easily seen directly using the transitivity of degree extensions.

**Proposition 1.12.** *Let $S$, $S'$ be regular schemes and $q : S' \to S$ be a flat morphism. Consider the two following cartesian squares of schemes :*

$$\begin{tikzcd} X' \arrow[r, "t"] \arrow[d] \arrow[dr, phantom, "\Theta"] & X \arrow[d] \\ Y' \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\Delta"] & Y \arrow[d] \\ S' \arrow[r, "q"'] & S. \end{tikzcd}$$

*We denote by $\Delta \bullet \Theta$ the cartesian square defined by the external arrows of this diagram. Then for all finite relative cycle $\alpha \in \mathrm{c}_0(X/S)$, we have the relation*

$$g_*(\Delta \bullet \Theta)^*(\alpha) = \Delta^* f_*(\alpha).$$

*Proof.* Using linearity, we can assume that $\alpha$ is a closed integral subscheme $Z$ of $X$. The cycle $f_*(\alpha)$ is supported in $f(Z)$. Thus we can assume $Y$ equals $f(Z)$ and $f$ is proper. Finally, using Lemma 1.7 we reduce to the classical projection formula of [Ful98, 1.7]. □

We state another projection formula involving intersection products that will be useful in the sequel.

**Proposition 1.13.** *Let $X$, $X'$ and $S$ be regular schemes and consider the diagram*

$$\begin{tikzcd} & S & \\ Y' \arrow[ur] \arrow[rr, "f"] \arrow[d] \arrow[drr, phantom, "\Delta"] & & Y \arrow[ul] \arrow[d, "p"] \\ X' \arrow[rr, "g"'] & & X \end{tikzcd}$$

*where $\Delta$ is cartesian and $p$ smooth. Let $\sigma \in \mathrm{c}_0(Y/X)$, $\epsilon \in \mathrm{c}_0(Y'/S)$.*

*Then, the following equation holds when the intersections involved are proper :*

$$f_*\big(\Delta^*(\sigma).\epsilon\big) = \sigma.f_*(\epsilon).$$

*Proof.* Let $V$ be the support of $\epsilon$. As $V$ is proper over $S$, the restriction $f|_V : V \to Y$ is proper using the arguments preceding Definition 1.10. Thus the formula is simply formula (10) of [Ser58], V.C.8. □

1.2. **Composition of finite correspondences.** Let $S$ be a regular scheme. Generalising the definition of [Voe00a], we introduce finite $S$-correspondences.

**Definition 1.14.** Let $X$ and $Y$ be two smooth $S$-schemes. We define the group of finite $S$-correspondences from $X$ to $Y$ as the abelian group

$$\mathrm{c}_S\,(X,Y) = \mathrm{c}_0(X \times_S Y/X)\,.$$

We adopt the following notation. If $X$ and $Y$ are two $S$-schemes, we put $XY = X \times_S Y$ and we denote by $p_X^{XY} : XY \to X$ the canonical projection. If $X$, $X'$, $Y$, $Y'$ are smooth schemes, we denote by $p_{(X)Y}^{(XX')YY'}$ the cartesian square

$$\begin{tikzcd} XX'YY' \arrow[r] \arrow[d] & XY \arrow[d] \\ XX' \arrow[r] & X \end{tikzcd}$$

induced by the canonical projections.

The following lemma shows that the composition law of finite correspondences is well defined.

**Lemma 1.15.** *Let $X$, $Y$, $Z$ be smooth $S$-schemes, $\alpha \in \mathrm{c}_S\,(X,Y)$ and $\beta \in \mathrm{c}_S\,(Y,Z)$.*

1. *The cycles $p_{(Y)Z}^{(XY)Z\,*}(\beta)$ and $p_{(X)Y}^{(X)Y(Z)\,*}(\alpha)$ of $XYZ$ intersect properly.*
2. *According to the first point, the intersection product in $XYZ$*
$$p_{(Y)Z}^{(XY)Z\,*}(\beta).p_{(X)Y}^{(X)Y(Z)\,*}(\alpha)$$
*is well defined. It is a finite relative cycle on $XYZ/X$.*

*Proof.* We can assume that $\alpha$ and $\beta$ are closed integral subschemes. Then we can assume also that $X$, $Y$ and $Z$ are irreducible, considering the particular component dominated by $\alpha$ respectively $\beta$.

Consider the following diagram :

$$\begin{tikzcd} \beta \times_Y \alpha \arrow[r] \arrow[d] & \beta \arrow[r] \arrow[d] & Z \\ \alpha \arrow[r] \arrow[d] & Y & \\ X. & & \end{tikzcd}$$

The vertical arrows are all finite equidimensional. In particular, $\beta \times_Y \alpha$ is finite equidimensional over $X$. But there is a canonical isomorphism

$$\beta \times_Y \alpha \to (X\beta) \times_{XYZ} (\alpha Z) = (X\beta) \cap (\alpha Z).$$

In particular, every component of $(X\beta) \cap (\alpha Z)$ is finite equidimensional over $X$. Thus, they are all of codimension $\dim(Y) + \dim(Z)$ in $XYZ$, which proves the first point. The second point then follows from the preceding remark. $\square$

**Definition 1.16.** Let $X$, $Y$ and $Z$ be smooth $S$-schemes, and $\alpha \in \mathrm{c}_S\,(X,Y)$ and $\beta \in \mathrm{c}_S\,(Y,Z)$. Consider the pushout
$p_{(X)Z\;*}^{(X)YZ} : \mathrm{c}_0(XYZ/X) \to \mathrm{c}_0(XZ/X)$ defined in 1.10.

We put
$$\beta\circ\alpha p_{(X)Z\;*}^{(X)YZ}\Big(p_{(Y)Z}^{(XY)Z*}(\beta).p_{(X)Y}^{(X)Y(Z)*}(\alpha)\Big),$$
which is a well defined finite $S$-correspondence from $X$ to $Z$ by the preceding lemma.

**Example 1.17.** Let $f : X \to Y$ be an $S$-morphism between smooth $S$-scheme. As $Y/S$ is separated, the $S$-graph $\Gamma_f$ of $f$ is a closed subscheme of $XY$. As the canonical projection $\Gamma_f \to X$ is an isomorphism, the cycle $[\Gamma_f]_{XY/X}$ belongs to $\mathrm{c}_S(X,Y)$. This allows us to define a map
$$\mathrm{Hom}_{\mathscr{S}m_S}(X,Y) \to \mathrm{c}_S(X,Y),$$
which is obviously injective.

We will use the same letter $f$ to denote the finite $S$-correspondence $[\gamma_f]_{XY/X}$. This identification is justified by following lemma, which shows that composition of $S$-morphisms coincides with composition of finite $S$-correspondences.

**Lemma 1.18.** *Let $X$, $Y$, $Z$ be smooth $S$-schemes. Then the following relations hold :*

(1) *For all $\alpha \in \mathrm{c}_S(X,Y)$, $\beta \in \mathrm{c}_S(Y,Z)$, $\gamma \in \mathrm{c}_S(Z,T)$, $\gamma\circ(\beta\circ\alpha) = (\gamma\circ\beta)\circ\alpha$.*

(2) *For all $\alpha \in \mathrm{c}_S(X,Y)$ and all $S$-morphisms $f : Y \to Z$,*
$$f\circ\alpha = (1_X \times_S f)_*(\alpha)$$
*using Definition 1.10 for the pushout.*

(3) *For all $\beta \in \mathrm{c}_S(Y,Z)$ and all $S$-morphisms $f : X \to Y$, considering the cartesian square $f_Z :$*
$$\begin{tikzcd} XZ \arrow[r] \arrow[d] & YZ \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
*, one has*
$$\beta\circ f = f_Z^*(\beta)$$
*using definition 1.6 for the pullback.*

(4) *For all $S$-morphisms $f : X \to Y$ and $g : Y \to Z$, $[\Gamma_g]\circ[\Gamma_f] = [\Gamma_{g\circ f}]$.*

*Proof.* (1). The idea to prove this relation is to show that one can compose the three correspondences by pulling them all back to $XYZT$, forming the intersection product in that scheme and then pushing down the result to $XT$. We carry out this procedure in detail :

$$\gamma\circ(\beta\circ\alpha) = \gamma\circ\left(p_{XZ\;*}^{XYZ}\left(p_{(Y)Z}^{(XY)Z*}(\beta).p_{(X)Y}^{(X)Y(Z)*}(\alpha)\right)\right)$$
$$= p_{XT\;*}^{XZT}\left(p_{(Z)T}^{(XZ)T*}(\gamma).p_{(X)Z}^{(X)Z(T)*}p_{XZ\;*}^{XYZ}\left(p_{(Y)Z}^{(XY)Z*}(\beta).p_{(X)Y}^{(X)Y(Z)*}(\alpha)\right)\right)$$
$$= p_{XT\;*}^{XZT}\left(p_{(Z)T}^{(XZ)T*}(\gamma).p_{XZT\;*}^{XYZT}p_{(X)YZ}^{(X)YZ(T)*}\left(p_{(Y)Z}^{(XY)Z*}(\beta).p_{(X)Y}^{(X)Y(Z)*}(\alpha)\right)\right) \quad (a)$$
$$= p_{XT\;*}^{XZT}p_{XZT\;*}^{XYZT}\left(p_{(XZ)T}^{(XYZ)T*}p_{(Z)T}^{(XZ)T*}(\gamma).p_{(X)YZ}^{(X)YZ(T)*}\left(p_{(Y)Z}^{(XY)Z*}(\beta).p_{(X)Y}^{(X)Y(Z)*}(\alpha)\right)\right) \quad (b)$$
$$= p_{XT\;*}^{XYZT}\left(p_{(Z)T}^{(XYZ)T*}(\gamma).\left(p_{(Y)Z}^{(XY)Z(T)*}(\beta).p_{(X)Y}^{(X)Y(ZT)*}(\alpha)\right)\right) \quad (c)$$
$$= p_{XT\;*}^{XYZT}\left(p_{(Z)T}^{(XYZ)T*}(\gamma).p_{(Y)Z}^{(XY)Z(T)*}(\beta).p_{(X)Y}^{(X)Y(ZT)*}(\alpha)\right) \quad (d)$$

with the following justifications :

(a) This is Proposition 1.12 for the cartesian squares
$$\begin{tikzcd} XYZT \arrow[r] \arrow[d] & XYZ \arrow[d] \\ XZT \arrow[r] \arrow[d] & XZ \arrow[d] \\ XT \arrow[r] & X \end{tikzcd}.$$

(b) This is the projection formula 1.13 for
$$\begin{tikzcd} & X & \\ XYZT \arrow[ur] \arrow[rr] \arrow[d] & & XZT \arrow[ul] \arrow[d] \\ XYZ \arrow[rr] & & XZ. \end{tikzcd}$$

(c) This is the functoriality of pullback and pushout, and the compatibility of intersection product with pullback.

(d) This notation is valid using the associativity of intersection product (cf [Ser58], V.C.3.b).

A similar computation works for the right hand side of the equality. This finishes the proof of (1).

(2). We can assume that $\alpha$ is a closed integral subscheme of $XY$. Then we must compute the following cycle :

$$\begin{aligned} f \circ \alpha &= p^{XYZ}_{XZ\,*}\left(p^{XYZ*}_{YZ}[\Gamma_f].p^{XYZ*}_{XY}(\alpha)\right) \\ &= p^{XYZ}_{XZ\,*}\left([X\Gamma_f].[\alpha Z]\right). \end{aligned}$$

The intersection involved in this cycle is particularly simple :

$$\begin{tikzcd} X\Gamma_f \times_{XYZ} \alpha Z \arrow[rr] \arrow[dr, "b"] \arrow[dd, hook] & & \alpha Z \arrow[dr] \arrow[dd, hook] & \\ & X\Gamma_f \times_{XY} \alpha \arrow[rr, "a"] \arrow[dd, hook] & & \alpha \arrow[dd, hook, "i_\alpha"] \\ X\Gamma_f \arrow[rr, hook, "i"] \arrow[dr, equal] & & XYZ \arrow[dr] & \\ & X\Gamma_f \arrow[rr, "\sim", "p"'] & & XY \end{tikzcd}$$

where $p$ is induced by the canonical isomorphism $\Gamma_f \to Y$, and $i$, $i_\alpha$ are the canonical closed immersions.

The front square in this cube is cartesian, so $a$ is an isomorphism. But the back and right squares are both cartesian, which implies that the left square is also cartesian. Hence $b$ is an isomorphism.

This implies that $W = X\Gamma_f \times_{XYZ} \alpha Z$ is isomorphic to $\alpha$. Thus it is an integral scheme. In particular, the intersection of $X\Gamma_f$ and $\alpha Z$ is reduced to the single component $W$. Moreover, the intersection multiplicity of $W$ is 1 using [Ful98], Prop. 7.2. Indeed, a base field is not needed here : we only need the comparison between Serre's and Samuel's intersection multiplicities (the Tor formula). This is [Ser58], V.C.4. We have obtained :

$$\begin{aligned} &[X\Gamma_f]_{XYZ}.[\alpha Z]_{XYZ} = [X\Gamma_f \times_{XYZ} \alpha Z]_{XYZ} \\ &= i_*([X\Gamma_f \times_{XYZ} \alpha Z]_{X\Gamma_f}) = i_*([X\Gamma_f \times_{XY} \alpha]_{X\Gamma_f}) = i_* p^*(\alpha). \end{aligned}$$

We now use the factorization

$$\begin{tikzcd} & X\Gamma_f \arrow[dl, "p"'] \arrow[dr, "i"] & \\ XY \arrow[rr, "1_X \times_S \gamma_f"'] & & XYZ \end{tikzcd}$$

where $\gamma_f : Y \to YZ$ is the graph morphism. Thus $i_* = (1_X \times_S \gamma_f)_* p_*$, which means $(1_X \times_S \gamma_f)_* = i_* p^*$ as $p$ is an isomorphism.

And finally : ${p^{XYZ}_{XZ}}_* ([X\Gamma_f].[\alpha Z]) = {p^{XYZ}_{XZ}}_* (1_X \times_S \gamma_f)_*(\alpha) = (1_X \times_S f)_* \alpha$.

(3). We compute

$$\beta \circ f = p^{(XY)Z*}_{(X)Z} \left( p^{(XY)Z}_{(Y)Z}(\beta).[\Gamma_f Z]_{XYZ} \right)$$

We consider the cartesian square
$$\begin{tikzcd} \Gamma_f Z \arrow[r, "p"] \arrow[d, "\iota"'] & XZ \arrow[d, "f \times_S 1_Z"] \\ XYZ \arrow[r, "p^{XYZ}_{YZ}"] & YZ. \end{tikzcd}$$

Using the definition of [Ser58], V.C.7,

$$p^{(XY)Z}_{(Y)Z}(\beta).[\Gamma_f Z]_{XYZ} = \iota_* \iota^* p^{XYZ*}_{YZ}(\beta) = \iota_* p^*(f \times_Z 1_Z)^*(\beta).$$

As $p$ is an isomorphism, $p^* = (p_*)^{-1}$. Therefore we conclude because $p^{XYZ}_{XZ} \circ \iota = p$.

(4). This follows either from point (2) and functoriality of pushout or point (3) and functoriality of pullback. □

The preceding lemma implies that the product $\circ$ of finite $S$-correspondences is associative, and that for any smooth $S$-scheme $X$ the $S$-morphism $1_X$, seen as a correspondence, is the neutral element.

**Definition 1.19.** Let $S$ be a regular scheme.

We denote by $\mathscr{S}m^{\mathrm{cor}}_S$ the category whose objects are smooth separated $S$-schemes of finite type and whose morphisms are the finite $S$-correspondences.

We denote by $\gamma : \mathscr{S}m_S \to \mathscr{S}m^{\mathrm{cor}}_S$ the faithful functor which is the identity on objects and sends an $S$-morphism to its graph (cf ex. 1.17).

If $X$ is a smooth $S$-scheme, we denote by $[X]$ the corresponding object of $\mathscr{S}m^{\mathrm{cor}}_S$.

This category is additive and for all smooth $S$-schemes $X$ and $Y$, $[X] \oplus [Y] = [X \sqcup Y]$.

### 1.3. Monoidal structure.

**Lemma 1.20.** *Let $X$, $X'$, $Y$, $Y'$ be smooth $S$-schemes.*

*Then for any $\alpha \in \mathrm{c}_S(X,Y)$ and $\beta \in \mathrm{c}_S(X',Y')$, the cycles ${p^{XY}_{XYX'Y'}}^*(\alpha)$ and ${p^{X'Y'}_{XYX'Y'}}^*(\beta)$ intersect properly, and the intersection cycle is a finite relative cycle on $XYX'Y'/XX'$.*

*Proof.* Let assume $\alpha$ et $\beta$ are closed integral subscheme. Consider the diagram :

$$\begin{tikzcd} X\beta \times_X \alpha \arrow[r] \arrow[d] & \alpha \arrow[d] \\ X\beta \arrow[r] \arrow[d] & X \\ XX'. & \end{tikzcd}$$

All vertical arrows are finite equidimensional. But $X\beta \times_X \alpha$ is isomorphic to $XY\beta \cap \alpha X'Y'$. Thus this scheme is finite equidimensional on $XX'$, which implies that the corresponding intersection is proper and concludes the proof. □

**Definition 1.21.** Let $X$, $X'$, $Y$, $Y$' be smooth $S$-schemes.

For all $\alpha \in \mathrm{c}_S(X,Y)$ and $\beta \in \mathrm{c}_S(X',Y')$ we put

$$\alpha \otimes_S^{tr} \beta = p_{XY}^{XYX'Y'*}(\alpha).p_{X'Y'}^{XYX'Y'*}(\beta).$$

From the preceding lemma, this is a well defined cycle and an element of $\mathrm{c}_S(XX',YY')$.

**Lemma 1.22.** *Suppose we are given finite $S$-correspondences :* $\alpha : X \to Y,\ \alpha' : Y \to Z,\ \beta : X' \to Y',\ \beta' : Y' \to Z'$. *Then,*

$$(\alpha' \circ \alpha) \otimes_S^{tr} (\beta' \circ \beta) = (\alpha' \otimes_S^{tr} \beta') \circ (\alpha \otimes_S^{tr} \beta).$$

*Proof.* As for the associativity of composition of finite correspondences, the proof consists of showing that one can first pull back all cycles to $XYZX'Y'Z'$, then take the intersection and push down the result to $XZX'Z'$. As in the proof of the first point of 1.18, we use the two projection formulas 1.12 and 1.13 and the functoriality of pushout and pullback. We leave the details to the reader. □

**Proposition 1.23.** *The category $\mathscr{S}m_S^{\mathrm{cor}}$ is monoidal symetric with tensor product*

$$[X] \otimes_S^{tr} [Y] = [X \times_S Y]$$

*for smooth $S$-scheme $X$ and $Y$ and tensor product of finite $S$-correspondences given by definition 1.21. The functor $\mathscr{S}m_S \xrightarrow{\gamma} \mathscr{S}m_S^{\mathrm{cor}}$ of definition 1.19 is monoidal; the tensor product on $\mathscr{S}m_S$ is the cartesian product over $S$.*

*Proof.* Commutativity is obvious using commutativity of the intersection product (cf [Ser58], V.C.3.a). Associativity is proved in the same way as associativity of composition product : using the projection formulas (cf prop. 1.12 and 1.13), one reduces to associativity of the intersection product.

For the last assertion, it suffices to prove the equality $f \otimes_S^{tr} 1_Y = f \times_S 1_Y$ for smooth $S$-schemes $X$, $X'$, $Y$ and an $S$-morphism $f : X \to X'$. Let $\Delta_Y$ be the diagonal of $Y/S$ and $\Gamma_f$ the $S$-graph of $f$. Following now the same line as in the proof of the third point of Lemma 1.18 we note that the intersection of $\Gamma_f YY$ and $XX'\Delta_Y$ is isomorphic to the graph of $f \times_S 1_Y$ which is isomorphic to $XY$ and hence reduced. This implies that the intersection multiplicities are 1 and gives the result. □

1.4. **A finiteness property.** Let $(X_i)_{i\in I}$ be a pro-object of affine smooth $S$-schemes. As we have seen in 1.1.3, this pro-object admits a limit $\mathcal{X}$ in the category of affine $S$-schemes. We assume that $\mathcal{X}$ is regular noetherian.

In this case, for any smooth $S$-scheme $Y$, we will put $\bar{\mathrm{c}}_S(\mathcal{X},Y) = \mathrm{c}_0(\mathcal{X} \times_S Y/\mathcal{X})$ to extend definition 1.14. Moreover, the projection morphisms $p_i : \mathcal{X} \to X_i$ induce by pullback a morphism $p_i^* : \mathrm{c}_S(X_i,Y) \to \bar{\mathrm{c}}_S(\mathcal{X},Y)$. These morphisms are obviously natural in $i \in I$.

Then Proposition 1.8 admits immediately the following corollary :

**Proposition 1.24.** *Consider the hypotheses and notations above. Then the morphism*

$$\text{(A)} \qquad \varinjlim_{i\in I^{op}} p_i^* : \varinjlim_{i\in \mathcal{I}^{op}} \mathrm{c}_S\left(X_i, Y\right) \to \bar{\mathrm{c}}_S\left(\mathcal{X}, Y\right)$$

*is an isomorphism.*

*Remark* 1.25. This proposition is implicitly used in the proof of Prop. 3.1.3 in [SV00], chap.5.

**1.26.** Note that we can define a product $\bar{\mathrm{c}}_S\left(\mathcal{X}, Y\right) \otimes_{\mathbb{Z}} \mathrm{c}_S\left(Y, Z\right) \to \bar{\mathrm{c}}_S\left(\mathcal{X}, Z\right), (\bar{\alpha}, \beta) \mapsto \beta \bar{\circ} \bar{\alpha}$, using the arguments of Lemma 1.15. Then, using the argument of the proof for the first point of Lemma 1.18, we obtain the relation $(\gamma \circ \beta)\bar{\circ}\bar{\alpha} = \gamma\bar{\circ}(\beta\bar{\circ}\bar{\alpha})$.

In particular, the abelian group $\bar{\mathrm{c}}_S\left(\mathcal{X}, Y\right)$ is functorial with respect to finite $S$-correspondences in $Y$. Finally, considering this functoriality, the isomorphism (A) is natural in $Y$ with respect to finite $S$-correspondences.

**1.27.** Suppose now that we are given a second pro-object $(X'_i)_{i\in I}$ of smooth affine $S$-schemes and a family of $S$-morphisms

$$f_i : X'_i \to X_i$$

which are compatible with transition morphisms.

Let $\mathcal{X}'$ be the projective limit of $(X'_i)_{i\in I}$, $p'_i : \mathcal{X}' \to X'_i$ the canonical projection and $\mathfrak{f} : \mathcal{X}' \to \mathcal{X}$ the projective limit of the $(f_i)_{i\in I}$. By considering suitable pullbacks of relative cycles, we obtain the commutative diagram

$$\begin{tikzcd}
\mathrm{c}_S\left(X_i, Y\right) \arrow[r, "p_i^*"] \arrow[d, "f_i^*"'] & \bar{\mathrm{c}}_S\left(\mathcal{X}, Y\right) \arrow[d, "\mathfrak{f}^*"] \\
\mathrm{c}_S\left(X'_i, Y\right) \arrow[r, "p_i'^*"] & \bar{\mathrm{c}}_S\left(\mathcal{X}', Y\right).
\end{tikzcd}$$

This commutative diagram shows that the isomorphism (A) is natural in $\mathcal{X}$ with respect to morphisms of pro-$S$-schemes.

