MIXED WEIL COHOMOLOGIES

DENIS-CHARLES CISINSKI AND FRÉDÉRIC DÉGLISE

Abstract. We define, for a regular scheme S and a given field of characteristic zero $\mathbf{K}$, the notion of $\mathbf{K}$-linear mixed Weil cohomology on smooth S-schemes by a simple set of properties, mainly: Nisnevich descent, homotopy invariance, stability (which means that the cohomology of $\mathbf{G}_m$ behaves correctly), and Künneth formula. We prove that any mixed Weil cohomology defined on smooth S-schemes induces a symmetric monoidal realization of some suitable triangulated category of motives over S to the derived category of the field $\mathbf{K}$. This implies a finiteness theorem and a Poincaré duality theorem for such a cohomology with respect to smooth and projective S-schemes (which can be extended to smooth S-schemes when S is the spectrum of a perfect field). This formalism also provides a convenient tool to understand the comparison of such cohomology theories.

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Introduction

Weil cohomologies were introduced by Grothendieck in the 1960's as the cohomologies defined on smooth and projective varieties over a field with enough good properties (mainly, the existence of cycle class maps, Künneth formula and Poincaré duality) to prove the Weil conjectures (i.e. to understand the L-functions attached to smooth and projective varieties over a finite field). According to the philosophy of Grothendieck, they can be seen as the fiber functors of the (conjecturaly) tannakian category of pure motives. From this point of view, a mixed Weil cohomology should define an exact tensor functor from the (conjectural) abelian category of mixed motives to the category of (super) vector spaces over a field of characteristic zero, such that, among other things, its restriction to pure motives would be a Weil cohomology.

The purpose of these notes is to provide a simple set of axioms for a cohomology theory to induce a symmetric monoidal realization functor of a suitable version of the triangulated category of mixed motives to the derived category of vector spaces over a field of characteristic zero. Such a compatibility with symmetric monoidal structures involves obviously a Künneth formula for our given cohomology. And the main result we get here says that this property is essentially sufficient to get a realization functor. Moreover, apart from the Künneth formula, our set of axioms is very close to that of Eilenberg and Steenrod in algebraic topology.

Let k be a perfect field and $\mathbf{K}$ a field of characteristic zero. Let $\mathfrak{V}$ be the category of smooth affine k-schemes. Consider a presheaf of commutative differential graded $\mathbf{K}$-algebras E on $\mathfrak{V}$. Given any smooth affine scheme X, any closed subset $Z \subset X$ such that $U = X - Z$ is affine, and any integer n, we put:

$$H^n_Z(X,E) = H^{n-1}\big(\mathrm{Cone}(E(X) \to E(U))\big)$$

Definition. A $\mathbf{K}$-linear *mixed Weil theory* is a presheaf of differential graded $\mathbf{K}$-algebras E on $\mathfrak{V}$ satisfying the following properties:

Dimension.— $\dim_{\mathbf{K}} H^i(\mathrm{Spec}\,(k)\,, E) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{otherwise;} \end{cases}$

Homotopy.— $\dim_{\mathbf{K}} H^i(\mathbf{A}^1_k, E) = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{otherwise;} \end{cases}$

Stability.— $\dim_{\mathbf{K}} H^i(\mathbf{G}_m, E) = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = 1, \\ 0 & \text{otherwise;} \end{cases}$

Excision.— Consider a commutative diagram of k-schemes

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

such that i and j are closed immersions, the schemes $X, Y, X - Z, Y - T$ are smooth and affine, f is étale and $f^{-1}(X - Z) = Y - T$, g is an isomorphism. Then the induced morphism

$$H^*_T(Y,E) \longrightarrow H^*_Z(X,E)$$

is an isomorphism;

Künneth formula.— For any smooth affine k-schemes X and Y, the exterior cup product induces an isomorphism

$$\bigoplus_{p+q=n} H^p(X,E) \otimes_{\mathbf{K}} H^q(Y,E) \xrightarrow{\sim} H^n(X \times_k Y, E) \ .$$

The easiest example of a mixed Weil theory the reader might enjoy to have in mind is algebraic de Rham cohomology over a field of characteristic zero. The homotopy axiom in this setting is rather called the *Poincaré lemma*.

We will prove that the excision axiom on a presheaf of differential graded $\mathbf{K}$-algebras E is equivalent to the following property :
Nisnevich Descent.— For any smooth affine scheme X, the cohomology groups of the complex $E(X)$ are isomorphic to the Nisnevich hypercohomology groups of X with coefficients in E_{Nis} under the canonical map.

Given a mixed Weil theory E and any smooth scheme X, we denote by $H^n(X,E)$ the Nisnevich hypercohomology groups of X with coefficients in E_{Nis}. According to the previous assertion, this extends the definition given above to the case where X is affine. We define for a $\mathbf{K}$-vector space V and an integer n

$$V(n) = \begin{cases} V \otimes_{\mathbf{K}} \mathrm{Hom}_{\mathbf{K}}(H^1(\mathbf{G}_m,E)^{\otimes n}, \mathbf{K}) & \text{if } n \geqslant 0, \\ V \otimes_{\mathbf{K}} H^1(\mathbf{G}_m,E)^{\otimes(-n)} & \text{if } n \leqslant 0. \end{cases}$$

Note that any choice of a generator of $H^1(\mathbf{G}_m,E)$ defines an isomorphism $V(n) \simeq V$. The introduction of these Tate twists allows us to make canonical constructions, avoiding the choice of a generator for $H^1(\mathbf{G}_m,E)$. Our main results can now be summarized as follows.