## 1.5. **Functoriality.**

1.5.1. *Base change.* Let $\tau : T \to S$ be a morphism of regular schemes.

If $X$ and $Y$ are smooth $S$-schemes, we identify the schemes $X_T \times_T Y_T$ and $X \times_T Y$ via the canonical isomorphism; we denote both schemes by $XY_T$.

Given smooth $S$-schemes $X$, $X'$, $Y$, $Y'$ we consider the following cartesian squares :

$$\begin{tikzcd}
XX'YY' \arrow[r] \arrow[d] \arrow[dr, phantom, "p^{(XX')YY'}_{(X)Y}"] & XY \arrow[d] \\
XX' \arrow[r] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
XX'YY'_T \arrow[r] \arrow[d] \arrow[dr, phantom, "q^{(XX')YY'}_{(X)Y}"] & XY_T \arrow[d] \\
XX'_T \arrow[r] & X_T
\end{tikzcd}
\qquad
\begin{tikzcd}
XY_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\tau_{XY}"] & XY \arrow[d] \\
X_T \arrow[r] & X
\end{tikzcd}$$

with the obvious projections.

For every finite $S$-correspondence $\alpha : X \to Y$ we put $\alpha_T = \tau_{XY}^*(\alpha)$ using Definition 1.6.

**Lemma 1.28.** *Let $X$ and $Y$ be smooth schemes in $\mathscr{S}m_S$. Then, for all $\alpha \in \mathrm{c}_S\left(X, Y\right)$ and $\beta \in \mathrm{c}_S\left(Y, Z\right)$, we have*

$$\beta_T \circ \alpha_T = (\beta \circ \alpha)_T.$$

*Proof.* Indeed we can do the following computation :

$$\begin{aligned}(\tau_{YZ}^*\beta)\circ(\tau_{XY}^*\alpha) &= q_{XZ\ *}^{XYZ}\big(q_{(Y)Z}^{(XY)Z^*}(\tau_{YZ}^*\beta).q_{(X)Y}^{(X)Y(Z)^*}(\tau_{XY}^*\alpha)\big) \\ &= q_{XZ\ *}^{XYZ}\Big(\tau_{XYZ}^*(p_{(Y)Z}^{(XY)Z^*}\beta).\tau_{XYZ}^*(p_{(X)Y}^{(X)Y(Z)^*}\alpha)\Big) \quad (1) \\ &= q_{XZ\ *}^{XYZ}\tau_{XYZ}^*\Big((p_{(Y)Z}^{(XY)Z^*}\beta).(p_{(X)Y}^{(X)Y(Z)^*}\alpha)\Big) \qquad (2) \\ &= \tau_{XZ}^*\left(p_{XZ\ *}^{XYZ}\Big((p_{(Y)Z}^{(XY)Z^*}\beta).(p_{(X)Y}^{(XY)Z^*}\alpha)\Big)\right). \quad (3)\end{aligned}$$

where equality (1) follows from the functoriality of pullback, equality (2) is compatibility of pullback with intersection product (cf [Ser58], V.C.7) and equality (3) is Proposition 1.12. □

**Definition 1.29.** Let $\tau : T \to S$ be a morphism of regular schemes. Using the preceding lemma, we define the base change functor

$$\begin{array}{rcl}\tau^* : \mathscr{S}m_S^{\mathrm{cor}} & \to & \mathscr{S}m_T^{\mathrm{cor}} \\ X/S & \mapsto & X_T/T \\ \mathrm{c}_S\,(X,Y)\ni\alpha & \mapsto & \alpha_T.\end{array}$$

We sum up the basic properties of base change for correspondences in the following lemma.

**Lemma 1.30.** (1) *The functor $\tau^*$ is symmetric monoidal.*

(2) *Let $\tau_0^* : \mathscr{S}m_S^{\mathrm{cor}} \to \mathscr{S}m_T^{\mathrm{cor}}$ be the classical base change functor on smooth schemes. The following diagram is commutative.*

$$\begin{array}{ccc}\mathscr{S}m_S & \xrightarrow{\gamma_S} & \mathscr{S}m_S^{\mathrm{cor}} \\ {\scriptstyle\tau_0^*}\downarrow & & \downarrow{\scriptstyle\tau^*} \\ \mathscr{S}m_T & \xrightarrow{\gamma_T} & \mathscr{S}m_T^{\mathrm{cor}}.\end{array}$$

(3) *If $\sigma : T' \to T$ is a morphism of regular schemes, we have a canonical isomorphism of functors*

$$(\tau\circ\sigma)^* \simeq \sigma^*\circ\tau^*.$$

*Proof.* (1). Let $\alpha \in \mathrm{c}_S\,(X,Y)$, $\beta\in\mathrm{c}_S\,(X',Y')$. Then,

$$\begin{aligned}(\alpha\otimes^{tr}\beta)_T &= \tau_{XX'YY'}^*\left(p_{XY}^{XX'YY'^*}(\alpha).p_{X'Y'}^{XX'YY'^*}(\beta)\right) \\ &= \left(\tau_{XX'YY'}^*p_{XY}^{XX'YY'^*}(\alpha)\right).\left(\tau_{XX'YY'}^*p_{X'Y'}^{XX'YY'^*}(\beta)\right) \\ &= \left(q_{XY}^{XX'YY'^*}(\alpha_T)\right).\left(q_{X'Y'}^{XX'YY'^*}(\beta)\right).\end{aligned}$$

(2). This point follows from the fact that for any $S$-morphism $f : X \to Y$, there is a canonical isomorphism $\Gamma_{f_T}\to\Gamma_f\times_S T$.

(3). Indeed there is a canonical isomorphism $X_{T'}\simeq (X_T)_{T'}$. Its naturality with respect to finite correspondences follows from the functoriality of pullback on cycles. □

1.5.2. *Restriction.* Let $\tau : T \to S$ be a smooth morphism of regular schemes.

Let $X$, $Y$ be smooth $T$-schemes. We denote by $\delta_{XY} : X \times_T Y \to X \times_S Y$ the canonical regular closed immersion, obtained from the diagonal immersion of $T/S$ by base change.

Let $\alpha \in \mathrm{c}_T\,(X,Y)$. We will consider the cycle $\delta_{XY*}(\alpha)$ as an element of $\mathrm{c}_S\,(X,Y)$ using Definition 1.10.

**Lemma 1.31.** *Let $X$, $Y$ and $Z$ be smooth $T$-schemes. The following relations hold :*

1. *For all $T$-morphisms $f : X \to Y$, $\delta_{XY*}\big([\Gamma_f]_T\big) = [\Gamma_f]_S$.*
2. *For all $\alpha \in \mathrm{c}_T\,(X,Y)$ and $\beta \in \mathrm{c}_T\,(Y,Z)$,*

$$\delta_{XZ*}(\beta \circ \alpha)(\delta_{YZ*}(\beta)) \circ (\delta_{XY*}(\alpha)).$$

*Proof.* In this proof, we stop mentioning the extensions of the schemes involved to simplify the notation.

The first assertion is obvious.

For the second assertion, we start by introducing the following notations :

$$\begin{tikzcd}
& X \times_T Y \arrow[dl, equal] & & X \times_T Y \times_T Z \arrow[ll, "q^{XYZ}_{XY}"'] \arrow[rr, "q^{XYZ}_{YZ}"] \arrow[dl, "a"] \arrow[dd, "\delta_{XYZ}"] \arrow[dr, "b"] & & Y \times_T Z \arrow[dr, equal] & \\
X \times_T Y \arrow[dr, "\delta_{XY}"'] & & X \times_T Y \times_S Z \arrow[ll, "p"'] \arrow[dr, "c"] & & X \times_S Y \times_T Z \arrow[rr, "q"] \arrow[dl, "d"'] & & Y \times_T Z \arrow[dl, "\delta_{YZ}"] \\
& X \times_S Y & & X \times_S Y \times_S Z \arrow[ll, "p^{XYZ}_{XY}"] \arrow[rr, "p^{XYZ}_{YZ}"'] & & Y \times_S Z, &
\end{tikzcd}$$

where all horizontal arrows are canonical projections and all the other arrows are canonical closed immersions.

The equality is obtained in the following way :

$$\begin{aligned}
\delta_{XZ*}\big(q^{XYZ}_{XZ}\big)_*\Big(q^{XYZ*}_{YZ}(\beta).q^{XYZ*}_{XY}(\alpha)\Big) &= {p^{XYZ}_{XZ}}_*\delta_{XYZ*}\Big(q^{XYZ*}_{YZ}(\beta).q^{XYZ*}_{XY}(\alpha)\Big) \\
&= {p^{XYZ}_{XZ}}_* d_* b_*\Big(b^* q^*(\beta).a^* p^*(\alpha)\Big) \\
&= {p^{XYZ}_{XZ}}_* d_*\Big(q^*(\beta).b_* a^* p^*(\alpha)\Big) \\
&= {p^{XYZ}_{XZ}}_* d_*\Big(q^*(\beta).d^* c_* p^*(\alpha)\Big) \\
&= {p^{XYZ}_{XZ}}_*\Big(d_* q^*(\beta).c_* p^*(\alpha)\Big) \\
&= {p^{XYZ}_{XZ}}_*\Big(p^{XYZ*}_{YZ}\delta_{YZ*}(\beta).p^{XYZ*}_{XY}\delta_{XY*}(\alpha)\Big)
\end{aligned}$$

using the functoriality of pullback and pushout and the projection formulas 1.13, 1.12. □

**Definition 1.32.** Let $\tau : T \to S$ be a smooth morphism of regular schemes.

Using the preceding lemma, we define a functor

$$\begin{array}{rcl}
\tau_\sharp : \mathscr{S}m^{\mathrm{cor}}_T & \to & \mathscr{S}m^{\mathrm{cor}}_S \\
(X \to T) & \mapsto & (X \to T \xrightarrow{\tau} S) \\
\mathrm{c}_T\,(X,Y) \ni \alpha & \mapsto & \delta_{XY*}(\alpha).
\end{array}$$

**1.33.** Note that from the first point of the preceding lemma, the restriction of $\tau$ to $\mathscr{S}m_T$ is the classical functor forgetting the base.

Moreover, for a sequence of smooth morphisms $R \xrightarrow{\sigma} T \xrightarrow{\tau} S$ between regular schemes, we clearly have

$$(\tau \circ \sigma)_\sharp = \tau_\sharp \circ \sigma_\sharp \ .$$

1.5.3. *Properties.*

**Proposition 1.34.** *Let $\tau : T \to S$ be a smooth morphism of finite type between regular schemes.*

(1) *The functor $\tau_\sharp$ is left adjoint to the functor $\tau^*$.*
(2) *For every smooth algebraic $T$-scheme $X$ (resp. $S$-scheme $Y$), the obvious morphism obtained by adjunction*

$$\tau_\sharp(\tau^* X \otimes_T Y) \to X \otimes_S \tau_\sharp Y$$

*is an isomorphism.*

*Proof.* For the first assertion, we only remark that for a smooth $T$-scheme $X$ (resp. $S$-scheme $Y$), $(\tau_\sharp X) \times_S Y \simeq X \times_T (\tau^* Y)$.

The second assertion is clear for the case of $\tau^*$ and $\tau_\sharp$ (for morphisms of schemes) and we only have to apply the second point of 1.30 and the first point of 1.33. $\square$

## 2. Sheaves with transfers

In this section $S$ will be a regular scheme, unless stated otherwise.

2.1. **Nisnevich topology.** We will consider the *Nisnevich topology* on the site $\mathscr{S}m_S$. Recall that a cover for the Nisnevich topology is a family of étale maps $p_i : Y_i \to X$ such that for any $x \in X$, there exists $y_i \in Y_i$ satisfying $p_i(y_i) = x$ and the induced map between the residue fields $\kappa(x) \to \kappa(y_i)$ is an isomorphism.

Among such covers, there is the family of covers induced by the *distinguished squares* of [MV99] which are cartesian squares

$$\begin{array}{ccc} W & \xrightarrow{k} & V \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{j} & X \end{array}$$

such that $j$ is an open immersion, $p$ is an étale morphism and the induced morphism $p^{-1}(X - U)_{red} \to (X - U)_{red}$ is an isomorphism. The family $(p, j)$ is indeed a Nisnevich cover. Moreover, recall from [MV99], prop. 1.4 that a presheaf $F$ on $\mathscr{S}m_S$ is a Nisnevich sheaf if and only if for any disinguished square as above, the square

$$\begin{array}{ccc} F(X) & \xrightarrow{p^*} & F(V) \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle k^*} \\ F(U) & \xrightarrow{-q^*} & F(W) \end{array}$$

is cartesian.

Let $X$ be a smooth $S$-scheme and $x$ a point of $X$. A *Nisnevich neighbourhood* of $x$ in $X$ is a pair $(V, y)$ where $V$ is an étale $X$-scheme and $y$ a point of $V$ over $x$ such that the induced morphism $\kappa(x) \to \kappa(y)$ is an isomorphism.

We let $\mathcal{V}^h_x(X)$ be the category of Nisnevich neighbourhoods of $x$ in $X$ with arrows the morphisms of pointed schemes. This category is non empty, essentially small and left filtered. We define the h-localisation of $X$ in $x$ as the pro-scheme $X^h_x = \underset{V\in\mathcal{V}^h_x(X)}{\tilde{\varprojlim}} V$.

Following the general notations of this article, we define the fiber of $F$ at the point $x$ of $X$ as the abelian group $F(X^h_x) = \varinjlim_{V\in\mathcal{V}^h_x(X)^{op}} F(V)$.

Standard arguments show that the functor from Nisnevich sheaves to abelian groups $F \mapsto F(X^h_x)$ is exact and commutes with arbitrary sums. Moreover, the family of fiber functors induced by a pointed smooth $S$-scheme $(X, x)$ is conservative for the category of Nisnevich sheaves over $\mathscr{S}m_S$.

*Remark* 2.1. Let $\mathcal{O}^h_{X,x}$ be the henselisation of the local ring of $X$ at $x$. Then $\operatorname{Spec}\left(\mathcal{O}^h_{X,x}\right)$ is the limit of the pro-object $X^h_x$.

**Definition 2.2.** We will denote by $\mathscr{P}_S$ (resp. $\mathscr{N}_S$) the category of presheaves (resp. sheaves for the Nisnevich topology) on $\mathscr{S}m_S$.

2.2. **Definition and examples.** Recall the canonical map $\gamma : \mathscr{S}m_S \to \mathscr{S}m^{\mathrm{cor}}_S$ of definition 1.19.

**Definition 2.3.** A presheaf with transfers $F$ over $S$ is an additive presheaf of abelian groups over $\mathscr{S}m^{\mathrm{cor}}_S$. We denote by $\mathscr{P}^{\mathrm{tr}}_S$ the corresponding category.

A sheaf with transfers over $S$ is a presheaf with transfers $F$ such that the functor $F \circ \gamma$ is a Nisnevich sheaf. We denote by $\mathscr{N}^{\mathrm{tr}}_S$ the full subcategory of $\mathscr{P}^{\mathrm{tr}}_S$ of sheaves with transfers.

Let $X$ be a smooth $S$-scheme. We denote by $\mathrm{L}_S[X]$ the presheaf on $\mathscr{S}m^{\mathrm{cor}}_S$ represented by $X$.

**Lemma 2.4.** *Let $X$ be a smooth $S$-scheme. The presheaf $\mathrm{L}_S[X]$ restricted to $\mathscr{S}m_S$ via $\gamma$ is an étale sheaf.*

*Proof.* Let $Y$ be a smooth algebraic $S$-scheme. As $Y$ is algebraic, it is sufficient to consider a surjective étale morphism $f : V \to Y$. We may assume $Y$ is irreducible by additivity.

Let $W = V \times_X V$ and consider the canonical projections $p, q : W \to V$. Using the third property of Lemma 1.18, we have to show the exactness of the sequence

$$0 \to \mathrm{c}_0(Y \times_S X/Y) \xrightarrow{f_X^*} \mathrm{c}_0(V \times_S X/V) \xrightarrow{p_X^* - q_X^*} \mathrm{c}_0(W \times_S X/W)\,.$$

As $f_X$ is faithfully flat, the pullback $f_X^*$ is injective.

Let $\alpha \in \mathrm{c}_0(V \times_S X/V)$ be a cycle such that $p_X^*(\alpha) = q_X^*(\alpha)$. Write $\alpha$ as a linear combination $\alpha = \sum_{i=1}^n \lambda_i.z_i$ with $z_i$ a point of $V \times_S X$ whose closure is finite and surjective over $Y$. As $p_X$ and $q_X$ are étale, we get by hypothesis

$$\sum_{i=1}^n \lambda_i. \sum_{x\in p_X^{-1}(z_i)} x = \sum_{j=1}^n \lambda_j. \sum_{y\in q_X^{-1}(z_j)} y.$$

Denote by $I$ the set $\{f_X(z_1), ..., f_X(z_n)\}$. Note that for any $w \in I$, if $i$ and $j$ are integers such that $f(z_i) = w = f(z_j)$ then $\lambda_i = \lambda_j$. Indeed, the equality above shows that the coefficient of $x = (z_i, z_j) \in W \times_S X$ is $\lambda_i$ and $\lambda_j$. For $w \in I$, we put $\lambda(w) = \lambda_i$ for any $i$ such that $w = f(z_i)$.

If we define $\beta = \sum_{w \in I} \lambda(w).w$, then $f_X^*(\beta) = \alpha$ as $f_X$ is étale. Finally, Lemma 1.9 shows that $\beta$ is an element of $\mathrm{c}_0(Y \times_S X/Y)$. □

2.3. **Associated sheaf with transfers.** Let $p : U \to X$ be an $S$-morphism of smooth $S$-schemes. We denote by $U_X^n$ the $n$-fold product of $U$ over $X$.

Consider the Čech simplicial scheme $\check{S}_*(U/X)$ associated to $U/X$ with the convention $\check{S}_n(U/X) = U_X^{n+1}$. We will denote by $\check{C}_*(U/X)$ the associated chain complex considered in the additive category generated by the category of schemes.

Applying the additive functor $\mathrm{L}_S[.]$ to this complex, we get a complex of sheaves with transfers $\mathrm{L}_S\left[\check{C}_*(U/X)\right]$ naturally augmented over $\mathrm{L}_S[X]$.

The following proposition is an obvious generalisation of prop. 3.1.3 in [Voe00b].

**Proposition 2.5.** *Let $X$ be a smooth $S$-scheme and $p : U \to X$ be a Nisnevich cover. The natural augmentation morphism $\mathrm{L}_S\left[\check{C}_*(U/X)\right] \to \mathrm{L}_S[X]$ is a quasi-isomorphism in the category of Nisnevich sheaves over $S$.*

*Proof.* We have only to check the assertion on the conservative family of points introduced in section 2.1. Let $(Y, y)$ be a pointed smooth $S$-scheme. We consider $\mathcal{O}_{Y,y}^h$, the henselian local ring of $Y$ at the point $y$ and put $\mathcal{Y} = \mathrm{Spec}\left(O_{Y,y}^h\right)$. Using Proposition 1.24 the canonical morphism

$$\mathrm{c}_S\left(Y_y^h, X\right) \to \bar{\mathrm{c}}_S\left(\mathcal{Y}, X\right)$$

defined on the fiber of $\mathrm{L}_S[X]$ at $(Y, y)$ (cf section 2.1) is an isomorphism.

Thus we have to show the exactness of the complex

$$C_* = \ldots \xrightarrow{d_{n*}} \bar{\mathrm{c}}_S\left(\mathcal{Y}, U_X^{n+1}\right) \to \ldots \xrightarrow{d_{0*}} \bar{\mathrm{c}}_S\left(\mathcal{Y}, U\right) \xrightarrow{p_*} \bar{\mathrm{c}}_S\left(\mathcal{Y}, X\right) \to 0.$$

Let $\mathcal{F}$ be the set, ordered by inclusion, of reduced closed subschemes of $\mathcal{Y} \times_S X$ which are finite equidimensional over $\mathcal{Y}$. Consider $Z \in \mathcal{F}$ and put

$$C_n^{(Z)} = \bar{\mathrm{c}}_S\left(\mathcal{Y}, Z \times_X U_X^{n+1}\right) \subset \bar{\mathrm{c}}_S\left(\mathcal{Y}, U_X^{n+1}\right).$$

Then $C_*^{(Z)}$ is a subcomplex of $C_*$. Moreover the complex $C_*^{(Z)}$ is increasing with respect to $Z$ and we have $C_* = \bigcup_{Z \in \mathcal{F}} C_*^{(Z)}$. Thus it suffices prove that the complex $C_*^{(Z)}$ is contractible.

According to the hypothesis, $Z$ is finite over $\mathcal{Y}$. As $\mathcal{Y}$ is henselian, $Z$ is a direct sum of local henselian schemes. Following [Ray70], the Nisnevich cover $p_Z : Z \times_X U \to Z$ admits a section $s : Z \to Z \times_X U$.

It is now a classical fact that the augmented Čech complex $\check{C}_*(Z \times_X U/Z) \to Z$ is contractible in the additive category generated by the category of schemes. An explicit homtopy is given by the collection of morphism for $n \geq -1$

$$s_n = s \times_X 1_{U_X^{n+1}} : Z \times_X U_X^{n+1} \to Z \times_X U_X^{n+2}.$$

The result follows by application of the functor $\mathrm{c}_S\left(\mathcal{Y}, .\right)$. □

Following the idea of the proof of Lemma 3.1.6 in [Voe00b], we obtain the next lemma.

**Lemma 2.6.** *Let $F$ be a presheaf with transfers. Denote by $\check{H}^0 F$ the 0-th Čech Nisnevich cohomology presheaf on $\mathscr{S}m_S$ associated with $F$ and $\eta : F \to \check{H}^0 F$ the canonical morphism.*

*Then there exists a unique pair $(\check{H}^0_{tr} F, \mu)$ such that*

(1) *$\check{H}^0_{tr} F$ is a presheaf with transfers satisfying $\check{H}^0_{tr} F \circ \gamma = \check{H}^0 F$.*
(2) *$\mu$ is a natural transformation $F \to \check{H}^0_{tr} F$ of presheaves with transfers that coincides with $\eta$.*

*Proof.* As $F$ is a presheaf with transfers, we have a canonical inclusion

$$F(X) \simeq \mathrm{Hom}_{\mathscr{P}^{\mathrm{tr}}_S}(\mathrm{L}_S[X], F) \subset \mathrm{Hom}_{\mathscr{P}_S}(\mathrm{L}_S[X], F).$$

Conversely, a natural transformation of presheaves on $\mathscr{S}m_S$

$$F \xrightarrow{\phi} \mathrm{Hom}_{\mathscr{P}_S}(\mathrm{L}_S[.], F)$$

is equivalent to a structure of presheaf with transfers on $F$ as soon as it respects the composition product (in which case, it is a monomorphism). The two structures are in one-to-one correspondence using the equation

$$\forall \alpha \in \mathrm{c}_S(Y, X),\ a \in F(X),\ F(\alpha).a = \phi_X(a)_Y.\alpha \tag{B}$$

We work with the natural transformation $\phi$ and not with the structure of a presheaf with transfers.

1) Suppose for the moment that we already defined $\check{H}^0_{tr} F$. Consider $\alpha \in \mathrm{c}_S(Y, X)$ and $a \in F_{\mathrm{Nis}}(X)$. Recall that

$$\check{H}^0 F(X) = \varinjlim_{U \to X} \mathrm{Ker}\big(F(U) \to F(U \times_X U)\big).$$

As the colimit is filtered, there exists a Nisnevich cover $U$ of $X$ such that $a$ can be lifted to an element $a_U \in F(U)$. Applying Lemma 2.5, $\mathrm{L}_S[U] \to \mathrm{L}_S[X]$ is an epimorphism. Hence there exist a Nisnevich cover $V$ of $Y$ and a correspondence $\alpha_U \in \mathrm{c}_S(V, U)$ such that $p \circ \alpha_U = \alpha|_V$.

We thus have obtained the following commutative diagram (the commutativity of the bottom square is the first condition appearing in the statement of the theorem)

$$\begin{tikzcd}
\check{H}^0 F(X) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{P}_S}\big(\mathrm{L}_S[X], \check{H}^0 F\big) \arrow[d] \\
\check{H}^0 F(U) \arrow[r] & \mathrm{Hom}_{\mathscr{P}_S}\big(\mathrm{L}_S[U], \check{H}^0 F\big) \\
F(U) \arrow[r] \arrow[u, "\eta"] & \mathrm{Hom}_{\mathscr{P}_S}(\mathrm{L}_S[U], F). \arrow[u]
\end{tikzcd}$$

This in turn can be translated into the following local equation which carcterizes $\check{H}^0_{tr} F$

$$\check{H}^0_{tr} F(\alpha|_V).a = \eta_Y(F(\alpha_U).a_U) \in F(V).$$

2) Conversely, we have to prove that the above equality does not depend on the choice of the cover $U$ because it then defines $\check{H}^0_{tr} F$.

Let $U$ be a Nisnevich cover of $X$ and $H = \mathrm{Hom}_{\mathscr{P}_S}\big(., \check{H}^0 F\big)$. As $H$ is left exact, Lemma 2.5 implies the following sequence is exact

$$0 \to H(\mathrm{L}_S[X]) \to H(\mathrm{L}_S[U]) \to H(\mathrm{L}_S[U \times_X U]).$$

Let us consider the arrow

$$F(X) \hookrightarrow \mathrm{Hom}_{\mathscr{P}_S}(\mathrm{L}_S[X], F) \xrightarrow{\eta_X} \mathrm{Hom}_{\mathscr{P}_S}\big(\mathrm{L}_S[X], \check{H}^0 F\big) = H(\mathrm{L}_S[X]).$$

As it is natural, it induces the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H(\mathrm{L}_S[X]) \arrow[r] & H(\mathrm{L}_S[U]) \arrow[r] & H(\mathrm{L}_S[U \times_X U]) \\ 0 \arrow[r] & Ker_U \arrow[r] \arrow[u, dashed] & F(U) \arrow[r] \arrow[u] & F(U \times_X U), \arrow[u] \end{tikzcd}$$

which is natural in $U$. Taking the limit over all Nisnevich covers $U$ of $X$, we obtain the arrow

$$\phi_X^{\mathrm{Nis}} : \check{H}^0 F(X) = \varinjlim_{U \to X} \big(Ker_U\big) \longrightarrow H(\mathrm{L}_S[X]) = \mathrm{Hom}_{\mathscr{P}_S}\big(\mathrm{L}_S[X], \check{H}^0 F\big)$$

which in turn defines transfers on $\check{H}^0 F$ according to equation (B) :

$$\check{H}^0_{tr} F(\alpha) : F(X) \to F(Y),\ a \mapsto \phi_X^{\mathrm{Nis}}(a)_Y.\alpha.$$

Consider $a \in \check{H}^0 F(X)$ and $\alpha \in \mathrm{c}_S(Y, X)$. As in the first step of the proof, choose a cover $U$ (resp. $V$) of $X$ (resp. $Y$) and liftings $a_U \in F(U)$, $\alpha_U \in \mathrm{c}_S(V, U)$. Then tautologically,

$$\check{H}^0_{tr} F(\alpha|_V).a = \eta_Y(F(\alpha_U).a_U).$$

From this local equation we now easily deduce the compatibility of $\phi^{\mathrm{Nis}}$ with the product of correspondences. By the very construction, $\phi^{\mathrm{Nis}}$ extends $\phi$. $\square$

The uniqueness statement in the preceding lemma implies the naturality with respect to $F$ of the transformation $F \to \check{H}^0_{tr} F$ and the following corollary.

**Corollary 2.7.** *With the notation of the previous lemma, the association*

$$a_{tr} : \mathscr{P}_S^{\mathrm{tr}} \to \mathscr{N}_S^{\mathrm{tr}}, F \mapsto \check{H}^0_{tr}\big(\check{H}^0_{tr} F\big)$$

*defines a functor left adjoint to the forgetful functor*

$$\mathcal{O}_{tr} : \mathscr{N}_S^{\mathrm{tr}} \hookrightarrow \mathscr{P}_S^{\mathrm{tr}}.$$

*Moreover, the following diagram commutes*

$$\begin{tikzcd} \mathscr{P}_S^{\mathrm{tr}} \arrow[r, "a_{tr}"] \arrow[d] & \mathscr{N}_S^{\mathrm{tr}} \arrow[d] \\ \mathscr{P}_S \arrow[r, "a_{\mathrm{Nis}}"] & \mathscr{N}_S. \end{tikzcd}$$

Recall that a *Grothendieck abelian category* is an abelian category which admits arbitrary direct sums, has a set of generators (any object is a quotient of a direct sums of the generators) and such that filtered inductive limits are exact.

**Proposition 2.8.** *The category $\mathscr{N}_S^{\mathrm{tr}}$ is an abelian Grothendieck category. It is complete (i.e. admits arbitrary small projective limits). The forgetful functor $\mathscr{N}_S^{\mathrm{tr}} \to \mathscr{N}_S$ admits a left adjoint $\mathrm{L}_S[.]$. An essentially small family of generators for $\mathscr{N}_S^{\mathrm{tr}}$ is given by the sheaves $\mathrm{L}_S[X]$ for a smooth $S$-scheme $X$.*

*Proof.* The existence of the right exact functor $a_{tr}$ implies that $\mathscr{N}_S^{\mathrm{tr}}$ is cocomplete (as is the category $\mathscr{P}_S^{\mathrm{tr}}$). As in the classical case, an inductive limit of sheaves with transfers is constructed by first computing it in the category of presheaves with transfers, then taking the associated sheaf with transfers. This description shows that $\mathscr{N}_S^{\mathrm{tr}}$ is a Grothendieck abelian category.

Consider a Nisnevich sheaf $F$. Then, classically

$$F = \varinjlim_{X/F \in \mathscr{S}m_S/F} \mathbb{Z}_S(X),$$

where $\mathbb{Z}_S(X)$ is the free abelian sheaf represented by a smooth $S$-scheme $X$ and the limit is taken over every morphism $\mathbb{Z}_S(X) \to F$ for such an $X$.

We simply put $\mathrm{L}_S[F] = \varinjlim_{X/F \in \mathscr{S}m_S/F} \mathrm{L}_S[X]$, the inductive limit being calculated in the category of sheaves with transfers. The construction of the functor $\mathrm{L}_S[.]$ implies the last assertion. □

Let $X$ be a smooth $S$-scheme. The graph morphism induces a morphism of sheaves $\eta_X : \mathbb{Z}_S(X) \to \mathcal{O}_{tr}\mathrm{L}_S[X]$. Using the description of the Ext groups for sheaves and for sheaves with transfers we deduce a canonical morphism, natural in $X$ and the sheaf with transfers $F$

$$\eta_X^i : \mathrm{Ext}^i_{\mathscr{N}_S^{\mathrm{tr}}}(\mathrm{L}_S[X], F) \to \mathrm{Ext}^i_{\mathscr{N}_S}(\mathbb{Z}_S(X), \mathcal{O}_{tr}F) = H^i(X; \mathcal{O}_{tr}F).$$

**Proposition 2.9.** *Using the notation introduced above, for every smooth $S$-scheme $X$, every sheaf with transfers $F$ over $S$, and every integer $i \in \mathbb{N}$, the morphism $\eta_X^i$ is an isomorphism.*

*Proof.* Using the Yoneda lemma, the property is clear for $i = 0$. Consider the case $i > 0$.

The category $\mathscr{N}_S^{\mathrm{tr}}$, being a Grothendieck abelian category, has enough injectives. In particular, the Ext groups with coefficients in $F$ are calculated by choosing an injective resolution of $F$ in $\mathscr{N}_S^{\mathrm{tr}}$. Consequently, it suffices to prove that for any sheaf with transfers $I$ which is injective in the category $\mathscr{N}_S^{\mathrm{tr}}$, the sheaf $\mathcal{O}_{tr}I$ is acyclic.

Following [Mil80], prop. III.2.11, this property is in turn equivalent to the vanishing of all the positive Čech cohomology groups $\check{H}^i(X; \mathcal{O}_{tr}I)$. But this now follows from Proposition 2.5. □

**Corollary 2.10.** *Let $F$ be a presheaf with transfers. Then for all integers $i \in \mathbb{N}$, the presheaf $H^i_{\mathrm{Nis}}(., F_{\mathrm{Nis}})$ has a canonical structure of a presheaf with transfers.*

2.4. **Closed monoidal structure.** Recall that we have defined in Proposition 1.23 a monoidal structure on $\mathscr{S}m_S^{\mathrm{cor}}$.

**Lemma 2.11.** *The category $\mathscr{N}_S^{\mathrm{tr}}$ admits a unique structure of a symmetric monoidal category with a right exact tensor product such that the graph functor*
*$\mathscr{S}m_S^{\mathrm{cor}} \to \mathscr{N}_S^{\mathrm{tr}}$ is monoidal.*

*Proof.* Let $F$ and $G$ be sheaves with transfers. Using Proposition 2.8, we can write

$$F = \varinjlim_{X/F \in \mathscr{S}m_S/F} \mathrm{L}_S\left[X\right], \quad G = \varinjlim_{Y/F \in \mathscr{S}m_S/G} \mathrm{L}_S\left[Y\right].$$

Necessarily, the tensor product of sheaves with transfers must satisfy

$$F \otimes_S^{tr} G = \varinjlim_{X/F, Y/G} (\mathrm{L}_S\left[X\right] \otimes_S^{tr} \mathrm{L}_S\left[Y\right]).$$

The axioms of a symmetric monoidal category then follow from the corresponding properties of the category $\mathscr{S}m_S^{\mathrm{cor}}$ and uniqueness is established as well. $\square$

**Definition 2.12.** We denote by $\otimes_S^{tr}$ the tensor product on $\mathscr{N}_S^{\mathrm{tr}}$ satisfying the conditions of the previous lemma.

*Remark* 2.13. We can express the difference between the tensor product with transfers and the usual tensor product of abelian sheaves. Indeed, for any sheaf with transfers $F$ we have an epimorphism of sheaves with transfers

$$\begin{array}{rcl} \bigoplus_{X \in \mathscr{S}m_S} F(X) \otimes_{\mathbb{Z}} \mathrm{L}_S\left[X\right] & \to & F \\ \bigoplus_{X \in \mathscr{S}m_S} F(X) \otimes_{\mathbb{Z}} \mathrm{c}_S\left(Y, X\right) & \to & F(Y) \\ \rho \otimes \alpha & \mapsto & \rho \circ \alpha, \end{array}$$

where we view $\rho$ (resp. $\alpha$) as a map $\mathrm{L}_S\left[X\right] \to F$ (resp. $\mathrm{L}_S\left[Y\right] \to \mathrm{L}_S\left[X\right]$).

Thus, as $\otimes_S^{tr}$ is right exact, we deduce an epimorphism of sheaves

$$\begin{array}{rcl} \bigoplus_{X, X' \in \mathscr{S}m_S} (F(X) \otimes_{\mathbb{Z}} \mathrm{L}_S\left[X\right]) \otimes (G(X') \otimes_{\mathbb{Z}} \mathrm{L}_S\left[X'\right]) & \to & F \otimes_S^{tr} G \\ \bigoplus_{X, X' \in \mathscr{S}m_S} F(X) \otimes_{\mathbb{Z}} G(X') \otimes_{\mathbb{Z}} \mathrm{c}_S\left(Y, X \times X'\right) & \to & (F \otimes_S^{tr} G)(Y) \\ \rho \otimes \mu \otimes \alpha & \mapsto & (\rho \circ \alpha) \otimes_S^{tr} (\mu \circ \alpha). \end{array}$$

In particular, for any pointed scheme $(Y, y)$, we have on the level of the fiber at $Y_y^h$ (cf section 2.1) an epimorphism of abelian groups

$$\begin{array}{rcl} \bigoplus_{X, X' \in \mathscr{S}m_S} F(X) \otimes_{\mathbb{Z}} G(X') \otimes_{\mathbb{Z}} \mathrm{c}_S\left(Y_y^h, X \times X'\right) & \to & (F \otimes_S^{tr} G)(Y_y^h) \\ \rho \otimes \mu \otimes \bar{\alpha} & \mapsto & (\rho \circ \bar{\alpha}) \otimes_S^{tr} (\mu \circ \bar{\alpha}). \end{array}$$

**Proposition 2.14.** *The monoidal category $\mathscr{N}_S^{\mathrm{tr}}$ is closed : the bifunctor $\otimes_S^{tr}$ admits a right adjoint $\underline{\mathrm{Hom}}_{\mathscr{N}_S^{\mathrm{tr}}}(.,.)$.*

*Proof.* Let $F$ and $G$ be sheaves with transfers. We put

$$\underline{\mathrm{Hom}}_{\mathscr{N}_S^{\mathrm{tr}}}\left(F, G\right)\left(X\right) = \mathrm{Hom}_{\mathscr{N}_S^{\mathrm{tr}}}\left(F \otimes_S^{tr} \mathrm{L}_S\left[X\right], G\right).$$

As a sheaf with transfers is an inductive limit of representable presheaves with transfers (cf Prop. 2.8), one easily obtains the expected adjoint property. $\square$

### 2.5. **Functoriality.**

**We fix a morphism $\tau : T \to S$ of regular schemes.**

2.5.1. *The abstract case.* Consider an abstract additive functor $\varphi : \mathscr{S}m_S^{\mathrm{cor}} \rightarrow \mathscr{S}m_T^{\mathrm{cor}}$ that sends a Nisnevich cover of an $S$-scheme to a Nisnevich cover of a $T$-scheme.

In this situation, we will define the following two functors :

(1) If $F$ is a sheaf with transfers over $S$, we define over $T$ the sheaf with transfers $\varphi(F) = \varinjlim_{X/F} \mathrm{L}_T\left[\varphi(X)\right]$.

(2) If $G$ is a sheaf with transfers over $T$, we define over $S$ the sheaf with transfers $\varphi'(G) = G \circ \varphi$.

Note there is an abuse of notation in (1). This is justified by the fact that the functor $\varphi$ on sheaves with transfers is an extension of the functor $\varphi$ on schemes via the associated represented sheaf with transfers functor.

The Yoneda lemma implies immediately that $\varphi'$ is right adjoint to $\varphi$.

Similarly, the same construction applies to the graph functor $\gamma_S : \mathscr{S}m_S \rightarrow \mathscr{S}m_S^{\mathrm{cor}}$. Indeed this functor respects tautologically the Nisnevich coverings and we obtain an extension on sheaves $\gamma_S : \mathscr{N}_S \rightarrow \mathscr{N}_S^{\mathrm{tr}}$ and a right adjoint which is the forgetful functor $\mathcal{O}_S^{tr} : \mathscr{N}_S^{\mathrm{tr}} \rightarrow \mathscr{N}_S$.

Going back to the hypothesis of the beginning, we suppose given in addition the commutative diagram of functors

$$\begin{array}{ccc} \mathscr{S}m_S & \xrightarrow{\gamma_S} & \mathscr{S}m_S^{\mathrm{cor}} \\ {\scriptstyle \varphi_0}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathscr{S}m_T & \xrightarrow{\gamma_T} & \mathscr{S}m_T^{\mathrm{cor}}. \end{array}$$

By hypothesis, $\varphi_0$ respects Nisnevich coverings and the same process gives a pair of adjoint functors

$$\varphi_0 : \mathscr{N}_S \rightarrow \mathscr{N}_T, \quad \varphi_0' : \mathscr{N}_T \rightarrow \mathscr{N}_S.$$

It is now obvious that these functors are related by the commutative diagrams

$$\begin{array}{ccc} \mathscr{N}_S & \xrightarrow{\gamma_S} & \mathscr{N}_S^{\mathrm{tr}} \\ {\scriptstyle \varphi_0}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathscr{N}_T & \xrightarrow{\gamma_T} & \mathscr{N}_T^{\mathrm{tr}}, \end{array} \qquad \begin{array}{ccc} \mathscr{N}_S & \xleftarrow{\mathcal{O}_S} & \mathscr{N}_S^{\mathrm{tr}} \\ {\scriptstyle \varphi_0'}\uparrow & & \uparrow{\scriptstyle \varphi'} \\ \mathscr{N}_T & \xleftarrow{\mathcal{O}_T} & \mathscr{N}_T^{\mathrm{tr}}. \end{array}$$

Finally, suppose that $\varphi$ is monoidal. Then the extension $\varphi : \mathscr{N}_S^{\mathrm{tr}} \rightarrow \mathscr{N}_T^{\mathrm{tr}}$ is again monoidal. In addition, we have a canonical isomorphism

$$\underline{\mathrm{Hom}}_{\mathscr{N}_S^{\mathrm{tr}}}\left(F, \varphi'(G)\right) \simeq \underline{\mathrm{Hom}}_{\mathscr{N}_T^{\mathrm{tr}}}\left(\varphi(F), G\right).$$

The same remark applies to the pair of adjoint functors $(\gamma_S, \mathcal{O}_S^{tr})$.

2.5.2. *Base change.* We first apply the abstract construction above to the monoidal functor

$$\tau^* : \mathscr{S}m_S^{\mathrm{cor}} \rightarrow \mathscr{S}m_T^{\mathrm{cor}}$$

defined in 1.29. This yields the base change functor $\tau^* : \mathscr{N}_S^{\mathrm{tr}} \rightarrow \mathscr{N}_T^{\mathrm{tr}}$ and its right adjoint $\tau_* = (\tau^*)' : \mathscr{N}_T^{\mathrm{tr}} \rightarrow \mathscr{N}_S^{\mathrm{tr}}$. The first one is monoidal and the second one coincides with the usual pushout for sheaves without transfers.

2.5.3. *Exceptional direct image.* Suppose now that the morphism $\tau : T \to S$ is a smooth morphism of regular schemes. We apply the abstract construction to the monoidal functor

$$\tau_\sharp : \mathscr{S}m_T^{\mathrm{cor}} \to \mathscr{S}m_S^{\mathrm{cor}}$$

defined in 1.32. This yields the twisted exceptional direct image functor $\tau_\sharp : \mathscr{N}_T^{\mathrm{tr}} \to \mathscr{N}_S^{\mathrm{tr}}$ which is monoidal.

*Remark* 2.15. When $\tau$ is étale, this functor is really the usual exceptional direct image $\tau_!$. Otherwise we need to twist this functor in order to get the fundamental equality $\tau_! = \tau_*$ when $\tau$ is smooth projective.

**Lemma 2.16.** *If $\tau : T \to S$ is smooth, there exists a canonical isomorphism of functors $\tau^* \simeq (\tau_\sharp)'$.*

*Proof.* Let $F$ be a sheaf with transfers over $S$. Accordign to the definitions, $\tau^* F$ is the sheaf associated with the presheaf with transfers over $T$

$$Y \mapsto \varinjlim_{X/F} \mathrm{c}_T\left(Y, X \times_S T\right).$$

The canonical isomorphism $Y \times_S X \to Y \times_T (X \times_S T)$ induces an isomorphism $\mathrm{c}_S\left(\tau_\sharp Y, X\right) \to \mathrm{c}_T\left(Y, X \times_S T\right)$. The definition of composition product and base change for finite correspondences shows that this isomorphism is natural in $X$ and $Y$ with respect to finite correspondences (the projections involved in the two ways of computing products in the above isomorphic groups coincide).

As $F = \varinjlim_{X/F} \mathrm{L}_S\left[X\right]$ in the category of sheaves with transfers, the result follows from the computation of inductive limits in the category of sheaves with transfers over $S$ (cf the proof of 2.8). □

In particular, when $\tau$ is smooth, $\tau^*$ is right adjoint to $\tau_\sharp$. Thus $\tau^*$ is exact (and commutes with inductive and projective limits). Moreover, $\tau^*$ coincides with the usual base change functor on sheaves without transfers.

We now set up the projection formula. Let $F$ (resp. $G$) be a sheaf with transfers over $S$ (resp. $T$). We consider the adjunction morphism deduced from the previous lemma

$$G \to \tau^* \tau_\sharp G.$$

Applying the functor $(\tau^* F) \otimes_S^{tr} (.)$ to this morphism we get

$$(\tau^* F) \otimes_S^{tr} G \to (\tau^* F) \otimes_S^{tr} (\tau^* \tau_\sharp G).$$

Using the monoidal property of $\tau^*$ and adjunction we get a morphism

$$\phi : \tau_\sharp((\tau^* F) \otimes_S^{tr} G) \to F \otimes_S^{tr} (\tau_\sharp G).$$

**Lemma 2.17.** *With the above hypotheses and notation, the morphism $\phi$ is an isomorphism.*

*Proof.* The morphism $\phi$ is natural in $F$ and $G$. As all functors involved commute with inductive limits, it is sufficient to check the isomorphism on representable sheaves $F = \mathrm{L}_S\left[X\right]$, $G = \mathrm{L}_T\left[Y\right]$. Then the morphism is reduced to the canonical isomorphism $(X \times_S T) \times_T Y \to X \times_S Y$ of $S$-schemes. □

2.5.4. *Pro-smooth morphisms.* Let $(T_i)_{i\in I}$ be a pro-object of smooth affine $S$-schemes. As in subsection 1.1.3, we write $T_i = \mathrm{Spec}_S(\mathcal{A}_i)$ and put $\mathcal{A} = \varinjlim_{i\in I^{op}} \mathcal{A}_i$. The scheme $\mathcal{T} = \mathrm{Spec}_S(\mathcal{A})$ is the projective limit of $(T_i)_{i\in I}$ in the category of affine $S$-schemes. We suppose it is regular noetherian. We denote by $\tau : \mathcal{T} \to S$ the canonical morphism[4].

First, we note that the functoriality constructed above for sheaves with transfers can also be constructed for presheaves with transfers. In particular, from the the functor $\tau^* : \mathscr{S}m^{\mathrm{cor}}_S \to \mathscr{S}m^{\mathrm{cor}}_{\mathcal{T}}$ we obtain the base change functor $\hat{\tau}^* : \mathscr{P}^{\mathrm{tr}}_S \to \mathscr{P}^{\mathrm{tr}}_{\mathcal{T}}$ and its right adjoint $\hat{\tau}_* : \mathscr{P}^{\mathrm{tr}}_{\mathcal{T}} \to \mathscr{P}^{\mathrm{tr}}_S$.

In fact, when $F$ is a sheaf with transfers over $S$, we have $\tau^*F = a_{tr}(\hat{\tau}^*F)$ using the associated sheaf with transfers of Corollary 2.7. For a sheaf with transfers $G$ over $\mathcal{T}$ we simply have $\tau_*G = \hat{\tau}_*G$.

Secondly, given a smooth scheme $\mathcal{X}$ over $\mathcal{T}$, as it is in particular of finite presentation, there exists $i \in I$ such that $\mathcal{X}/\mathcal{T}$ descends to a finite presentation scheme $X_i/T_i$. That is, $\mathcal{X} = X_i \times_{T_i} \mathcal{T}$. For any $j \to i$, we put $X_j = X_i \times_{T_i} T_j$. Using now [GD66], 17.7.8, by enlarging $i$, we can assume $X_i/T_i$ is smooth. We finally have

$$\mathcal{X} = \varprojlim_{j\in I/i} X_j$$

where every $X_j$ is smooth over $S$.

**Proposition 2.18.** *If the hypotheses described above are satisfied, we have a canonical isomorphism*

$$\hat{\tau}^*F(\mathcal{X}) \simeq \varinjlim_{j\in I/i^{op}} F(X_j)$$

*for any presheaf with transfers $F$ over $S$.*

*Proof.* According to the definition,

$$\hat{\tau}^*F(\mathcal{X}) = \varinjlim_{U/F} \mathrm{c}_{\mathcal{T}}\left(\mathcal{X}, U \times_S \mathcal{T}\right)$$

where the limit runs over all morphisms $\mathrm{L}_S[U] \to F$ of sheaves with transfers for a smooth $S$-scheme $U$.

Note that using the notation introduced in 1.1.3, we have a canonical isomorphism $\mathrm{c}_{\mathcal{T}}\left(\mathcal{X}, U \times_S T\right) \simeq \bar{\mathrm{c}}_S\left(\mathcal{X}, U\right)$. Proposition 1.24 now implies

$$\bar{\mathrm{c}}_S\left(\mathcal{X}, U\right) = \bar{\mathrm{c}}_S\left(\varprojlim_{j\in I/i} X_j, U\right) \simeq \varinjlim_{j\in I/i^{op}} \mathrm{c}_S\left(X_j, U\right),$$

the isomorphism being functorial in $U$ with respect to finite $S$-correspondences by subsection 1.26. Finally, we can conclude as we have

$$\varinjlim_{U/F}\varinjlim_{j\in I/i^{op}} \mathrm{c}_S\left(X_j, U\right) = \varinjlim_{j\in I/i^{op}}\varinjlim_{U/F} \mathrm{c}_S\left(X_j, U\right) = \varinjlim_{j\in I/i^{op}} F(X_j).$$

□

[4]In general, $\tau$ is not necessarily formally smooth but only regular, that is, the fibers of $\tau$ are geometrically regular.

Suppose now that we are given a $\mathcal{T}$-morphism $\mathfrak{f} : \mathcal{X}' \to \mathcal{X}$. This morphism is of finite presentation; hence there exists $i \in I$ such that $\mathfrak{f}$ descends to $T_i$. That is, there exist schemes of finite presentation $X_i/T_i$ and $X_i'/T_i$ and a $T_i$-morphism $f_i : X_i' \to X_i$ such that $\mathfrak{f} = f_i \times_{T_i} \mathcal{T}$.

We put $f_j = f_i \times_{T_i} T_j$. Again using [GD66], 17.7.8, we may assume $X_i'$ and $X_i$ to be smooth over $T_i$.

Then using subsection 1.27, the isomorphism of the preceding proposition is functorial with respect to $(f_j)_{j \in I/i}$. As a consequence, we obtain the following proposition.

**Proposition 2.19.** *Suppose that the hypotheses described before Proposition 2.18 are satisfied. Then for any sheaf with transfers $F$ over $S$, $\hat{\tau}^*F$ is a sheaf with transfers. In particular,* $\tau^*F(\mathcal{X}) \simeq \varinjlim_{j \in I/i^{op}} F(X_j)$.

Indeed, using the characterisation of a Nisnevich sheaf from 2.1, this is a consequence of the following lemma and the exactness of filtered inductive limits.

**Lemma 2.20.** *Consider a distinguished square of smooth $\mathcal{T}$-schemes*

$$\begin{tikzcd} \mathcal{W} \arrow[r, "\mathfrak{v}"] \arrow[d, "\mathfrak{g}"'] \arrow[dr, phantom, "\Delta"] & \mathcal{V} \arrow[d, "\mathfrak{f}"] \\ \mathcal{U} \arrow[r, "\mathfrak{u}"'] & \mathcal{X}. \end{tikzcd}$$

*Then there exist $i \in I$ and a distinguished square of smooth schemes over $T_i$*

$$\begin{tikzcd} W_i \arrow[r, "v_i"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\Delta_i"] & V_i \arrow[d, "f_i"] \\ U_i \arrow[r, "u_i"'] & X_i. \end{tikzcd}$$

*such that $\Delta = \Delta_i \times_{T_i} \mathcal{T}$.*

*Proof.* We have already seen just before the above proposition that we can find $i \in I$ and a square of smooth $T_i$-schemes

$$\begin{tikzcd} W_i \arrow[r, "v_i"] \arrow[d, "g_i"'] \arrow[dr, phantom, "\Delta_i"] & V_i \arrow[d, "f_i"] \\ U_i \arrow[r, "u_i"'] & X_i. \end{tikzcd}$$

For any $j \to i$, we put $\Delta_j = \Delta_i \times_{T_i} T_j$, $Z_j = (X_j - U_j)_{red}$ and $T_j = (V_j \times_{T_j} Z_j)_{red}$. Then $\Delta$ is the projective limit of the $\Delta_j$.

By finding a suitable $j \to i$, we can assume :

(1) this square is cartesian, that is the morphism $W_j \to U_j \times_{X_j} V_j$ is an isomorphism (cf [GD66], 8.10.5(i)),
(2) the morphism $T_j \to V_j$ induced by $f_j$ is an isomorphism (cf *loc. cit.*),
(3) the morphism $u_j$ is an open immersion (cf [GD66], 8.10.5(i)),
(4) the morphism $f_j$ is étale (cf [GD66], 17.7.8(ii)).

□

In the course of section 4, we will need the following strengthening of the preceding proposition, relying on the same lemma :

**Lemma 2.21.** *Suppose that the hypotheses introduced before Proposition 2.18 are satisfied. We denote by $\check{H}^0_{tr}$ the functor constructed in Lemma 2.6, either for presheaves with transfers over $S$ or over $\mathcal{T}$. Then we have a canonical isomorphism of functors $\mathscr{P}^{\mathrm{tr}}_S \to \mathscr{P}^{\mathrm{tr}}_{\mathcal{T}}$ :*

$$\hat{\tau}^* \check{H}^0_{tr} \simeq \check{H}^0_{tr} \hat{\tau}^*.$$

*Proof.* Let $F$ be a presheaf with transfers over $S$ and $\mathcal{X}$ be a smooth $S$-scheme. Fix $i \in I$ and a smooth $T_i$-scheme $X_i$ such that $\mathcal{X} = X_i \times_{T_i} \mathcal{T}$. We put $X_j = X_i \times_{T_i} T_j$.

For any (noetherian) scheme $A$, we let $\mathcal{D}_A$ be the subcategory of $A$-schemes $W$ such that there exists a distinguished square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & A \end{tikzcd}$$

such that $W = U \sqcup V$ as an $A$-scheme. This category is left filtered, as any Nisnevich covering admits a refinement of this form. Then

$$\hat{\tau}^* \check{H}^0_{tr} F(\mathcal{X}) \ \varinjlim_{j \in I/i^{op}} \ \varinjlim_{W \in \mathcal{D}^{op}_{X_j}} \ \mathrm{Ker}\big(F(W) \to F(W \times_{X_j} W)\big).$$

Moreover the preceding lemma says precisely that the inclusion functor

$$\bigsqcup_{j \in I/i} \mathcal{D}_{X_j} \to \mathcal{D}_{\mathcal{X}}, W_j \mapsto W_j \times_{X_j} \mathcal{X}$$

is surjective, hence final. This implies that

$$\begin{aligned} \check{H}^0_{tr} \hat{\tau}^* F(\mathcal{X}) &\ \varinjlim_{j \in I/i^{op}} \ \varinjlim_{W_j \in \mathcal{D}_{X_j}} \ \mathrm{Ker}\big(\hat{\tau}^* F(W_j \times_{X_j} \mathcal{X}) \to \hat{\tau}^* F(W_j \times_{X_j} W_j \times_{X_j} \mathcal{X})\big) \\ &\ \varinjlim_{j \in I/i^{op}} \ \varinjlim_{W_j \in \mathcal{D}_{X_j}} \ \varinjlim_{k \in I/j^{op}} \ \mathrm{Ker}\big(F(W_j \times_{X_j} X_k) \to F(W_j \times_{X_j} X_k)\big) \end{aligned}$$

where the second equality follows from Proposition 2.18 and the exactness of filtered inductive limits. The lemma then follows. □

## 3. Homotopy equivalence for finite correspondences

3.1. **Definition.** Consider a regular scheme $S$.

**Definition 3.1.** Let $X$ and $Y$ be smooth $S$-schemes. Consider two correspondences $\alpha, \beta \in \mathrm{c}_S(X, Y)$.

A homotopy from $\alpha$ to $\beta$ is a correspondence $H \in \mathrm{c}_S\left(\mathbb{A}^1 \times X, Y\right)$ such that

(1) $H \circ i_0 = \alpha$
(2) $H \circ i_1 = \beta$

where $i_0$ (resp. $i_1$) is the closed immersion $X \to \mathbb{A}^1_X$ corresponding to the point 0 (resp. the point 1) of $\mathbb{A}^1_X$.

The existence of a homotopy between two correspondences is obviously a reflexive and symmetric relation. However, transitivity fails. We thus adopt the following definition :

**Definition 3.2.** Let $X$ and $Y$ be smooth $S$-schemes and $\alpha, \beta \in \mathrm{c}_S(X,Y)$.

We say $\alpha$ is homotopic to $\beta$, denoted by $\alpha \sim_h \beta$, if there exists a sequence of correspondences $\gamma_0, ..., \gamma_n \in \mathrm{c}_S(X,Y)$ such that $\gamma_0 = \alpha$, $\gamma_n = \beta$ and for every integer $0 \leq i < n$, there exists a homotopy from $\gamma_i$ to $\gamma_{i+1}$.

The relation $\sim_h$ is obviously additive and compatible with the composition law of finite correspondences.

**Definition 3.3.** For two smooth $S$-schemes $X$ and $Y$, we denote by $\pi_S(X,Y)$ the quotient of the abelian group $\mathrm{c}_S(X,Y)$ by the homotopy relation $\sim_h$.

We denote by $\pi\mathscr{S}m_S^{\mathrm{cor}}$ the category with objects smooth $S$-schemes and with morphisms the equivalence classes of finite $S$-correspondences for the relation $\sim_h$.

3.2. **Compactifications.** The purpose of this section is to present a tool (the good compactifications) which allows us to compute the equivalence classes of finite correspondences for the homotopy relation.

3.2.1. *Definition.*

**Definition 3.4.** Let $S$ be a regular scheme and $X$ be an algebraic $S$-curve.

(1) A compactification of $X/S$ is a proper normal curve $\bar{X}/S$ containing $X$ as an open subscheme.
(2) Let $\bar{X}/S$ be a compactification of $X/S$. Put $X_\infty = \bar{X} - X$, seen as a reduced closed subscheme of $\bar{X}$. We say the compactification $\bar{X}/S$ of $X/S$ is *good* if $X_\infty$ is contained in an open subscheme of $\bar{X}$ which is affine over $S$.

When considering a given compactification $\bar{X}/S$ of a curve $X/S$, we will always put $X_\infty = \bar{X} - X$.

*Remark* 3.5. If $\bar{X}/S$ is a good compactification of $X/S$, $X_\infty$ is finite over $S$ as it is proper and affine over $S$. If $S$ is irreducible, $X_\infty$ is surjective over $S$ and Chevalley's theorem (cf [GD61], II.6.7.1) implies that $S$ is affine.

**Definition 3.6.** We call closed pair any pair$(X,Z)$ such that $X$ is a scheme and $Z$ is a closed subscheme of $X$.

A morphism of closed pairs $(f,g) : (Y,T) \to (X,Z)$ is a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r] & X \end{tikzcd}$$

which is cartesian on the corresponding topological spaces. The morphism is said to be cartesian if it is cartesian as a square of schemes.

Let $(X,Z)$ be a closed pair such that $X$ is an $S$-curve. A good compactification of $(X,Z)$ over $S$ is an $S$-scheme $\bar{X}$ which is a good compactification for both $X/S$ and $(X-Z)/S$.

3.2.2. *The case of a base field.* We suppose here $S$ is the spectrum of a field $k$.

**Proposition 3.7.** *Let $C/k$ be a quasi-affine regular algebraic curve. There exists a projective regular curve $\bar{C}$ over $k$ such that for all closed subschemes $Z$ of $C$ nowhere dense in $C$, $\bar{C}$ is a good compactification of $(X, Z)$ over $k$.*

*Proof.* We can restrict to the case $C$ is affine and integral. As $C/k$ is algebraic, we can find a closed immersion $C \to \mathbb{A}^n_k$. Let $\bar{C}$ be the reduced closure of $C$ in $\mathbb{P}^n_k$. It is an integral projective curve over $k$.

Consider the normalisation $\tilde{C}$ of $\bar{C}$. Then $\tilde{C}$ is finite over $\bar{C}$. Is is then a proper algebraic $k$-curve. As it is normal, it is then a projective regular curve over $k$ from [GD61], 7.4.5 and 7.4.10.

The curve $C$ is a dense open subscheme of $\bar{C}$. As it is normal it is again a dense open subscheme of $\tilde{C}$.

Let $Z$ be a closed subscheme of $C$ of dimension 0. Then $(\tilde{C} - C) \sqcup Z$ is a finite closed subset of $\tilde{C}$. As $\tilde{C}/k$ is projective, it admits an open affine neighbourhood in $\tilde{C}$. □

3.2.3. *Semi-local case.*

**Theorem 3.8** (Walker)**.** Let $k$ be an infinite field.

*Let $(X, Z)$ be a closed pair such that $X$ is a smooth affine $k$-scheme and $Z$ is nowhere dense in $X$. Let $\{x_1, ..., x_n\}$ be a finite set of points of $X$.*

*Then there exist*

1. *a smooth affine $k$-scheme $S$,*
2. *an open affine neighbourhood of $x_1, ..., x_n$ in $X$,*
3. *a smooth $k$-morphism $f : U \to S$ of relative dimension* 1

*such that $(U, U \cap Z)$ admits a good compactification over $S$.*

*Proof.* For the reader's convenience, we include the following proof which follows the outline of [Wal96], Remark 4.13.

1) Reduction : We can assume that $X$ is irreducible.

Moreover, we can assume all the $x_i$ are closed, taking specialisations if necessary. If we can find a good compactification in a neighbourhood $V_i$ of each $x_i$ separately, we can define a good compactification in a neighbourhood of all the $x_i$ by first reducing the neighbourhoods $V_i$ such that they become disjoint, then taking their disjoint union. We are thus reduced to the case of a single point $x_1 = x$.

Finally, as we can enlarge $Z$, we may assume that it is a divisor in $X$.

2) Construction of $S$ : Let $r$ be the dimension of $Z$.

As $X$ is an affine algebraic $k$-scheme, we can find a closed immersion $X \hookrightarrow \mathbb{A}^n_k$. We identify $X$ with its image in $\mathbb{A}^n_k$ under this embedding. Let us denote by :

1. $\bar{X}$ (resp. $\bar{Z}$) the reduced closure of $X$ (resp. $Z$) in $\mathbb{P}^n_k$
2. $\dot{X} = \bar{X} - X$ the intersection of $\bar{X}$ with the hyperplane at infinity, a scheme of dimension less than $r$.

If necessary, we may increase $n$ by considering an embedding $\mathbb{A}_k^n \hookrightarrow \mathbb{A}_k^{n'}$.

We find $f$ by considering the orthogonal projection of $\mathbb{A}_k^n$ with center in *general position* among the linear subvarieties of $\mathbb{A}_k^n$ of codimension $r$.

*Parametrisation of the orthogonal projections $\mathbb{A}_k^n \to \mathbb{A}_k^r$.*
These projections are parametrized by the points of $\mathbb{A}_k^{nr}$. Indeed, let $\lambda$ be a point of $\mathbb{A}_k^{nr}$ and $\kappa(\lambda)$ its residue field. It is in fact an element $(\lambda_{i,j})_{\substack{1\le i\le r\\ 1\le j\le n}}$ of $\kappa(\lambda)^{nr}$.

We associate to $\lambda$ the linear projection $p_\lambda : \mathbb{A}^n_{\kappa(\lambda)} \to \mathbb{A}^r_{\kappa(\lambda)}$ defined as the spectrum of the $\kappa(\lambda)$-linear morphism

$$\begin{array}{rcl} \kappa(\lambda)[t_1,...,t_r] & \to & \kappa(\lambda)[X_1,...,X_n] \\ t_i & \mapsto & \sum_{j=1}^n X_j - \lambda_{i,j}. \end{array}$$

We denote by $L_\lambda$ the center of this projection. It is the intersection of the $r$ hyperplanes corresponding to the zeroes of each projection of $\mathbb{A}^r_{\kappa(\lambda)}$ to $\mathbb{A}^1_{\kappa(\lambda)}$ composed with $p_\lambda$.

Moreover if $\dot{L}_\lambda$ denotes the boundary of $L_\lambda$ as a reduced sub-scheme of $\mathbb{P}^n_{\kappa(\lambda)}$, we can extend the morphism $p_\lambda$ to a morphism $\bar{p}_\lambda : \mathbb{P}^n_{\kappa(\lambda)} - \dot{L}_\lambda \to \mathbb{P}^r_{\kappa(\lambda)}$. These notations being fixed, we state the following lemma which allows us to construct $f$ :

**Lemma 3.9.** *Let $\Omega_n$ be the open subset of $\mathbb{A}_k^{nr}$ defined by the points $\lambda$ such that :*

1. *$p_\lambda|_{Z_{\kappa(\lambda)}}$ is finite,*
2. *$\dot{X}_{\kappa(\lambda)} \cap \dot{L}_\lambda$ is a finite set of closed points,*
3. *$p_\lambda$ is smooth at all the points of $X_{\kappa(\lambda)} \cap p_\lambda^{-1}(p_\lambda(x))$.*

*Then, for $n$ large enough, $\Omega_n$ is dense in $\mathbb{A}_k^{nr}$.*

*Proof.* It is easy to see that $\Omega_n$ is open. To prove that it is dense, we proceed in two steps :

i) Let us first assume that $x$ is a *rational point* of $X$. Then we may further assume $x = 0$.

The first condition defines a dense subset, as $Z$ is closed in $\mathbb{A}_k^n$ of dimension $r$. The second condition defines a dense subset as the intersection in $\mathbb{P}_k^n$ of the projective subvariety $\dot{X}$, of dimension less than $r$, with a linear subvariety of codimension $r$ in general position is finite.

For the third condition we only need to assume that the intersection of $L_\lambda$ and $X$ is transversal in 0. We finally use the following theorem of [MAJ73], exposé XI, théorème 2.1 :

**Theorem 3.10.** *The intersection in $\mathbb{A}_k^n$ of $X$ with $r$ hypersurfaces of degree* 2 *containing* 0 *in general position is transversal.*

Via the Veronese embedding of $\mathbb{A}_k^n$ in $\mathbb{A}_k^{n^2}$, a linear subvariety of $\mathbb{A}_k^n$ corresponds to a quadric in $\mathbb{A}_k^{n^2}$ and the preceding theorem can be applied to our case, replacing $n$ by $n^2$.

ii) *General case.* Let $k'/k$ be a finite extension such that the fiber of $x$ in $X \otimes_k k'$ consists of rational points $x'_i$. For each $i$, the preceding lemma gives a dense open subset $\Omega'_{n,i}$ of $\mathbb{A}^{nr}_{k'}$. As $\mathbb{A}^r_{k'}/\mathbb{A}^r_k$ is faithfully flat and he three conditions of the lemma satisfy faithfully flat descent, the direct image of $\cap_i \Omega'_{n,i}$ in $\mathbb{A}^{nr}_k$ is contained in $\Omega_n$. This implies that $\Omega_n$ is dense. □

As $k$ is infinite, $\Omega_n$ admits a rational point $\lambda$. We set $\dot{L} = \dot{X} \cap \dot{L}_\lambda$, which is a finite $k$-scheme. Let $p : X \to \mathbb{A}^r_k$ (resp. $\bar{p} : \bar{X} - \dot{L} \to \mathbb{P}^r_k$) be the restriction of $p_\lambda$ (resp. $\bar{p}_\lambda$).

To extend $\bar{p}$ into a projective morphism we consider $\tilde{X}$, the closure of the graph of $\bar{p}$ in $\bar{X} \times_k \mathbb{P}^r_k$. Then $\bar{X} - \dot{L}$ is a dense open subscheme of $\tilde{X}$ and the canonical projection $\tilde{p} : \tilde{X} \to \mathbb{P}^r_k$ extends $\bar{p}$. As $\bar{X}/k$ is projective, $\tilde{p}$ is projective. We have obtained the following diagram :

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "p"'] & \bar{X} - \dot{L} \arrow[r, hook] \arrow[d, "\bar{p}"] & \tilde{X} \arrow[dl, "\tilde{p}"] \\ A^r_k \arrow[r, hook] & \mathbb{P}^r_k . & \end{tikzcd}$$

3) <u>Construction of the compactification</u> :

As the square in the above diagram is cartesian and $L$ is finite over $k$, the fibers of $\tilde{p}$ in $\tilde{X} - X$ above $\mathbb{A}^r_k$ are finite.

Hence there exists an open affine neighbourhood $S$ of $p(x)$ in $\mathbb{A}^r_k$ such that $\tilde{p}^{-1}(S) \cap (\tilde{X} - X)$ is finite over $S$. Reducing $S$ if necessary, we can assume that $p^{-1}(S) \to S$ is smooth using the third condition imposed on $\Omega_n$ in the preceding lemma.

Finally we put $U = p^{-1}(S)$ and we denote by $f : U \to S$ the restriction of $p$ to $U$. Then the morphism $f$ is smooth of relative dimension 1. Moreover the restriction $Z \cap U \to S$ of $p$ is finite by the first condition imposed on the points of $\Omega_n$.

We set $\bar{U} = \tilde{p}^{-1}(S)$; the restriction $\bar{f} : \bar{U} \to S$ of $\tilde{p}$ is projective. By the choice of $S$, $\bar{U} - U$ is finite over $S$.

To conclude the proof we use the following lemma, which shows that by reducing $S$ near $p(x)$ we may assume that $(\bar{U} - U) \sqcup Z \cap U$ admits an affine neighbourhood :

**Lemma 3.11.** *Let $\bar{p} : \bar{U} \to S$ be a projective curve and $F$ be a closed subscheme of $\bar{U}$ such that $F/S$ is finite. Let $x$ be a point of $F$ and $s = \bar{p}(x)$.*

*Then there exist an open affine neighbourhood $S'$ of $s$ in $S$ and an effective divisor $D$ in $\bar{X}$ such that :*

(1) $F_{S'} \subset \bar{U}_{S'} - D$.
(2) $\bar{U}_{S'} - D_{S'}$ *is affine.*

*Proof.* Let $F_s$ be the fiber of $F$ over $s$. As a set, $F_s$ is finite. As $\bar{U}/S$ is projective there exists for $i$ large enough a section $f$ in $\Gamma(\bar{U}, \mathcal{O}_{\bar{U}}(i))$ whose divisor $D$ is disjoint from $F_s$. Thus there exists an open affine neighbourhood $S'$ of $s$ in $S$ such that $D$ is disjoint from $F_{S'}$; this gives the first condition. As $S'$ is affine and $D_{S'}$ is the divisor associated to a global section of a very ample fiber bundle over $\bar{U}_{S'}$, the scheme $\bar{U}_{S'} - D_{S'}$ is affine. □

□

### 3.3. **Relative Picard group.**

**Definition 3.12.** Let $(X, Z)$ be a closed pair.

We denote by $\mathrm{Pic}\,(X, Z)$ the group of couples $(\mathcal{L}, s)$ where $\mathcal{L}$ is an invertible sheaf on $X$ and $s : \mathcal{O}_Z \xrightarrow{\sim} \mathcal{L}|_Z$ is a trivialisation of $\mathcal{L}$ over $Z$, modulo isomorphisms of invertible sheaves compatible with the trivialisation.

The group structure is induced by the tensor product of $\mathcal{O}_X$-modules.

There is a canonical morphism $\mathrm{Pic}\,(X, Z) \to \mathrm{Pic}(X)$.

**Definition 3.13.** let $X$ be a scheme and $Z$ a closed subscheme of $X$.

(1) If $Z'$ is closed subscheme of $Z$, we define the restriction morphism from $Z$ to $Z'$

$$\begin{array}{ccc} \mathrm{Pic}\,(X, Z) & \xrightarrow{r_{Z'}} & \mathrm{Pic}\,(X, Z') \\ (\mathcal{L}, s) & \mapsto & (\mathcal{L}, s|_{Z'}). \end{array}$$

(2) Let $(f, g) : (Y, W) \to (X, Z)$ be a morphism of closed pairs (cf def. 3.6). We define the pullback morphism

$$\begin{array}{rcl} (f, g)^* : \mathrm{Pic}\,(X, Z) & \to & \mathrm{Pic}\,(Y, W) \\ (\mathcal{L}, s) & \mapsto & (f^*(\mathcal{L}), g^*(s))\,. \end{array}$$

**3.14.** Let $S$ be a regular affine scheme. We consider a smooth quasi-affine curve $X/S$ and suppose that it admits a good compactification $\bar{X}/S$.

Let $\alpha$ be a relative cycle on $X/S$. Then, considered as a codimension 1 cycle of $\bar{X}$, it corresponds to an invertible sheaf $\mathcal{L}(\alpha)$ on $\bar{X}$ whose isomorphism class is unique. Moreover, if $Z$ is the support of $\alpha$, this sheaf has a canonical trivialisation on $\bar{X} - Z$. Let $s(\alpha)$ be its restriction to $X_\infty$. We have thus defined a canonical morphism

$$\mathrm{c}_0(X/S) \xrightarrow{\lambda_{X/S}} \mathrm{Pic}\left(\bar{X}, X_\infty\right), \alpha \mapsto (\mathcal{L}(\alpha), s(\alpha)).$$

**Lemma 3.15.** *Consider the above notations.*

(1) *Let $S'$ be a regular scheme and $\tau : S' \to S$ be a flat morphism. Put $X' = X \times_S S'$, $\bar{X}' = \bar{X} \times_S S'$, and let $\Delta : X'/S' \to X/S$ be the morphism induced by $\tau$. We consider $(\bar{f}, f_\infty) : (\bar{X}', X'_\infty) \to (\bar{X}, X_\infty)$, the cartesian morphism of closed pairs induced by $\tau$. Then the following diagram is commutative :*

$$\begin{array}{ccc} \mathrm{c}_0(X/S) & \xrightarrow{\lambda_{X'/S'}} & \mathrm{Pic}\left(\bar{X}, X_\infty\right) \\ {\scriptstyle \Delta^*}\downarrow & & \downarrow{\scriptstyle (\bar{f}, f_\infty)^*} \\ \mathrm{c}_0(X'/S') & \xrightarrow{\lambda_{X/S}} & \mathrm{Pic}\left(\bar{X}', X'_\infty\right). \end{array}$$

(2) *Let $Z$ be a closed subscheme of $X$ and suppose that $\bar{X}$ is a good compactification of $(X, Z)$ over $S$. Then the following diagram commutes :*

$$\begin{array}{ccc} \mathrm{c}_0(X - Z/S) & \xrightarrow{\lambda_{X-Z/S}} & \mathrm{Pic}\left(\bar{X}, X_\infty \sqcup Z\right) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle r_{(Y \times_S X_\infty)}} \\ \mathrm{c}_0(X/S) & \xrightarrow{\lambda_{X/S}} & \mathrm{Pic}\left(\bar{X}, X_\infty\right). \end{array}$$

*Proof.* The second point is obvious by construction.

For the first point, let $f : X' \to X$ be the flat morphism induced by $\tau$. Let $\alpha$ be a finite relative cycle on $\bar{X}/S$. Suppose that $\alpha$ is the class of a closed subscheme $Z$ in $X$. Then by Proposition 1.7, $\Delta^*\alpha$ is the cycle associated to the closed subscheme $f^{-1}(Z)$ of $X'$. Thus the conclusion follows from the construction of $\lambda$. □

**Proposition 3.16.** *Consider the notation of the previous lemma and let $Y$ be a smooth affine $S$-scheme. Then the morphism $\lambda_{Y \times_S X/Y}$ factors through the homotopy relation. The induced morphism*

$$\pi_S(Y, X) \to \mathrm{Pic}\left(Y \times_S \bar{X}, Y \times_S X_\infty\right)$$

*is an isomorphism.*

*Proof.* Let $i_0 : Y \to \mathbb{A}^1_Y$ (resp. $i_1 : Y \to \mathbb{A}^1_Y$) be the zero section (resp. unit section) of $\mathbb{A}^1_Y/Y$.

Note that $i_0$ and $i_1$ are inclusions of a cycle associated to a principal Cartier divisor. Then the pullback maps $c_0\left(\mathbb{A}^1_Y \times_S X/\mathbb{A}^1_Y\right) \to c_0(Y \times_S X/Y)$ induced by $i_0$ and $i_1$ coincide with the operation of intersecting with divisors defined in [Ful98], 2.3 (see also remark 2.3 of *loc. cit.*). This allows us to extend the first case of the previous lemma to the case where $\tau$ is $i_0$ or $i_1$. Finally, using the homotopy invariance of the Picard group for regular schemes, $\lambda$ indeed factors through the homotopy relation of $S$-correspondences.

To show that the induced morphism is an isomorphism, we construct its inverse. It suffices to treat the case $Y = S$. Let $(\mathcal{L}, s)$ be an element of $\mathrm{Pic}\left(\bar{X}, X_\infty\right)$. Consider an open affine neighbourhood $V$ of $X_\infty$ in $\bar{X}$. The trivialisation $s$ of $\mathcal{L}$ then extends to a trivialisation $\tilde{s}$ of $\mathcal{L}$ over $V$. To the pseudo-divisor $(\mathcal{L}, \bar{X} - V, \tilde{s})$ is associated a unique Cartier divisor $D(\mathcal{L}, \bar{X} - V, \tilde{s})$ on $\bar{X}$ following [Ful98], Lemma 2.2. Let $\alpha$ be the associated cycle. The support of $\alpha$ lies in $(\bar{X} - V)$. Moreover, as $X/S$ is quasi-affine and $\bar{X}/S$ is proper, $V$ is dense in all the fibers of the curve $\bar{X}/S$ which implies $\bar{X} - V$ is finite over $S$. Finally, the support of $\alpha$ is finite over $S$ and $\alpha$ is in fact a finite relative cycle on $X/S$.

We prove now that the homotopy class of $\alpha$ in $c_S(S, X)$ does not depend on the choice of $\tilde{s}$. Suppose given two extensions $\tilde{s}_0$ and $\tilde{s}_1$ of $s$ to $V$. Let $\alpha_0$ and $\alpha_1$ be the respective cycles obtained in the process described above. Define $\mathcal{L}'$ as the pullback of $\mathcal{L}$ along the morphism $\pi : \mathbb{A}^1_{\bar{X}} \to \bar{X}$. For $i = 0, 1$, we obtain a trivialisation $\pi^*\tilde{s}_i$ of $\mathcal{L}'$ over $\mathbb{A}^1_V$. Let $H$ be the cycle associated to the pseudo-divisor $D(\mathcal{L}', \mathbb{A}^1_{\bar{X}-V}, t\pi^*\tilde{s}_0 + (1-t)\pi^*\tilde{s}_1)$, where $t$ is the canonical parameter of $\mathbb{A}^1_{\bar{X}}$. Then, using the beginning of the proof, we obtain $H \circ i_0 = \alpha_0$ and $H \circ i_1 = \alpha_1$. □

*Remark* 3.17. The previous proposition is a particular case of the computation of the Suslin singular homology of the curve $X/S$ in [SV96], Th. 3.1.

## 3.4. **Constructing useful correspondences up to homotopy.**

3.4.1. *Factorisations.*

**Proposition 3.18.** *Let $S$ be an affine regular scheme, and $(X, Z)$ a closed pair such that $X$ is a smooth affine $S$-curve. Put $U = X - Z$ and denote* *by $i : U \to X$ the canonical open immersion. Suppose that $(X, Z)$ admits a good compactification $\bar{X}$ over $S$.*

*Let $\mathcal{L}(1_X)$ be the invertible sheaf corresponding to $1_X \in \mathrm{c}_S(X, X)$ in the notation of 3.14.*

*The following conditions are equivalent :*

(1) *For any smooth affine $S$-scheme $Y$, the morphism*

$$\pi_S(Y, U) \xrightarrow{i\circ} \pi_S(Y, X)$$

*is surjective.*

(2) *The morphism*

$$\pi_S(X, U) \xrightarrow{i\circ} \pi_S(X, X)$$

*is surjective.*

(3) *The invertible sheaf $\mathcal{L}(1_X)|_{X \times_S Z}$ is trivial.*

*Proof.* Conditions (1) and (2) are equivalent to the existence of a section of $i$ up to homotopy. Thus the proposition is implied by the following more precise lemma :

**Lemma 3.19.** *Consider the hypothesis of the preceding proposition. Let $Y$ be a smooth affine $S$-scheme and $\beta : Y \to X$ a finite $S$-correspondence. The following conditions are equivalent :*

(1) *There exists a finite $S$-correspondence $\alpha$ that makes the following diagram of $S$-correspondences commutative up to homotopy*

$$\begin{tikzcd} & X - Z \arrow[dr, "i"] & \\ Y \arrow[ur, dashed, "\alpha"] \arrow[rr, "\beta"] & & X. \end{tikzcd}$$

(2) *The invertible sheaf $\mathcal{L}(\beta)|_{Y \times_S Z}$ is trivial, with the notation of 3.14.*

*Moreover, the finite $S$-correspondences which satisfy condition (1) are in one-to-one correspondence with the trivialisations of $\mathcal{L}(\beta)|_{Y \times_S Z}$.*

We use Proposition 3.16 applied first to the affine curve $X/S$ and secondly to the quasi-affine curve $U/S$.

(2) $\Rightarrow$ (1) : Consider a trivialisation $s$ of $\mathcal{L}(\beta)|_{Y \times_S Z}$. Then the class of the pair $(\mathcal{L}(\beta), s(\beta) \oplus s)$ in $\mathrm{Pic}\big(Y \times_S \bar{X}, Y \times_S X_\infty \sqcup Y \times_S Z\big)$ defines a finite $S$-correspondence $\alpha$ which, according to Lemma 3.15, satisfies $i \circ \alpha = \beta$ as required.

(1) $\Rightarrow$ (2) : Conversely, the finite $S$-correspondence $\alpha$ corresponds to an element of $\mathrm{Pic}\big(Y \times_S \bar{X}, Y \times_S X_\infty \sqcup Y \times_S Z\big)$ which is the class of the pair $(\mathcal{L}(\alpha), s(\alpha))$. Thus, as $i \circ \alpha = \beta$, there exists an isomorphism $\phi : \mathcal{L}(\beta) \to \mathcal{L}(\alpha)$ that makes the following diagram commutative

$$\begin{tikzcd} \mathcal{L}(\beta)|_{Y \times_S X_\infty} \arrow[rr, "\phi|_{Y \times_S X_\infty}"] \arrow[dr, "s(\beta)"'] & & \mathcal{L}(\alpha)|_{Y \times_S X_\infty} \arrow[dl, "s(\alpha)|_{Y \times_S X_\infty}"] \\ & \mathcal{O}_{Y \times_S X_\infty}. & \end{tikzcd}$$

Then $s(\alpha)|_{Y \times_S Z} \circ \phi^{-1}|_{Y \times_S Z}$ is indeed a trivialisation of $\mathcal{L}(\beta)|_{Y \times_S Z}$.

The last point of the lemma is clear from the proof. $\square$

**Example 3.20.** As an easy application of this proposition, we consider two open subschemes $X$ and $U$ of the affine line $\mathbb{A}^1_k$ over a field $k$ such that $U \subset X$. Put $Z = (X - U)_{red}$.

Then the open immersion $i : U \to X$ admits a section in $\pi\mathscr{S}m^{\mathrm{cor}}_k$ as $\mathbb{P}^1_k$ is a good compactification of $(X, Z)$, $X$ is affine and $\mathrm{Pic}\,(X \times_k Z) = 0$.

Moreover, choosing a trivialisation of $\mathcal{L}(1_{\mathbb{A}^1_k})$ once and for all, we define trivialisations for all open immersions $i : U \to X$ which are functorial with respect to open immersions in $X$ and $U$.

3.4.2. *Local section of open immersions in $\pi\mathscr{S}m^{\mathrm{cor}}_k$.* The following proposition is directly inspired by Proposition 4.17 of [Voe00a] :

**Proposition 3.21.** *Let $k$ be a field, $X$ a smooth $k$-scheme, $U$ a dense open subscheme of $X$, and $x$ a point of $X$. Then there exist*

(1) *an open neighbourhood $V$ of $x$ in $X$,*
(2) *a finite $k$-correspondence $\alpha : V \to U$,*

*such that the following diagram is commutative up to homotopy*

$$\begin{tikzcd} & V \arrow[dl, dashed, "\alpha"'] \arrow[d, "j"] \\ U \arrow[r, "i"] & X, \end{tikzcd}$$

*where $i$ and $j$ are the obvious open immersions.*

*Proof.* Suppose first that $k$ is infinite.

Put $Z = (X - U)_{red}$. Using Theorem 3.8, there exist an affine smooth $k$-scheme $S$, an affine open neighbourhood $V$ of $x$ in $X$, a smooth morphism $f : V \to S$ of relative dimension 1, and an $S$-scheme $\bar{X}$ such that $\bar{X}/S$ is a good compactification of $(V, V \cap Z)$.

From the commutative diagram

$$\begin{tikzcd} V \cap U \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & ; \end{tikzcd}$$

we see that the theorem holds for $V$, if it holds for $X$. Thus we can assume $X = V$, which implies that $(X, Z)$ has a good compactification $\bar{X}$ over $S$.

Let $\mathcal{L}(1_X)$ be an invertible sheaf over $X \times_S \bar{X}$ which corresponds to $1_X \in \pi_S(X, X)$ according to Proposition 3.16. As $Z$ is affine and closed in the proper curve $\bar{X}/S$, it is finite over $S$. The scheme $\mathrm{Spec}\,(\mathcal{O}_{X,x}) \times_S Z$ is finite over the local scheme $\mathrm{Spec}\,(\mathcal{O}_{X,x})$, hence it is semi-local. This implies that $\mathrm{Pic}(\mathrm{Spec}\,(\mathcal{O}_{X,x}) \times_S Z) = 0$. In particular, $\mathcal{L}(1_X)$ is trivial over $\mathrm{Spec}\,(\mathcal{O}_{X,x}) \times_S Z$. Thus there exists an open neighbourhood $V$ of $x$ in $X$ such that $\mathcal{L}(1_X)$ is trivial over $V \times_S Z$.

From Lemma 3.19, applied to $Y = V$ and to the finite $S$-correspondence $V \xrightarrow{j} X$, there exist a finite $S$-correspondence $\alpha : V \to U$ which makes the following diagram commutative :

$$\begin{tikzcd} & V \arrow[dl, bend right, "\alpha"'] \arrow[d, "j"] \\ U \arrow[r, "i"] & X. \end{tikzcd}$$

Let $\tau : S \to k$ be the canonical morphism. As $\tau$ is smooth, the restriction functor $\tilde{\tau}_\sharp$ of definition 1.32 is well defined. Applying this functor to the preceding diagram, we see that the $k$-finite correspondence $\tilde{\tau}_\sharp(\alpha)$ is appropriate.

When $k$ is finite, we consider $L = k(t)$. We put $X_L = X \times_k \operatorname{Spec}(L)$ and similarly for any $k$-scheme. The point $x$ corresponds canonically to a point of $X_L$ still denoted by $x$.

Applying the preceding case to the open immersion $i_L : U_L \to X_L$ and to the point $x$, we find a neighbourhood $\Omega$ of $x$ in $X_L$ and a finite $L$-correspondence $\alpha : \Omega \to U_L$ such that $i_L \circ \alpha$ is the open immersion $\Omega \to X_L$. As $x$ comes from a point of $X$, we can always find an open neighbourhood $V$ of $x$ in $X$ such that $V_L \subset \Omega$. The following diagram

$$\begin{tikzcd} & V_L \arrow[dl, "\alpha|_{V_L}"'] \arrow[d, "j_L"] \\ U_L \arrow[r, "i_L"] & X_L. \end{tikzcd}$$

is commutative in $\pi\mathscr{S}m_L^{\mathrm{cor}}$, with $j : V \to X$ the canonical immersion.

Applying Proposition 1.8, we obtain a canonical isomorphism

$$\mathrm{c}_L\left(V_L, Y_L\right) = \varinjlim_{W \subset \mathbb{A}^1_k} \mathrm{c}_k\left(V \times_k W, Y\right),$$

for any $k$-scheme $Y$, where the limit runs over the non empty open subschemes $W$ of $\mathbb{A}^1_k$. It is functorial in $Y$.

In particular, we can lift both the finite $L$-correspondence $\alpha|_{V_L}$ and the homotopy making the above diagram commutative for a sufficiently small $W$ in $\mathbb{A}^1_k$. We thus obtain a finite $k$-correspondence $\alpha_0 : V \times_k W \to U$ such that the diagram over $k$

$$\begin{tikzcd} & V \times_k W \arrow[dl, "\alpha_0"'] \arrow[d, "j \times_k p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

is commutative up to homotopy, with $p : W \to \operatorname{Spec}(k)$ the canonical projection.

Finally, we factor out $j \times_k p$ as $V \times_k W \xrightarrow{1 \times_k p} V \xrightarrow{j} X$. Example 3.20 gives a section of the open immersion $W \to \mathbb{A}^1_k$, which shows $1 \times_k p$ admits a section in $\pi\mathscr{S}m_k^{\mathrm{cor}}$ and concludes the proof. $\square$

**Corollary 3.22.** *Let $k$ be a field, $X$ a smooth $k$-scheme and $U$ a dense open subscheme of $X$. Then there exist*

(1) *an open covering $p : W \to X$ of $X$,*
(2) *a finite $k$-correspondence $\alpha : W \to U$*

*such that the following diagram is commutative up to homotopy*

$$\begin{tikzcd} & W \arrow[dl, dashed, "\alpha"'] \arrow[d, "j"] \\ U \arrow[r, "i"] & X, \end{tikzcd}$$

*where $i$ and $j$ are the canonical open immersions.*

*Proof.* We simply apply the preceding lemma to every point of $X$ and use its quasi-compactness. $\square$

3.4.3. *Homotopy excision.* The following proposition is one of the central points in our interpretation of Voevodsky's theory. It is a generalisation of Lemma 4.6 of [Voe00a].

**Theorem 3.23.** *Let $S$ be an affine regular scheme. Consider a distinguished square (cf section 2.1) of smooth affine $S$-schemes*

$$\begin{array}{ccc} W & \xrightarrow{l} & V \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & C. \end{array}$$

*We put $Z = C - U$ and $T = V - W$ with their reduced structure and assume that there exist good compatifications $\bar{C}/S$ of $(C, Z)$ and $\bar{V}/S$ of $(V, T)$ which fit into the commutative square*

$$\begin{array}{ccc} V & \hookrightarrow & \bar{V} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ C & \hookrightarrow & \bar{C} \end{array}$$

*and satisfy $V_\infty \subset \bar{f}^{-1}(C_\infty)$. Assume finally* $\mathrm{Pic}\,(C \times_S Z) = 0$. *Then the complex*

$$0 \to [W] \xrightarrow{h-l} [U] \oplus [V] \xrightarrow{(j,f)} [C] \to 0$$

*is contractible in the additive category $\pi\mathscr{S}m_S^{\mathrm{cor}}$.*

*Proof.* In the following lemma, we will construct the chain homotopy between the complex above and the zero complex. Indeed, with the notations of this lemma, the chain homotopy is given by the two morphisms

$$\begin{tikzcd} & {[U] \oplus [V]} \arrow[dl, "(\gamma, -\beta)"'] & & {[C]} \arrow[dl, "\alpha"] \\ {[W]} & & {[U] \oplus [V].} & \end{tikzcd}$$

The necessary relations are stated and proved in the lemma.

**Lemma 3.24.** *With the hypotheses and notation of the preceding theorem, there exists finite correspondences*

$$\begin{array}{ccc} W & \xleftarrow{\beta} & V \\ {\scriptstyle \gamma}\uparrow & & \\ U & \xleftarrow{\alpha} & C \end{array}$$

*that satisfy the following relations in $\pi\mathscr{S}m_k^{\mathrm{cor}}$ :*

$$\begin{cases} j \circ \alpha = 1_C & (1) \\ l \circ \beta = 1_V & (2) \\ \alpha \circ f = h \circ \beta & (3) \\ l \circ \gamma = 0 & (4) \\ h \circ \gamma = 1_U - \alpha \circ j & (5) \\ \gamma \circ h = 1_W - \beta \circ l & (6) \end{cases}$$

We first apply Proposition 3.16 to the morphism $1_C$, as an element of $\pi_S(C, C)$. It corresponds to the class of a pair$(\mathcal{L}(1_C), s(1_C))$ in $\mathrm{Pic}\left(C\bar{C}, CC_\infty\right)$ [5].

[5]In this proof, we sometimes omit the symbol $\times_S$ when it simplifies the notation

By hypothesis Pic $(C \times_S Z) = 0$, hence the invertible sheaf $\mathcal{L}(1_C)$ is trivial on $C \times_S Z$. Let $t$ be a trivialisation. We define $\alpha$ corresponding to the following element in Pic $\big(C\bar{C}, CC_\infty \sqcup CZ\big)$

$$\alpha \qquad \leftrightarrow \qquad (\mathcal{L}(1_C), s(1_C) + t).$$

Relation (1) simply follows from Lemma 3.19 as in the preceding applications.

Using again Proposition 3.16, the morphism $1_V$, as an element of $\pi_S(V, V)$, corresponds to the class of an element $(\mathcal{L}(1_V), s(1_V))$ in Pic $\big(V\bar{V}, VV_\infty\big)$.

By construction, the sheaf $\mathcal{L}(1_C)$ (resp. $\mathcal{L}(1_V)$) corresponds to the diagonal $\Delta_C$ (resp. $\Delta_V$) of $C/k$ (resp. $V/k$) seen as a closed subscheme of $C \times_S \bar{C}$ (resp. $V \times_S \bar{V}$). Since the morphism $g = f \times_X Z : T \to Z$ is an isomorphism, we obtain

$$(f \times_S g)^{-1}(\Delta_X \cap (X \times_S Z)) = \Delta_V \cap (V \times_S T)$$

which finally gives

$$(f \times_S g)^*(\mathcal{L}(1_C)|_{X \times_S Z}) = \mathcal{L}(1_V)|_{V \times_S T}.$$

In particular, the section $\tau = (f \times_S g)^*(t)$ is a trivialisation of $\mathcal{L}(1_V)$ on $V \times_S T$. Let us define $\beta$ corresponding to the following element in Pic $\big(V\bar{V}, VV_\infty \sqcup VT\big)$

$$\beta \qquad \leftrightarrow \qquad (\mathcal{L}(1_V), s(1_V) + \tau).$$

Relation (2) is again a consequence of Lemma 3.19.

It remains to construct $\gamma$. We consider the invertible sheaf $\mathcal{M} = (1_C \times_S \bar{f})^*\mathcal{L}(1_C)$ on $C \times_S \bar{V}$. It corresponds to the divisor $D = (1_C \times_S \bar{f})^{-1}(\Delta_C)$. Let $u$ be the canonical trivialisation of $\mathcal{M}$ on $C \times_S V - D$. As $g$ is an isomorphism, $v = (1_C \times_S g)^* t$ is a trivialisation of $\mathcal{M}|_{C \times_S Z}$. Note that $1 + uv^{-1}$ is a regular invertible section of $\mathcal{O}_{C\bar{V}}$ over $CV_\infty \sqcup CT$. We define $\gamma$ corresponding to the class of the following element in Pic $\big(C\bar{V}, CV_\infty \sqcup CT\big)$ :

$$\gamma \qquad \leftrightarrow \qquad (\mathcal{O}_{C\bar{V}}, 1 + uv^{-1}).$$

By construction and Lemma 3.15, $l \circ \gamma$ corresponds to the pair $(\mathcal{O}_{C\bar{V}}, 1)$, which is the zero correspondence. This is relation (4).

Consider an open affine neighbourhood $\Omega$ of $C_\infty \sqcup Z$ in $\bar{C}$. Put $\Omega_0 = \bar{f}^{-1}(\Omega)$ and let $\nu : \Omega_0 \to \Omega$ be the finite morphism induced by $\bar{f}$. Then $\Omega_0$ is an open affine neighbourhood of $V_\infty \sqcup T$. Thus the invertible regular function $1 + uv^{-1}$ admits an extension $w$ to $U \times_S \Omega_0$. Following the computation of [Ful98], 1.4, we see that the correspondence $h \circ \gamma$ corresponds via the isomorphism of prop. 3.16 to the following element in Pic $\big(U\bar{C}, UC_\infty \sqcup UT\big)$:

$$\Big(\mathcal{O}_{U\bar{C}}, N(w)|_{UC_\infty \sqcup UT}\Big),$$

where $N$ is the norm associated to the extension ring corresponding to $U\Omega_0/U\Omega$. As $w|_{UV_\infty} = 1$, we easily obtain that $N(w)|_{UC_\infty} = 1$. A more detailed computation shows moreover $N(w)|_{UT} = s(1_U).t^{-1}$, as $g$ is an isomorphism and $f$ is étale.

The finite correspondence $1_U - \alpha \circ j$ corresponds to the pair

$$\Big(\mathcal{L}(1_U) \otimes (\mathcal{L}(1_C)|_{U\bar{C}})^{-1}, s(1_U).(s(1_C) + t)^{-1}\Big).$$

Thus relation (5) is now clear.

Finally, using again Lemma 3.15, $\gamma \circ h$ corresponds to the pair

$$\Big(\mathcal{O}_{W \times_S \bar{V}}, 1 + s(1_V).\tau^{-1}\Big).$$

Indeed, by definition, the pullback of $v$ over $W \times_S \bar{V}$ is $\tau$.

Relation (6) now follows, since the finite correspondence $1_W - \beta \circ l$ corresponds to the pair

$$\Big(\mathcal{L}(1_W) \otimes (\mathcal{L}(1_V)|_{W\bar{V}})^{-1}, s(1_W).(s(1_V) + \tau)^{-1}\Big).$$

Only the relation (3) remains. We consider the trivialisation $s(1_V)$ (resp. $\tau$)) of the invertible sheaf $\mathcal{L}(1_V)$ over $VV_\infty$ (resp. $VT$). As $\Omega_0$ is an affine neighbourhood of $V_\infty \sqcup T$, the trivialisation $s(1_V)$ (resp. $\tau$) admits an extension $w_1$ (resp. $w_2$) to $V \times_S \Omega_0$. Using a computation we have already seen, establishing relation (3) is equivalent to showing that the following two elements of Pic $\big(V\bar{C}, VC_\infty \sqcup VZ\big)$ are equal :

$$\Big((f \times_S 1_{\bar{C}})^* \mathcal{L}(1_C), (f \times_S 1_{\bar{C}})^*(s(1_C) + t)\Big)$$
$$\Big((1_V \times_S \bar{f})_* \mathcal{L}(1_V), N'(w_1 + w_2)|_{VC_\infty \sqcup VZ}\Big).$$

We have denoted by $N'$ the norm associated to the finite extension $V\Omega_0/V\Omega$. Using again that $g$ is an isomorphism and $f$ is étale, we obtain $N'(w_2)|_{VZ} = (f \times_S 1_{\bar{C}})^*(t)$.

But the equality $1_C \circ f = f \circ 1_V$ implies that the following pairs coincide

$$\Big((f \times_S 1_{\bar{C}})^* \mathcal{L}(1_C), (f \times_S 1_{\bar{C}})^*(s(1_C))\Big)$$
$$\Big((1_V \times_S \bar{f})_* \mathcal{L}(1_V), N'(w_1)|_{VC_\infty}\Big),$$

and this concludes the proof. □

To finish, we give a simple example where we can construct compactifications that appear in the above theorem. Suppose we are only given only the distinguished square in the hypothesis of the preceding proposition, and assume that $S$ is the spectrum of a field $k$.

Then, according to Proposition 3.7, there exists a smooth projective curve $\bar{C}/k$ which is a good compactification of $(C, Z)$.

The morphism $V \xrightarrow{f} C \to \bar{C}$ is quasi-affine. Applying Zariski's main theorem (cf [GD63], chap. III, 4.4.3), it can be factored as $V \xrightarrow{\tilde{j}} \tilde{V} \xrightarrow{\tilde{f}} \bar{C}$ where $\tilde{j}$ is an open immersion and $\tilde{f}$ a finite morphism.

As $\tilde{V}/k$ is algebraic, its normalisation $\bar{V}$ is finite over $\tilde{V}$, and still contains $V$ as an open subscheme since $V$ is normal. Thus $\bar{V}$ is a good compactification of $(V, Z)$ and we have the following commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & \bar{V} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ C & \longrightarrow & \bar{C}. \end{array}$$

## 4. Homotopy sheaves with transfers

### 4.1. Homotopy invariance.

**Definition 4.1.** Let $S$ be a scheme.

A presheaf $F$ on $\mathscr{S}m_S$ is said to be homotopy invariant if for all smooth $S$-schemes $X$, the morphism induced by the canonical projection $F(X) \to F(\mathbb{A}^1_X)$ is an isomorphism.

When $S$ is regular, we denote by $\mathrm{H}\mathscr{N}_S^{\mathrm{tr}}$ (resp. $\mathrm{H}\mathscr{P}_S^{\mathrm{tr}}$) the category of sheaves (resp. presheaves) with transfers over $S$ which are homotopy invariant. Such sheaves (resp. presheaves) will simply be called homotopy sheaves (resp. homotopy presheaves).

The following lemma relates homotopy presheaves to correspondences up to homotopy.

**Lemma 4.2.** *Let $F$ be a presheaf with transfers over a regular scheme $S$. The following conditions are equivalent :*

(1) *$F$ is homotopy invariant.*
(2) *For all smooth $S$-schemes $X$, considering $s_0 : X \to \mathbb{A}^1_X$ (resp. $s_1 : X \to \mathbb{A}^1_X$) the zero (resp. unit) section of $\mathbb{A}^1_X$, $s_0^* = s_1^*$.*
(3) *$F$ can be factored through the canonical morphism $\mathscr{S}m_S^{\mathrm{cor}} \to \pi\mathscr{S}m_S^{\mathrm{cor}}$.*

*Proof.* For a smooth $S$-scheme $X$, we denote by $p_X : \mathbb{A}^1_X \to X$ (resp. $\mu_X : \mathbb{A}^1_X \times_X \mathbb{A}^1_X \to \mathbb{A}^1_X$) the canonical projection (resp. multiplication) of the ringed $X$-scheme $\mathbb{A}^1_X$.

The lemma now follows easily from the relations $p_X \circ s_0 = p_X \circ s_1 = 1_X$ and the fact that $\mu_X$ defines a homotopy from $s_0$ to $s_1$. □

**4.3.** In particular, a homotopy presheaf (resp. homotopy sheaf) is nothing but a presheaf on $\pi\mathscr{S}m_S^{\mathrm{cor}}$ (resp. a presheaf whose restriction to $\mathscr{S}m_S$ is a Nisnevich sheaf).

As a corollary, the forgetful functor $\mathrm{H}\mathscr{P}_S^{\mathrm{tr}} \to \mathscr{P}_S^{\mathrm{tr}}$ admits a left adjoint $\hat{h}_0\,(.)$ constructed as follows. Let $F$ be a presheaf with transfers, and define $\hat{h}_0\,(F)\,(X)$ as the cokernel of the morphism $F(\mathbb{A}^1_X) \xrightarrow{s_0^*-s_1^*} F(X)$. The preceding lemma implies that $\hat{h}_0\,(F)$ is homotopy invariant and has the adjunction property.

Consider now a sheaf with transfers $F$. We denote by $h_0^{(1)}F$ the sheaf with transfers associated to the presheaf $\hat{h}_0\,(F)$ using Corollary 2.7. In general, this sheaf is not homotopy invariant - unless $S$ is the spectrum of a perfect field (see 4.14). For a natural integer $n$, we denote by $h_0^{(n)}$ the $n$-th composition power of $h_0^{(1)}$. We deduce a sequence of morphisms

$$F \to h_0^{(1)}F \to \dots \to h_0^{(n)}F \to \dots$$

and define

$$h_0(F) = \varinjlim_{n\in\mathbb{N}} h_0^{(n)}F,$$

where the limit is taken in the category of sheaves with transfers.

**Proposition 4.4.** *Let $S$ be a regular scheme and $F$ a sheaf with transfers over $S$.*

*Then the sheaf with transfers $h_0(F)$ defined above is homotopy invariant. Moreover, the functor $h_0 : \mathscr{N}_S^{\mathrm{tr}} \to \mathrm{H}\mathscr{N}_S^{\mathrm{tr}}$ is left adjoint to the obvious forgetful functor.*

*Proof.* Let $X$ be a smooth $S$-scheme, $s_0$ and $s_1$ the zero and unit sections of $\mathbb{A}^1_X/X$. According to the preceding lemma, we have to show that $s_0^* = s_1^*$ on $h_0(F)(\mathbb{A}^1_X)$. Let $\mathbf{x}$ be an element of

$$h_0(F)(\mathbb{A}^1_X) = \varinjlim_{n \in \mathbb{N}} h_0^{(n)} F(\mathbb{A}^1_X).$$

By definition, it is represented by a section $x_n$ in $h_0^{(n)} F(\mathbb{A}^1_X)$ for an integer $n \in \mathbb{N}$.

The transition morphism of level $n$ in the above inductive limit can be factored out as

$$h_0^{(n)} F \xrightarrow{a} \hat{h}_0\left(h_0^{(n)} F\right) \xrightarrow{b} h_0^{(n+1)} F.$$

From what we saw before, the sheaf $\hat{h}_0\left(h_0^{(n)} F\right)$ is homotopy invariant. Thus $s_0^*(ax_n) = s_1^*(ax_n)$. As $a$ is a natural transformation, we deduce that $as_0^*(x_n) = as_1^*(x_n)$, thus $bas_0^*(x_n) = bas_1^*(x_n)$ and $s_0^*(\mathbf{x}) = s_1^*(\mathbf{x})$. □

4.2. **Fibers along function fields.** In this subsection, we fix a field $k$. To simplify the notation, we write $\mathscr{S}m_k$ (resp. $\pi\mathscr{S}m_k^{\mathrm{cor}}$) in stead of $\mathscr{S}m_{\mathrm{Spec(k)}}$ (resp. $\pi\mathscr{S}m^{\mathrm{cor}}_{\mathrm{Spec(k)}}$).

4.2.1. *Open immersions.* The following proposition is analogous to Cor. 4.19 of [Voe00a]; its proof uses the same arguments.

**Proposition 4.5.** *Let $F$ be a presheaf over $\pi\mathscr{S}m_k^{\mathrm{cor}}$.*

*Let $G$ be one of the following presheaves over $\mathscr{S}m_k$ :*

1. *the Zariski sheaf $F_{\mathrm{Zar}}$ over $\mathscr{S}m_k$ associated to $F$,*
2. *the 0-th Čech cohomology presheaf $\check{H}^0 F$ over $\mathscr{S}m_k$ associated with $F$ for the Nisnevich topology.*

*Then for any smooth $k$-scheme $X$, any dense open subscheme $U$ of $X$, the restriction morphism $G(X) \to G(U)$ is a monomorphism.*

*Proof.* Consider $a \in G(X)$ such that $a|_U = 0$. We shall show that $a = 0$. We may assume that there exists an element $b \in F(X)$ such that $a$ is the image of $b$ by the canonical morphism $F(X) \to G(X)$. Indeed, there exists a Nisnevich covering (in the first case, even a Zariski covering) of $X$ such that $a|_W$ can be lifted along the morphism $F(W) \to G(W)$. The open scheme $W \times_X U$ of $W$ is still dense and we have $a|_{W \times_X U} = 0$ in $G(W \times_X U)$, hence we can replace $X$ by $W$ and make the above assumption.

Moreover, in both cases there exists by hypothesis a Nisnevich covering $W \xrightarrow{p} U$ such that $b|_W = 0$.

As $W$ is a Nisnevich covering of $U$, there exist a dense open subscheme $U_0$ of $U$ and an open subscheme $W_0$ of $W$ such that $p$ induces an isomorphism between $W_0$ and $U_0$. Thus, $b|_{U_0} = 0$.

Applying Corollary 3.22, we find a Zariski cover $W'$ of $X$ and a finite $k$-correspondence $\alpha : W' \to U_0$ such that the diagram
$$\begin{tikzcd} & W' \arrow[dl, "\alpha"'] \arrow[d] \\ U_0 \arrow[r] & X \end{tikzcd}$$
commutes up to homotopy.

Applying $F$ to this diagram, we thus obtain that $b|_{W'} = 0$ in $F(W')$ which implies $a = 0$. $\square$

**Corollary 4.6.** *Let $F$ be a homotopy sheaf over $k$. Consider a smooth $k$-scheme $X$ and a dense open subscheme $U$ of $X$.*

*Then the restriction morphism $F(X) \to F(U)$ is a monomorphism.*

4.2.2. *Generic points.* Let $X$ be a smooth $S$-scheme, and $x$ be a generic point of $X$. The local ring $\mathcal{O}_{X,x}$ of $X$ in $x$ is a field. Thus it is henselian.

If we let $\mathcal{V}_x(X)$ be the category of open neighbourhoods of $X$, and define the localisation of $X$ in $x$ as the pro-object $X_x = \widetilde{\varprojlim}_{U \in \mathcal{V}_x(X)} U$. Its limit is $\operatorname{Spec}(\mathcal{O}_{X,x})$. As $\mathcal{O}_{X,x} = \mathcal{O}^h_{X,x}$, we have a canonical isomorphism $F(X_x) = F(X^h_x)$.

Note that $\mathcal{O}_{X,x}$ is a separable field extension of $k$ of finite type. We call such an extension a function field. We let $\mathscr{E}_k$ be the category of function fields with arrows the $k$-algebra morphisms.

When $E/k$ is a function field, we put
$$\mathcal{M}^{sm}(E/k) = \{A \subset E \mid \operatorname{Spec}(A) \in \mathscr{S}m_k, \operatorname{Frac}(A) = E\}$$
as an ordered set, the order coming from inclusion. This set is in fact non empty and right filtering.

We define the pro-scheme $(E) = \widetilde{\varprojlim}_{A \in \mathcal{M}^{sm}(E/k)^{op}} \operatorname{Spec}(A)$. Thus, according to our general conventions, for any presheaf $F$ over $\mathscr{S}m_k$,
$$F(E) = \varinjlim_{A \in \mathcal{M}^{sm}(E/k)} F(\operatorname{Spec}(A)).$$

Moreover, for any $A \in \mathcal{M}^{sm}(E/k)$, if $x$ denotes the generic point of $X = \operatorname{Spec}(A)$, we have canonical isomorphism $F(E) = F(X^h_x) = F(X_x)$ as $\operatorname{Spec}(E)$ is the limit of all the pro-schemes $(E)$, $X^h_x$ and $X_x$. In particular, the morphism $F \mapsto F(E)$ from Nisnevich sheaves to abelian groups is a fiber functor.

The following proposition due to Voevodsky shows the fiber functors defined above form a conservative family of "fiber functors" for homotopy sheaves.

**Proposition 4.7.** *Let $F$, $G$ be homotopy sheaves over $k$, and $\eta : F \to G$ be a morphism of sheaves with transfers.*

*If for any field $E/k$ in $\mathscr{E}_k$ the induced morphism $\eta_E : F(E) \to G(E)$ is a monomorphism (resp. an isomorphism), then $\eta$ is a monomorphism (resp. an isomorphism).*

*Proof.* Indeed, it is sufficient to apply the next lemma to the morphism $\eta$.

**Lemma 4.8.** *Let $F$, $G$ be presheaves over $\pi\mathscr{S}m_k^{\mathrm{cor}}$ and $\eta : F \to G$ be a natural transformation.*

*The following conditions are equivalent :*

(1) *The morphism $\eta_{\mathrm{Zar}} : F_{\mathrm{Zar}} \to G_{\mathrm{Zar}}$ between the associated Zariski sheaves over $\mathscr{S}m_k$ is a monomorphism (resp. isomorphism).*

(2) *For all extension $E/k$ in $\mathscr{E}_k$, $\eta_E : F(E) \to G(E)$ is a monomorphism (resp. isomorphism).*

*Proof.* Clearly, (1) implies (2).

For the converse we consider $N$, the kernel of $\eta$ in the category of presheaves with transfers. It is homotopy invariant. Let $X$ be a smooth irreducible $k$-scheme with residue field $E$. Clearly, we have a canonical isomorphism

$$N(E) = \varinjlim_{U \subset X} N_{\mathrm{Zar}}(U)$$

where the limit runs over the open dense subschemes of $X$. Then Proposition 4.5 implies that the canonical morphism

$$N_{\mathrm{Zar}}(X) \to \varinjlim_{U \subset X} N_{\mathrm{Zar}}(U) = N(E)$$

is a monomorphism. But $N(E)$ is the kernel of $\eta_E : F(E) \to G(E)$, hence $N(E) = 0$ and $N(X) = 0$. We now conclude the proof by applying the same reasoning to the cokernel of $\eta$. ☐

### 4.3. **Associated homotopy sheaf.**

4.3.1. *Čech cohomology of curves.* Let $k$ be a field and $C/k$ be an algebraic curve. We introduce the following property for the curve $C$ :

(N) For all finite extensions $L/k$, $\mathrm{Pic}\,(C \otimes_k L) = 0$.

*Remark* 4.9. If this property is true for $C$, it is true for any open subscheme of $C$. If $C$ is affine with function ring $A$, property (N) is equivalent to the property that for any finite extension $L/k$, the ring $A \otimes_k L$ is factorial; cf [GD66], 21.7.6 and 21.7.7.

Note this property implies that for any closed subscheme $Z$ of $C$ nowhere dense, $\mathrm{Pic}\,(C \times_k Z) = 0$. We deduce from that fact the following proposition which is in fact a generalisation of [Voe00a], 5.4 :

**Proposition 4.10.** *Let $k$ be a field. Consider $C/k$, a smooth affine curve satisfying property (N), and $F$ a presheaf over $\pi\mathscr{S}m_k^{\mathrm{cor}}$.*

*Then for all integers $n \geq 0$, the $n$-th Čech cohomology group of $C$ with coefficients in $F$ for the Nisnevich topology is*

$$\check{\mathrm{H}}^n(C; F) \begin{cases} F(C) & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* First we remark that for any Nisnevich covering $W \to C$ there exists a distinguished square

$$\begin{array}{ccc} U \times_X V & \xrightarrow{l} & V \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j} & X \end{array}$$

with $U$ and $V$ affine such that the covering $U \sqcup V \to C$ is a refinement of $W \to C$. Indeed, we may assume that $W$ is affine. As $W/C$ is a Nisnevich covering, there exists a dense open subset $U$ of $X$ such that $W \times_C U \to U$ admits a section. As this morphism is étale, we have $W \times_C U = U \sqcup U'$. Put $Z = (C - U)_{red}$. Then $W \times_C Z$ is a finite set of closed points of $W$. As $W \times_C Z \to Z$ is a Nisnevich covering, any point of $Z$ has a preimage in $W \times_C Z$ which is isomorphic to it ; that it $W \times_C Z = Z \sqcup Z'$. If we now put $V = W - Z'$, then $V$ is affine as $W$ is regular and $Z'$ is a finite set of points; we have obtained our distinguished square.

Consider now a distinguished square as above. As $C/k$ satisfies property (N), Theorem 3.23 implies that the complex

$$0 \to F(C) \xrightarrow{j^*+f^*} F(U) \oplus F(V) \xrightarrow{h^*-l^*} F(U \times_X V) \to 0$$

is contractible.

This implies that the Čech cohomology group associated with the covering $U \sqcup V/C$ is $F(C)$ in degree 0 and 0 in other degrees; this concludes the proof of the proposition by the remark at the beginning of the proof. □

**Corollary 4.11.** *Let $C/k$ be a smooth curve satisfying property (N) and $F$ be a presheaf over $\pi\mathscr{S}m_k^{\mathrm{cor}}$. Then for all integers $n > 0$, we have*

$$\mathrm{H}^n(C; F_{\mathrm{Nis}}) = 0.$$

*Proof.* Indeed, the Nisnevich cohomology of $C/k$ vanishes in dimension strictly greater than 1, and the Čech cohomology coincides with the usual cohomology in degree 1. □

*Remark* 4.12. In the hypothesis of this proposition it is not only sufficient, but also necessary that $C/k$ satisfies property (N).

Let us assume $C/k$ satisfies $\mathrm{H}^1(C; F_{\mathrm{Nis}}) = 0$ for every presheaf $F$ over $\pi\mathscr{S}m_k^{\mathrm{cor}}$. Let $\mathbb{G}_m$ be the sheaf over $\mathscr{S}m_k$ represented by $\mathbb{G}_m$. It has a canonical structure of a sheaf with transfers : let $X$ and $Y$ be smooth $k$-schemes, and $\alpha$ be a finite $k$-correspondence from $X$ to $Y$. We assume $X$ is integral and $\alpha$ is an integral closed subscheme of $X \times_k Y$. Let $\kappa(X)$ and $\kappa(Z)$ be the respective function fields of $X$ and $Z$. Then $\kappa(Z)/\kappa(X)$ is a finite extension, as $Z \to X$ is finite surjective. Let $N_{\kappa(Z)/\kappa(X)}$ be the associated norm morphism. Then we construct $\alpha^*$ by the commutative diagram :

$$\begin{tikzcd}
\mathcal{O}_Y(Y)^\times \arrow[r] \arrow[rr, bend left, "\alpha^*"] & \mathcal{O}_Z(Z)^\times \arrow[r, dashed] & \mathcal{O}_X(X)^\times \\
& k(Z)^\times \arrow[u, hook] \arrow[r, "N_{k(Z)/k(X)}"] & k(X)^\times . \arrow[u, hook]
\end{tikzcd}$$

The dotted arrow exists as $\mathcal{O}_X(X)$ is integral. These transfers are compatible with the composition of finite correspondences using the property of the norm homomorphism.[6]

Let $L/k$ be a finite extension, $j : \mathrm{Spec}\,(L) \to \mathrm{Spec}\,(k)$ the canonical morphism. Then the sheaf with transfers $j_*j^*\mathbb{G}_m$ is still homotopy invariant and we have $H^1(C; j_*j^*\mathbb{G}_m) = \mathrm{Pic}\,(C \otimes_k L)$.

[6] It is also a consequence of [Dég05], 6.5 and 6.6 applied to $\mathbb{G}_m = A^0(.; K_*^M)_1$.

4.3.2. *The 0-th Čech cohomology presheaf.* Recall from Lemma 2.6 that for any presheaf with transfers $F$ over $k$, the 0th Čech cohomology presheaf associated to $F$ for the Nisnevich topology has a canonical structure of a presheaf with transfers. We denote by $\check{H}^0_{tr}F$ this presheaf with transfers. Recall that for any smooth $k$-scheme $X$, $\Gamma(X;\check{H}^0_{tr}F) = \check{H}^0(X;F)$

The following proposition is the very point where our proof of the technical results concerning homotopy sheaves differs from that of [Voe00a] (especially 4.26 and 5.5).

**Proposition 4.13.** *Let $k$ be field and $F$ be a presheaf over $\pi\mathscr{S}m^{\mathrm{cor}}_k$.*
*Then the presheaf $\check{H}^0_{tr}F$ is homotopy invariant.*

*Proof.* Let $X/k$ be a smooth scheme. If $s : X \to \mathbb{A}^1_X$ is the 0-section, we have to prove in fact that $s^* : \check{H}^0_{tr}F(\mathbb{A}^1_X) \to \check{H}^0_{tr}F(X)$ is a monomorphism.

We may assume $X$ is irreducible. Applying Proposition 4.5, we find that the morphism $\check{H}^0_{tr}F(X) \to \check{H}^0_{tr}F(U)$ is a monomorphism for any nonempty open subscheme $U$ of $X$. Thus in the commutative diagram below

$$\begin{tikzcd} \check{H}^0_{tr}F(\mathbb{A}^1_X) \arrow[r] \arrow[d, "s^*"'] & \varinjlim_{U\subset X} \check{H}^0_{tr}F(\mathbb{A}^1_U) \arrow[d, "\sigma"] \\ \check{H}^0_{tr}F(X) \arrow[r] & \varinjlim_{U\subset X} \check{H}^0_{tr}F(U) \end{tikzcd}$$

where $U$ runs over the nonempty open subschemes of $X$, the horizontal arrows are injective and we only have to prove that $\sigma$ is injective.

Denote by $E$ the function field of $X$, and let $\tau : \mathrm{Spec}\,(E) \to \mathrm{Spec}\,(k)$ be the canonical morphism. We let $\hat{\tau}^* : \mathscr{P}^{\mathrm{tr}}_k \to \mathscr{P}^{\mathrm{tr}}_E$ be the base change functor for presheaves with transfers (cf section 2.5.4).

Let $s_E : \mathrm{Spec}\,(E) \to \mathbb{A}^1_E$ be the 0-section. Then from Proposition 2.18 and the remark that follows about functoriality, we deduce that the morphism

$$\sigma : \varinjlim_{U\subset X} \check{H}^0_{tr}F(\mathbb{A}^1_U) \to \varinjlim_{U\subset X} \check{H}^0_{tr}F(U)$$

is isomorphic to

$$s_E^* : \hat{\tau}^*\check{H}^0_{tr}F(\mathbb{A}^1_E) \to \hat{\tau}^*\check{H}^0_{tr}F(\mathrm{Spec}\,(E)).$$

Let us recall that from Lemma 2.21 we have $\hat{\tau}^*\check{H}^0_{tr} = \check{H}^0_{tr}\hat{\tau}^*$. To conclude that $\sigma$ is an isomorphism, we apply Proposition 4.10 to the curve $\mathbb{A}^1_E/E$ and to the homotopy presheaf $\hat{\tau}^*F$ over $E$. $\square$

**Corollary 4.14.** *Let $k$ be a field. For any presheaf over $\pi\mathscr{S}m^{\mathrm{cor}}_k$, the sheaf $F_{\mathrm{Nis}}$ is homotopy invariant.*

*In particular, the functor $a_{tr} : \mathscr{P}^{\mathrm{tr}}_k \to \mathscr{N}^{\mathrm{tr}}_k$ of Corollary 2.7 induces an exact functor $a_{Htr} : \mathrm{H}\mathscr{P}^{\mathrm{tr}}_k \to \mathrm{H}\mathscr{N}^{\mathrm{tr}}_k$ which is left adjoint to the inclusion functor $\mathrm{H}\mathscr{N}^{\mathrm{tr}}_k \hookrightarrow \mathrm{H}\mathscr{P}^{\mathrm{tr}}_k$.*

**Corollary 4.15.** *The category $\mathrm{H}\mathscr{N}^{\mathrm{tr}}_k$ is a Grothendieck abelian category which admits arbitrary limits. The inclusion functor $\mathrm{H}\mathscr{N}^{\mathrm{tr}}_k \hookrightarrow \mathscr{N}^{\mathrm{tr}}_k$ is exact.*

Let us consider the notations introduced before Proposition 4.4 in the case where the base is a field $k$. Then the corollary above implies that $h_0^{(1)} = h_0$.

The generators of $\mathrm{H}\mathscr{N}_k^{\mathrm{tr}}$ are the elements of the (essentially small) family $\big(h_0\mathrm{L}[X]\big)_{X\in\mathscr{S}m_k}$.

Note finally that we also obtain the analogue of the results of [Voe00a] in the case of presheaves with transfers for the Zariski topology :

**Corollary 4.16.** *Let $F$ be a homotopy presheaf over a field $k$.*
*Then the canonical morphism $F_{\mathrm{Zar}} \to F_{\mathrm{Nis}}$ is an isomorphism.*

*Proof.* By the preceding results, we know $F_{\mathrm{Nis}}$ is a homotopy invariant sheaf over $k$. Thus the result follows from Lemma 4.8, applied to the morphism of homotopy presheaves with transfers $F \to F_{\mathrm{Nis}}$. □

## 5. Homotopy invariance of cohomology

The aim of this section is to prove the following theorem :

**Theorem 5.1** (Voevodsky)**.** *Let $k$ be a perfect field and $F$ be a homotopy sheaf over $k$. Then the Nisnevich cohomology presheaf $H^*(.;F)$ is homotopy invariant over $\mathscr{S}m_k$.*

### 5.1. **Lower grading.**

**Definition 5.2.** Let $S$ be a regular scheme and $F$ be a homotopy presheaf with transfers over $S$. We associate to $F$ the homotopy presheaf with transfers $F_{-1}$ over $S$ such that for all smooth $S$-schemes $X$,

$$F_{-1}(X) = \mathrm{coker}\left(F(\mathbb{A}^1 \times X) \to F(\mathbb{G}_m \times X)\right).$$

With this definition, we always have a split short exact sequence

$$0 \to F(\mathbb{A}^1 \times X) \xrightarrow{j^*} F(\mathbb{G}_m \times X) \to F_{-1}(X) \to 0$$

using the homotopy invariance of $F$; a canonical retraction of $j^*$ is induced by the morphism $\mathbb{A}^1 \times X \to X \xrightarrow{s_1} \mathbb{G}_m \times X$, given by projection followed by the unit section of $\mathbb{G}_m \times X/X$. Using this canonical splitting, we may assume that $F_{-1}(X) \subset F(\mathbb{G}_m \times X)$.

As a consequence, we deduce that if $F$ is a homotopy sheaf then $F_{-1}$ is a homotopy sheaf.

### 5.2. **Local purity.**

5.2.1. *Relative closed pairs.* In the definition below, we introduce the analogue of the definitions in 3.6 over a base.

**Definition 5.3.** Let $S$ be a scheme. A closed pair over $S$ is a pair $(X, Z)$ such that $X$ is a smooth $S$-scheme and $Z$ is a closed subscheme of $X$. We will say that $(X, Z)$ is *smooth* (resp. *has codimension* $n$) if $Z$ is smooth (resp. $Z$ is of pure codimension $n$ in $X$).

A morphism of closed pair $(Y, T) \to (X, Z)$ is a pair of morphisms $(f, g)$ which fits into the commutative diagram of schemes

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, hook] & X. \end{tikzcd}$$

which is cartesian on the corresponding topological spaces.

We will say $(f,g)$ is *cartesian* (resp. *excisive*) if the preceding square is cartesian in the category of schemes (resp. $g_{red} : T_{red} \to Z_{red}$ is an isomorphism).

*Remark* 5.4. Given a closed immersion $i : Z \to X$ into a smooth $S$-scheme, we will usually identify the schemes $Z$ with the closed subscheme $i(Z)$ of $X$ when no confusion can arise.

**5.5.** The following method gives a general process to construct excisive morphisms.

Let $S$ be a scheme. Consider two closed pairs $(X,Z)$ and $(X',Z)$ over $S$ such that $X$ and $X'$ are étale over $S$.

Let $\Delta$ be the diagonal of $Z$ over $S$. It is a closed subscheme of $Z \times_S Z$ and thus we identify it as a closed subscheme of $X \times_S X'$. Similarly, $\Delta$ is a closed subset of $X \times_S Z$ and $Z \times_S X'$.

**Lemma 5.6.** *We adopt the hypotheses and notations above.*

*Define the set* $\Omega = X \times_S X' - [(X \times_S Z - \Delta) \cup (Z \times_S X' - \Delta)]$*, and consider the canonical projections* $X \xleftarrow{p} \Omega \xrightarrow{q} X'$ *.*

*Then* $\Omega$ *is an open subscheme of* $X \times_S X'$ *and contains* $\Delta$ *as a closed subscheme. Thus, identifying* $\Delta$ *with* $Z$*,* $(\Omega, Z)$ *is a closed pair over* $S$ *such that* $\Omega/S$ *is étale. Moreover, the projections* $p$*,* $q$ *induce cartesian excisive morphisms*

$$(X,Z) \xleftarrow{\ p\ } (\Omega,Z) \xrightarrow{\ q\ } (X',Z).$$

*Proof.* We only have to prove that $\Omega$ is open in $X \times_S X'$.

Consider the closed immersion $\iota : \Delta \to X \times_S Z$. Identifying $\Delta$ with $Z$, $\iota$ is a section of the étale morphism $f \times_S Z : X \times_S Z \to Z$. In particular, $\iota$ is an open immersion and $X \times_S Z - \Delta$ is a closed subscheme of $X \times_S X'$. By symmetry, we get the conclusion. $\square$

*Remark* 5.7. The reader can check that the preceding construction is functorial with respect to cartesian étale morphisms $(Y,T) \to (X,Z)$ and $(Y',T) \to (X',Z)$ of closed pairs over $S$.

Let $X$ be a $S$-scheme and $s_0 : X \to \mathbb{A}^n_X$ be the 0-section. We will often consider the closed pair $(\mathbb{A}^n_X, s_0(X))$ which we will always denote by $(\mathbb{A}^n_X, X)$.

**Definition 5.8.** Let $S$ be a scheme and $(X,Z)$ a closed pair over $S$.

A parametrisation of $(X,Z)$ over $S$ is a cartesian étale morphism $(f,g) : (X,Z) \to (\mathbb{A}^{c+n}_S, \mathbb{A}^n_S)$ for a pair of integers $(n,c)$.

Suppose given a parametrisation $(X,Z) \to (\mathbb{A}^{c+n}_S, \mathbb{A}^n_S)$. Then $(X,Z)$ is smooth of codimension $c$. Moreover, $X$ has pure dimension $n$ over $S$. In particular, the integer $(n,c)$ are uniquely determined by $(X,Z)$.

Conversely, when the closed pair $(X,Z)$ is smooth, for any point $s$ of $Z$ there always exist an open neighbourhood $U$ of $s$ in $X$ and a parametrisation of $(U, Z \cap U)$ over $S$. We will loosely speak of a local parametrisation of $(X,Z)$ at $s$.

**5.9.** The following process is the geometric base for the local purity theorem.

Suppose given a closed pair $(X, Z)$ and a parametrisation $(u, v) : (X, Z) \to (\mathbb{A}^{c+n}_S, \mathbb{A}^n_S)$. We associate to this parametrisation a commutative square

$$\begin{tikzcd} Z \arrow[r, "s_0"] \arrow[d] & \mathbb{A}^c_Z \arrow[d, "1\times_S v"] \\ X \arrow[r] & \mathbb{A}^{c+n}_S \end{tikzcd}$$

where $s_0$ is the 0-section of $\mathbb{A}^c_Z$. This gives two closed pairs $(X, Z)$ and $(\mathbb{A}^c_Z, Z)$ over $\mathbb{A}^{n+c}_S$.

From the preceding lemma, we obtain cartesian excisive morphisms

$$(X, Z) \to (\Omega, Z) \leftarrow (\mathbb{A}^c_Z, Z).$$

*Remark* 5.10. Consider in addition a cartesian étale morphism $(Y, T) \xrightarrow{(f,g)} (X, Z)$. Then we associate to the induced parametrisation of $(Y, T)$ over $S$ a closed pair $(\Pi, T)$ which fits into the commutative diagram of closed pairs over $S$

$$\begin{tikzcd} (Y, T) \arrow[d, "(f,g)"'] & (\Pi, T) \arrow[l] \arrow[r] \arrow[d] & (\mathbb{A}^c_T, T) \arrow[d, "(1\times_S g, g)"] \\ (X, Z) & (\Omega, Z) \arrow[l] \arrow[r] & (\mathbb{A}^c_Z, Z) \end{tikzcd}$$

We note that this process allows us to deduce the following structure theorem for points in the Nisnevich topology :

**Corollary 5.11.** *Let $S$ be a scheme and $(X, Z)$ a smooth pair over $S$.*

*Then for any point $s$ of $Z$, there exists an isomorphism*

$$X^h_s \simeq (\mathbb{A}^c_Z)^h_s$$

*of pro-schemes over $S$ which is the identity on $Z^h_s$. The integer $c$ is the codimension of $Z$ in $X$ at $s$.*

*Proof.* Indeed, as we can find a local parametrisation of $(X, Z)$ at $s$, the preceding construction gives an open neighbourhood $U$ of $s$ in $X$ and excisive morphisms

$$(U, Z \cap U) \to (\Omega, Z \cap U) \leftarrow (\mathbb{A}^c_{Z\cap U}, Z \cap U).$$

This implies that $\Omega$ is a Nisnevich neighbourhood of $s$ in $U$ (resp. $\mathbb{A}^c_{Z\cap U}$), hence in $X$ (resp. $\mathbb{A}^c_Z$). This concludes the proof. □

5.2.2. *The case of homotopy presheaves.* Let $(X, Z)$ be a pair over a regular scheme $S$. For any point $s \in Z$, we have a canonical isomorphism

$$Z^h_s = \tilde{\varprojlim_{V\in\mathcal{V}^h_s(X)}} V \times_X Z.$$

It is natural to consider the pro-object

$$X^h_s - Z^h_s = \tilde{\varprojlim_{V\in\mathcal{V}^h_s(X)}} (V - V \times_X Z).$$

We thus have canonical morphisms of pro-objects (pro-immersions) :

$$X^h_s - Z^h_s \xrightarrow{\iota} X^h_s \leftarrow Z^h_s.$$

For any presheaf over $\mathscr{S}m_S$, we consider the induced morphism

$$\iota^* : F(X_s^h) \to F(X_s^h - Z_s^h),$$

and denote by $F(X_s^h - Z_s^h)/F(X_s^h)$ the cokernel of $\iota^*$. Note this is a little abuse of notation, as $\iota^*$ is not necessarily a monomorphism.

**Proposition 5.12.** *Let $S$ be a regular excellent scheme, $(X, Z)$ be a smooth closed pair over $S$ and $F$ be a homotopy presheaf over $S$.*

*Let $s$ be a point of $X$ such that $Z$ is of codimension 1 in $X$ at $s$. Then any local parametrisation of $(X, Z)$ at $s$ induces a canonical isomorphism*

$$F(X_s^h - Z_s^h)/F(X_s^h) \simeq F_{-1}(Z_s^h).$$

*Proof.* Indeed, Corollary 5.11 shwos that a local parametrisation of $(X, Z)$ at $s$ induces a canonical isomorphism $X_s^h \to (\mathbb{A}^1_Z)_s^h$ that is the identity on $Z_s^h$. Thus we are reduced to the case of the closed pair $(\mathbb{A}^1_Z, Z)$.

Let $V$ be a Nisnevich neighbourhood of $s$ in $Z$. Then $\mathbb{A}^1_V$ is a Nisnevich neighbourhood of $s$ in $\mathbb{A}^1_Z$. Thus we get a canonical morphism

$$\varinjlim_{V \in \mathcal{V}_s^h(Z)} F(\mathbb{A}^1_V - V)/F(V) \xrightarrow{(*)} \varinjlim_{W \in \mathcal{V}_s^h(\mathbb{A}^1_Z)} F(W - W_Z)/F(W).$$

**Lemma 5.13.** *Let $S$ be a regular excellent scheme and $Z$ a smooth $S$-scheme.*

*For every point $s$ in $Z$, the canonical morphism described above*

$$F_{-1}\big(\mathbb{A}^1_{Z_s^h}\big) \xrightarrow{(*)} F\big((\mathbb{A}^1_Z)_s^h - Z_s^h\big)/F\big((\mathbb{A}^1_Z)_s^h\big)$$

*is an isomorphism.*

Let $\mathcal{Z} = \operatorname{Spec}\left(\mathcal{O}_{Z,s}^h\right)$ be the limit of $Z_s^h$, and $\tau : \mathcal{Z} \to S$ the canonical morphism. Note that $\mathcal{Z}$ is regular and noetherian. As filtered inductive limits are exact, we obtain using Proposition 2.18 a canonical isomorphism

$$(\hat{\tau}^* F)_{-1}(\mathbb{A}^1_{\mathcal{Z}}) \simeq F_{-1}\big(\mathbb{A}^1_{Z_s^h}\big).$$

Moreover, we can write

$$F\big((\mathbb{A}^1_Z)_s^h - Z_s^h\big)/F\big((\mathbb{A}^1_Z)_s^h\big) = \varinjlim_{V \in \mathcal{V}_s^h(Z)} \varinjlim_{W \in \mathcal{V}_s^h(\mathbb{A}^1_V)} F(W - W_Z)/F(W).$$

This expression, together with Proposition 2.18, gives us a canonical isomorphism

$$\hat{\tau}^* F(\mathbb{A}^1_{\mathcal{Z}})_s^h - \mathcal{Z})/\hat{\tau}^* F(\mathbb{A}^1_{\mathcal{Z}}) \simeq F\big((\mathbb{A}^1_Z)_s^h - Z_s^h\big)/F\big((\mathbb{A}^1_Z)_s^h\big).$$

Thus we are reduced to the case where $S = \mathcal{Z}$ is a local henselian scheme with closed point $s$. Indeed, the two isomorphisms just constructed are compatible with the morphism $(*)$ (cf the remark after Proposition 2.18). Note also that $\mathcal{Z}$ is still a regular excellent scheme (cf [GD66], 18.6.10 and 18.7.6).

We consider the category $\mathcal{I}$ of étale morphisms $V \xrightarrow{g} \mathbb{A}^1_S$ such that $V$ is affine and $g^{-1}(S) \to S$ is an isomorphism, with arrows the $\mathbb{A}^1_S$-morphisms. Then $\mathcal{I}$ is a final subcategory of $\mathcal{V}_s^h\left(\mathbb{A}^1_S\right)$. Indeed, let $V \xrightarrow{f} \mathbb{A}^1_S$ be a Nisnevich neighbourhood of $s$ in $\mathbb{A}^1_S$. As $S$ is henselian, the morphism $g : V_S \to S$ induced by $f$ admits a section. Thus there exists an open subscheme $V'$ of

$V$ such that $V' \cap V_S = S$. As we can always reduce $V'$ in a neighbourhood of $s$, we can assume $V'$ is affine, that is, $V' \in \mathcal{I}$.

Let $V \xrightarrow{f} \mathbb{A}^1_S$ be an object in $\mathcal{I}$. To conclude the proof of the lemma, we will prove that the morphism

$$F(\mathbb{A}^1_S - S)/F(\mathbb{A}^1_S) \to F(V - V_S)/F(V)$$

induced by $f$ is an isomorphism.

Zariski's main theorem implies that there exist an $S$-scheme $\bar{V}$ and morphisms

$$\begin{array}{ccc} V & \xrightarrow{k} & \bar{V} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \mathbb{A}^1_S & \xrightarrow{j} & \mathbb{P}^1_S \end{array}$$

such that $\bar{f}$ is finite and $k$ is an open immersion. Replacing $\bar{V}$ by the reduced closure of $V$ in $\bar{V}$, we can assume that $V$ is dense in $\bar{V}$. As $S$ is excellent and $V$ is normal, we can assume furthermore that $\bar{V}$ is normal, replacing it by its normalisation.

We claim that $\bar{V}/S$ is a good compactification of $(V, S)$. Let $W = \bar{f}(\bar{V} - V)$ as a reduced closed subscheme of $\mathbb{P}^1_S$, and $W_s \subset \mathbb{P}^1_s$ be the special fiber of $W$. Then $W_s$ is a finite set, as it is nowhere dense in $\mathbb{P}^1_s$ : we can find a regular function $h$ on $\mathbb{P}^1_s$ such that $D(h) \cap (W_s \cup \{0\}) = \emptyset$. Let $l$ be an extension of $h$ to $\mathbb{P}^1_S$. As the projection $\bar{V} \to S$ is proper, we necessarily have $W \cap D(l) = \emptyset$. Thus, $W \cup \{0_S\}$ is contained in the affine open subset $V(h) = \mathbb{P}^1_S - D(h)$. Then $(\bar{V} - V) \cup V_S$ is contained in the affine open subset $\bar{f}^{-1}(D(h))$ as required.

Finally, when we apply Theorem 3.23 to the square

$$\begin{array}{ccc} V - V_S & \xrightarrow{l} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{A}^1_S - S & \xrightarrow{j} & \mathbb{A}^1_S \end{array}$$

we find that the complex

$$0 \to F(\mathbb{A}^1_S) \xrightarrow{j^* + f^*} F(\mathbb{A}^1_S - S) \oplus F(V) \xrightarrow{(h^*, -l^*)} F(V - V_S) \to 0$$

is split exact. This implies the desired result by an easy diagram chase. □

5.2.3. *The case of sheaves.* For any scheme $X$, we let $X_{\mathrm{Nis}}$ be the site of étale $X$-schemes with the Nisnevich topology, and $\tilde{X}_{\mathrm{Nis}}$ be the topos associated to $X_{\mathrm{Nis}}$. We also denote by $\mathbb{Z}.\tilde{X}_{\mathrm{Nis}}$ the category of abelian sheaves on $X_{\mathrm{Nis}}$.

When $f : Y \to X$ is any morphism of schemes we have, following [MAJ73], exp. IV, a pair of adjoint functors

$$(f_*, f^*) : \tilde{Y}_{\mathrm{Nis}} \to \tilde{X}_{\mathrm{Nis}}$$

with $f^*$ exact. When we restrict our attention to the category of abelian sheaves, the functor $f_*$ can be classically derived on the right and induces a functor

$$Rf_* : D^b(\mathbb{Z}.\tilde{Y}_{\mathrm{Nis}}) \to D^b(\mathbb{Z}.\tilde{X}_{\mathrm{Nis}}).$$

Recall that for any $q \in \mathbb{N}$, and any (abelian) Nisnevich sheaf $F_Y$ on $Y$, $R^q f_* F_Y$ is the Nisnevich sheaf associated to the presheaf $U/X \mapsto H^q(Y \times_X$

$U; F_Y)$. This implies that $f_*$ is exact whenever $f$ is finite as a finite scheme over a local henselian scheme is a disjoint union of local henselian schemes.

Let $S$ be a regular scheme and $F$ be a homotopy sheaf over $S$. For any smooth $S$-scheme $X$, we will denote by $F_X$ the restriction of $F$ to $X_{\mathrm{Nis}}$.

*Remark* 5.14. (1) For any $U$ in $X_{\mathrm{Nis}}$, we obviously have

$$H^n(U; F_X) = H^n(X; F).$$

(2) When $f : Y \to X$ is a smooth morphism, we have $f^* F_X = F_Y$.

More generally, let $(X_i)_{i \in I}$ be a pro-object of étale $X$-schemes affine over $\mathbb{Z}$, and let $\mathcal{X}$ be its limit.

We can consider $X_\bullet = (X_i)_{i \in I}$ as a pro-object of smooth affine $S$-schemes. Let $\tau : \mathcal{X} \to S$ be the canonical morphism. We have defined the sheaf with transfers $\tau^* F$ over $\mathcal{X}$ in section 2.5. Recall that Proposition 2.19 implies that it is homotopy invariant.

Let now $\tau_X : \mathcal{X} \to X$ the canonical morphism. Then, as another application of Proposition 2.19, we obtain $\tau_X^* F_X = (\tau^* F)_{\mathcal{X}}$.

Note that the coherence of the Nisnevich topos together with [MAJ73], VI, implies that

$$H^n(X_\bullet; F) = H^n(\mathcal{X}; \tau^* F).$$

Let $(X, Z)$ be a smooth closed pair over $S$ of codimension 1, $j : X - Z \to X$ and $i : Z \to X$ the canonical immersions.

Applying Corollary 4.6, the adjunction morphism $F_X \to j_* j^* F_X$ is a monomorphism. Let $C$ be its cokernel in $\mathbb{Z}.\tilde{X}_{\mathrm{Nis}}$. Then the adjunction morphism $C \to i_* i^* C$ is an isomorphism as for any point $s \in X - Z$, $C(X_s^h) = 0$.

**Definition 5.15.** With the above notations, we define $F_{(X,Z)}$ to be the Nisnevich sheaf on $Z_{\mathrm{Nis}}$ equal to $i^* C$.

Thus, by the above construction, we have a canonical exact sequence of sheaves on $X_{\mathrm{Nis}}$

$$\text{(C)} \qquad 0 \to F_X \to j_* F_U \to i_* F_{(X,Z)} \to 0.$$

For any cartesian morphism $(f, g) : (Y, T) \to (X, Z)$ there is an induced morphism

$$F_{(X,Z)} \to g_* F_{(Y,T)}$$

which makes the above exact sequence functorial.

**Lemma 5.16.** *Let $F$ be a homotopy sheaf over a regular scheme $S$, and $(X, Z)$ be a smooth codimension 1 closed pair over $S$.*

*Then for any $s \in Z$, we have a canonical isomorphism*

$$F_{(X,Z)}(Z_s^h) = F(X_s^h - Z_s^h)/F(X_s^h)$$

*using the notations preceding Proposition 5.12.*

*Proof.* By definition, $F_{(X,Z)}(Z_s^h) = (i_* F_X)(X_s^h)$. Hence the result follows by taking fibers along $X_s^h$ in the exact sequence (C). $\square$

**Lemma 5.17.** *Let $F$ be a homotopy sheaf over a regular scheme $S$, and $(f,g) : (Y,T) \to (X,Z)$ be an excisive morphism of smooth codimension 1 closed pairs over $S$. Then the canonical morphism $F_{(X,Z)} \to g_* F_{(Y,T)}$ is an isomorphism.*

*Proof.* Let $s$ be a point of $Z$. It suffices to check the assertion by evaluating the sheaves on $Z_s^h$. As $(f,g)$ is excisive, $Y$ is a Nisnevich neighbourhood of $s$ in $X$. Thus $f$ induces an isomorphism $Y_t^h \to X_s^h$, with $t$ being the point of $Y$ such that $g(t) = s$. The result now follows from the preceding lemma. □

Let $F$ be a homotopy sheaf over $S$ and $Z$ be a smooth $S$-scheme. For any étale $Z$-scheme $V$, let $\mathcal{I}_V$ be the category with objects the pairs $(U,\eta)$ such that $U$ is an étale $\mathbb{A}^1_Z$-scheme and $\eta : V \to U_Z$ is a $Z$-morphism. The arrows in $\mathcal{I}_V$ are the $X$-morphisms in $U$ compatible with $\eta$. Then, by contruction of the pullback on sheaves over small Nisnevich sites, $F_{(\mathbb{A}^1_Z,Z)}$ is the sheaf associated to the presheaf

$$G : V/Z \mapsto \varinjlim_{(U,\eta)\in\mathcal{I}_V} F(U - U_Z)/F(U).$$

There is an obvious morphism $F_{-1}|_Z \to G$ of presheaves over $Z_{\mathrm{Nis}}$, which induces a morphism of sheaves over $Z_{\mathrm{Nis}}$

$$\epsilon_{(\mathbb{A}^1_Z,Z)} : F_{-1,Z} \to F_{(\mathbb{A}^1_Z,Z)}.$$

**Lemma 5.18.** *The morphism $\epsilon_{(\mathbb{A}^1_Z,Z)} : F_{-1}|_Z \to F_{(\mathbb{A}^1_Z,Z)}$ described above is an isomorphism.*

*Proof.* It suffices to check the assertion on the fibres. Using the computation of Lemma 5.16 and the homotopy invariance of $F_{-1}$, this is just Lemma 5.13. □

The two previous lemmas imply the main result of this section :

**Corollary 5.19.** *Let $(X,Z)$ be a closed pair over $S$ and $\rho : (X,Z) \to (\mathbb{A}^{n+1}_S, \mathbb{A}^n_S)$ a parametrisation over $S$.*

*Let $(X,Z) \xrightarrow{p} (\Omega,Z) \xleftarrow{q} (\mathbb{A}^1_Z,Z)$ be the morphisms constructed in paragraph 5.9. Let $T$ be a smooth $S$-scheme. For any smooth $S$-scheme $Y$, we put $Y' = Y \times_S T$ ; this defines an endomorphism of smooth $S$-schemes.*

*Then, all the morphisms in the sequence*

$$F_{(X',Z')} \xleftarrow{(p')^*} F_{(\Omega',Z')} \xrightarrow{(q')^*} F_{(\mathbb{A}^1_{Z'},Z')} \xrightarrow{\epsilon_{(\mathbb{A}^1_{Z'},Z')}} F_{-1}|_{Z'}$$

*are isomorphisms.*

We thus have associated to the parametrisation $\rho$ an isomorphisms of sheaves over $Z_{\mathrm{Nis}}$,

$$\epsilon_{\rho,T} : F_{(X\times_k T, Z\times_k T)} \to F_{-1}|_{Z\times_k T}.$$

This isomorphism is obviously functorial in the smooth $S$-scheme $T$, using the naturality of $\phi_T$ with respect to $T$.

*Remark* 5.20. This isomorphism is functorial in $\rho$ in a suitable sense, but we will not use this functoriality. Note moreover that by using deformation to the normal cone, we can show at this point that $\epsilon_\rho$ does not depend on the choice of $\rho$. This could be used to construct such an isomorphism for any smooth closed pair of codimension 1 without requiring the existence of a global parametrisation.

5.3. **Localisation long exact sequences.** Let $S$ be a regular scheme.

Let $(X_0, Z_0)$ be a smooth closed pair over $S$ of codimension 1 and $\rho : (X_0, Z_0) \to (\mathbb{A}^{n+1}_S, \mathbb{A}^n_S)$ a parametrisation over $S$. Let $T$ be a smooth $S$-scheme, and put $(X, Z) = (X_0 \times_S T, Z_0 \times_S T)$.

Let $j : U \to X$ and $i : Z \leftarrow X$ be the canonical closed embeddings. The isomorphism constructed in Corollary 5.19 induces a canonical exact sequence of sheaves on $X_{\mathrm{Nis}}$

$$0 \to F_X \to j_*F_U \to i_*F_{-1}|_Z \to 0.$$

Recall that $i_*$ is exact. Thus, for any étale $X_0$-scheme $V_0$, taking cohomology on $V = V_0 \times_S T$ we get a localisation long exact sequence,

$$\text{(D)} \qquad \begin{aligned} ... \to & H^{n-1}(V; F_X) \to H^{n-1}(V; j_*F_U) \to H^{n-1}(V_Z; F_{-1}|_Z) \\ & \to H^n(V; F_X) \to H^n(V; j_*F_U) \to ... \end{aligned}$$

This sequence is functorial in $V_0$ with respect to étale $X_0$-morphisms and in $T$ with respect to any $S$-morphisms.

*Remark* 5.21. We could also have considered the closed pair $(V_0, V_0 \times_X Z)$ and the induced parametrisation $(V_0, V_0 \times_X Z) \to (\mathbb{A}^{n+1}_S, \mathbb{A}^n_S)$. By the very construction, the sequence obtained is exactly the above sequence.

To conclude, we note that if we know the vanishing of $R^m j_*$ for $m > 0$, this exact sequence as the form

$$... \to H^{n-1}(V; F) \xrightarrow{j^*} H^{n-1}(V - V_Z; F) \to H^{n-1}(V_Z; F_{-1}) \to H^n(V; F_X) \to ...$$

which is in fact the localisation exact sequence associated to $(V, V_Z)$.

5.4. **Proof.** Let now $S$ be the spectrum of a perfect field $k$.

The proof of Voevodsky proceeds by induction on the dimension. He inductively proves both the homotopy invariance of $H^n(.; F)$ and the existence of the localisation long exact sequence (for smooth divisors) in dimension less than $n$; according to what we have seen above, this amounts to proving the vanishing of $R^m j_*$ for $m < n$, and for any open immersion $j$ of the complement of a smooth divisor.

**Reduction step 1**.–

*Let $n$ be a positive integer. Suppose that for any smooth $k$-scheme $X$, any point $s \in X$, and any integer $0 < i \leq n$, $H^i(\mathbb{A}^1_{X^h_s}; F) = 0$. Then for any smooth $k$-scheme $X$ and any integer $0 < i \leq n$, $H^i(\mathbb{A}^1_X; F) = H^i(X; F)$.*

Indeed, we let $\pi : \mathbb{A}^1_X \to X$ be the canonical projection and apply the Leray spectral sequence to $\pi$ and to $F_{\mathbb{A}^1_X}$ :

$$E_2^{p,q} = H^p(X; R^q\pi_*F_{\mathbb{A}^1_X}) \Rightarrow H^{p+q}(\mathbb{A}^1_X; F_{\mathbb{A}^1_X}).$$

Now it suffices to note that the hypothesis implies $R^q\pi_*F_{\mathbb{A}^1_X} = 0$ if $0 < q \leq n$. Indeed, for any point $s \in X$, $R^q\pi_*F_{\mathbb{A}^1_X}(X^h_s) = H^q(\mathbb{A}^1_{X^h_s}; F)$.

**Reduction step 2**.– The following step is exactly the point where we need the base to be a perfect field.

*Let $n$ be a positive integer. For a closed pair $(X, Z)$ over $k$, we consider the property*
*$P(X, Z)$ : the induced morphism $H^n(\mathbb{A}^1_{X^h_s}; F) \to H^n(\mathbb{A}^1_{X^h_s - Z^h_s}; F)$ is a monomorphism.*

*If the property $P(X, Z)$ holds for any smooth closed pair $(X, Z)$ of codimension 1, then for any smooth $k$-scheme $X$, and any point $s \in X$, $H^n(\mathbb{A}^1_{X^h_s}; F) = 0$.*

We start by showing that under the assumption, $P(X, Z)$ holds for any closed pair $(X, Z)$ over $k$. For this, we use precisely the following lemma of Voevodsky (cf [Voe00a], Lem. 4.31) :

**Lemma 5.22.** *Let $X$ be a smooth $k$-scheme over a perfect field $k$ and $Z$ a nowhere dense closed subscheme of $X$.*

*Then for every point $s \in X$, there exists an open neighbourhood $U$ of $x$ in $X$ and an increasing sequence of closed subschemes $\emptyset = Y_0 \subset ... \subset Y_r$ of $U$ for $r > 0$ such that*

(1) *for any $1 \leq i \leq r$, $Y_i - Y_{i-1}$ is a smooth divisor of $U - Y_{i-1}$.*
(2) *$Z \cap U \subset Y_r$.*

*Proof.* We assume that $X$ is connected and use induction on the dimension $n \geq 1$ of $X$. The case $n = 1$ is trivial, as $k$ is perfect.

Let $n \geq 2$ and $X$ be a smooth $n$-dimensional scheme. Necessarily, there exist an open subscheme $U_0$ of $X$ and a morphism $p : U_0 \to Y$ to a smooth $k$-scheme $Y$ such that $p$ is smooth of relative dimension 1.

Let $Z_{sing}$ be the singular locus of $Z$. As $k$ is perfect, $\dim(Z_{sing}) < \dim(Z) < n$. Let $T$ be the reduced closure of $p(Z_{sing})$ in $Y$. We thus have $\dim(T) \leq n - 2$. As $Y$ has pure dimension $n - 1$, $T$ is nowhere dense in $Y$. By the induction hypothesis applied to $Y$ and $T$ in a neighbourhood of $p(s)$, there is a neighbourhood $V$ of $p(s)$ in $Y$ and an increasing sequence $Y'_0 \subset ... \subset Y'_r$ of closed subschemes of $V$ satisfying conditions 1 and 2 for $T$ and $V$.

Put $U = p^{-1}(V)$, $Y_i = p^{-1}(Y'_i)$ for $0 \leq i \leq r$ and $Y_{r+1} = Y_r \cup (Z \cap U)$. Then the sequence $Y_0 \subset ... \subset Y_{r+1}$ of closed subschemes of $U$ satisfies conditions (1) and (2) for $Z$ and $U$. □

With this lemma, we easily obtain $P(X, Z)$ for any pair $(X, Z)$, as $P(X, T) \Rightarrow P(X, Z)$ if $Z \subset T \subset X$ and $P(X, Z)$ is a local property on $X$.

Fix now a smooth $k$-scheme $X$ and a point $s \in X$. Let $E$ be the quotient field [7] of $\mathcal{O}^h_{X,s}$. The pro-object $(E) = \varprojlim_{Z \subset X} X^h_s - Z^h_s$ of étale $X$-schemes has the scheme $\operatorname{Spec}(E)$ as limit. The property $P(X, Z)$ for any $Z$ implies that the canonical morphism

$$H^n(\mathbb{A}^1_{X^h_s}; F) \to H^n(\mathbb{A}^1_{(E)}; F)$$

[7] The extension field $E/k$, though of finite transcendence degree, is not necessarily of finite type.

is a monomorphism. Let $\tau : \mathrm{Spec}\,(E) \to \mathrm{Spec}\,(k)$ be the canonical morphism. Then Remark 5.14 implies $H^n(\mathbb{A}^1_{(E)}; F) = H^n(\mathbb{A}^1_E; \tau^* F)$. Thus finally, this reduction step follows from Corollary 4.11.

We are now ready to prove the following assertions by induction on $n \geq 1$:

(i) For any smooth closed pair $(X, Z)$ of codimension 1, $j : X - Z \to X$ the open immersion, we have $R^m j_*(F_{X-Z}) = 0$ for all 0<m<n.

(ii) For any smooth closed pair $(X, Z)$ of codimension 1 with a given parametrisation $\rho$, $j : X - Z \to X$ the open immersion, and $V$ an étale $X$-scheme, the localisation exact sequence (D) induces an exact sequence

$$H^{n-1}(V - V_Z; F) \to H^{n-1}(V_Z; F_{-1}) \to H^n(V; F) \xrightarrow{j^*} H^n(V - V_Z; F)$$

functorial in $X$ with respect to étale morphisms (the parametrisation of an étale $X$-scheme being the parametrisation induced by $\rho$).

(iii) $H^n(.; F)$ is homotopy invariant.

Suppose that $n = 1$. Let $(X, Z)$ be a smooth pair of codimension 1 and $\rho$ a parametrisation of $(X, Z)$. Let $T = \mathrm{Spec}\,(k)$ or $\mathbb{A}^1_k$, $(X', Z') = (X \times_k T, Z \times_k T)$, $U' = X' - Z'$ and $j : U' \to X'$ the canonical immersion. The localisation exact sequence (D) associated with these data is

$$0 \to F(X' - Z') \xrightarrow{j^*} F(X') \to F_{-1}(X') \to H^1(X'; F) \xrightarrow{\nu} H^1(X'; j_* F_{X'-Z'}).$$

Applying the Leray spectral sequence for $j$ and $F_{X'-Z'}$, we obtain a canonical morphism $H^1(X'; j_* F_{X'-Z'}) \xrightarrow{b} H^1(X' - Z'; F_{X'-Z'})$ which is a monomorphism and such that $j^* = b \circ \nu$. This implies that $\mathrm{Ker}(\nu) = \mathrm{Ker}(j^*)$, hence we obtain the sequence of property (ii) for $(X', Z')$. Functoriality in $X$ with respect to étale morphisms now follows from the functoriality of the sequence (D).

Let $s$ be a point of $Z$. We now consider the limit of this exact sequence by replacing $X$ by an arbitrary Nisnevich neighbourhood of $s$ in $X$. The functoriality in $T$ of the sequence (D) implies that the following diagram, in which the lines are exact, is commutative :

$$\begin{tikzcd}
F(X^h_s - Z^h_s) \arrow[r] \arrow[d, "\sim"'] & F_{-1}(Z^h_s) \arrow[rr] \arrow[d, "\sim"] & & 0 \\
F(\mathbb{A}^1_{X^h_s - Z^h_s}) \arrow[r] & F_{-1}(\mathbb{A}^1_{Z^h_s}) \arrow[r] & H^1(\mathbb{A}^1_{X^h_s}; F) \arrow[r, "(1)"] & H^1(\mathbb{A}^1_{X^h_s - Z^h_s}; F).
\end{tikzcd}$$

Thus (1) is a monomorphism. As any smooth closed pair $(X, Z)$ locally admits a parametrisation, we are done by reduction steps (2) and (1).

We now prove the result for $n > 1$ using the induction hypothesis. To prove (i), we consider $s \in X$ and show that for any $0 < q < n$, the fiber of $R^q j_* F_U$ at $X^h_s$ is zero.

By the induction hypothesis and Corollary 2.10, $H = H^m(.; F)$ is a homotopy presheaf over $k$. Then Proposition 5.12 implies that we have an isomorphism

$$R^q j_* F_U(X^h_s) = H(X^h_s - Z^h_s) = H(X^h_s - Z^h_s)/H(X^h_s) \simeq H_{-1}(Z^h_s).$$

Let $E$ be the quotient field of $\mathcal{O}_{Z_s^h}$. Put $(E) = \tilde{\varprojlim_{T \subset Z}} Z_s^h - T_s^h$. The canonical morphism $(E) \to Z_s^h$ is a pro-immersion. Thus, applying Corollary 4.6, the induced morphism $H_{-1}(Z_s^h) \to H_{-1}(E)$ is a monomorphism. Indeed, $Z_s^h$ is a point and, though $E$ is not necessarily of finite type over $k$, it is the filtering union of its sub-$k$-extensions $E'$ of finite type. Thus $F \mapsto F(E)$ is still a fiber functor for the Nisnevich topology on $\mathscr{S}m_k$.

Let $\tau : \mathrm{Spec}\,(E) \to \mathrm{Spec}\,(k)$ be the canonical morphism. We obtain finally the following inclusion

$$H_{-1}(E) \subset H(\mathbb{G}_m \times (E)) = H^q(\mathbb{G}_m \times (E); F) = H^q(\mathbb{G}_{m,E}; \tau^* F)$$

using remark 5.14. Since the latter group is zero by Corollary 4.11, we are done for (i).

For (ii) it is now sufficient to apply the same reasoning than in the case $n = 1$. Indeed property (i) and the Leray spectral sequence for $j$ give the edge monomorphism $H^n(X; R^0 j_* F_X) \xrightarrow{b} H^n(X - Z; F_{X-Z})$.

For (iii) now, we consider a smooth closed pair $(X, Z)$ of codimension 1 and a point $s \in Z$. It admits a parametrisation in a neighbourhood $V$ of $s$, which induces a parametrisation of $(\mathbb{A}^1_V, \mathbb{A}^1_{V \cap Z})$. This parametrisation being fixed, we can consider the exact sequence of property (ii) for any étale $V$-scheme. If we take the colimit of these sequences with respect to the Nisnevich neighbourhoods of $s$ in $V$, we obtain the following exact sequence

$$H^{n-1}(\mathbb{A}^1_{Z_s^h}; F_{-1}) \to H^n(\mathbb{A}^1_{X_s^h}; F) \to H^n(\mathbb{A}^1_{X_s^h - Z_s^h}; F).$$

This concludes the proof using the induction hypothesis, as we can now use again reduction steps (2) and (1).

□

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