Theorem 1. *The cohomology groups $H^q(X,E)$ have the following properties.*

1. Finiteness.— *For any smooth k-scheme X, the $\mathbf{K}$-vector space $\oplus_n H^n(X,E)$ is finite dimensional.*
2. Cycle class map.— *For any smooth k-scheme X, there is a natural map which is compatible with cup product*
$$H^q(X,\mathbf{Q}(p)) \longrightarrow H^q(X,E)(p)$$
(where $H^q(X,\mathbf{Q}(p))$ is motivic cohomology, as defined by Voevodsky; in particular, we have $H^{2n}(X,\mathbf{Q}(n)) = CH^n(X)_{\mathbf{Q}}$).
3. Compact support.— *For any smooth k-scheme X, there are cohomology groups $H^q_c(X,E)$, which satisfies all the usual functorialities of a cohomology with compact support, and there are natural maps*
$$H^q_c(X,E) \longrightarrow H^q(X,E)$$
which are isomorphisms whenever X is projective.
4. Poincaré duality.— *For any smooth k-scheme X of pure dimension d, there is a natural perfect pairing of finite dimensional $\mathbf{K}$-vector spaces*
$$H^q_c(X,E)(p) \otimes_{\mathbf{K}} H^{2d-q}(X,E)(d-p) \longrightarrow \mathbf{K}\,.$$

Theorem 2 (Comparison)**.** *Let E' be a presheaf of commutative differential graded $\mathbf{K}$-algebras satisfying the dimension, homotopy, stability and excision axioms and such that for any smooth k-scheme X, the Künneth map*

$$\bigoplus_{p+q=n} H^p(X,E') \otimes_{\mathbf{K}} H^q(Y,E') \longrightarrow H^n(X \times_k Y, E')$$

is an isomorphism for $Y = \mathbf{A}^1_k$ or $Y = \mathbf{G}_m$ (e.g. E' might be a mixed Weil theory). Then a morphism of presheaves of differential graded $\mathbf{K}$-algebras $E \longrightarrow E'$ is a quasi-isomorphism (locally for the Nisnevich topology) if and only if the map $H^1(\mathbf{G}_m, E) \longrightarrow H^1(\mathbf{G}_m, E')$ is not trivial[1].

Remark this comparison theorem is completely similar to the classical comparison theorem of Eilenberg-Steenrod.

Theorem 3 (Realization). *There is a symmetric monoidal triangulated functor*

$$R : DM_{gm}(k)_{\mathbf{Q}} \longrightarrow D^b(\mathbf{K})$$

(where $DM_{gm}(k)_{\mathbf{Q}}$ is Voevodsky's triangulated category of mixed motives over k, and $D^b(\mathbf{K})$ denotes the bounded derived category of the category of finite dimensional $\mathbf{K}$-vector spaces) such that for any smooth k-scheme X, one has the following canonical identifications (where $M^\vee$ denotes the dual of M):

$$R(M_{gm}(X)^\vee) \simeq R(M_{gm}(X))^\vee \simeq \mathbf{R}\Gamma(X, E)\,.$$

Moreover, for any object M of $DM_{gm}(k)_{\mathbf{Q}}$, and any integer p, one has

$$R(M(p)) = R(M)(p)\,.$$

These statements are proved using the homotopy theory of schemes of Morel and Voevodsky. We work in the stable homotopy category of motivic symmetric spectra with rational coefficients, denoted by $D_{\mathbf{A}^1}(\operatorname{Spec}(k)\,, \mathbf{Q})$. We associate canonically to a mixed Weil theory E a commutative ring spectrum $\mathcal{E}$ such that for any smooth k-scheme X and integers p and q, we get a natural identification

$$H^q(X, E)(p) = H^q(X, \mathcal{E}(p))\,.$$

We also consider the triangulated category $D_{\mathbf{A}^1}(\operatorname{Spec}(k)\,, \mathcal{E})$, which might be thought of as the category of 'motives with coefficients in $\mathcal{E}$' (this is simply the localization of the category of $\mathcal{E}$-modules by stable $\mathbf{A}^1$-equivalences). We obviously have a symmetric monoidal triangulated functor

$$D_{\mathbf{A}^1}(\operatorname{Spec}(k)\,, \mathbf{Q}) \longrightarrow D_{\mathbf{A}^1}(\operatorname{Spec}(k)\,, \mathcal{E}) \quad , \qquad M \longmapsto \mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} M\,.$$

If $D(\mathbf{K})$ is the unbounded derived category of the category of $\mathbf{K}$-vector spaces, the result hidden behind Theorems 1 and 2 is

Theorem 4 (Tilting). *The homological realization functor*

$$D_{\mathbf{A}^1}(\operatorname{Spec}(k)\,, \mathcal{E}) \longrightarrow D(\mathbf{K}) \quad , \qquad M \longmapsto \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M)$$

is an equivalence of symmetric monoidal triangulated categories.

To obtain Theorem 3, we interpret the cycle class map as a map of ring spectra $\mathbf{HQ} \longrightarrow \mathcal{E}$, and use a result of Röndigs and Østvær which identifies $DM(k)_{\mathbf{Q}}$ with the homotopy category of modules over the motivic cohomology spectrum $\mathbf{HQ}$. Note that, by Theorem 4, the homological realization functor of Theorem 3 is essentially the derived base change functor $M \longmapsto \mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} M$. This means that the theory of motivic realization functors is part of (a kind of) tilting theory.

Most of our paper is written over a general regular base S rather than just a perfect field. The first reason for this is that a big part of this machinery works

[1] The main point here is in fact that the map $H^1(\mathbf{G}_m, E) \longrightarrow H^1(\mathbf{G}_m, E')$ controls the compatibility with cycle class maps, which in turns ensures the compatibility with Poincaré duality.

mutatis mutandis over a regular base[2] once we are ready to pay the price of slightly weaker or modified results (we essentially can say interesting things only for smooth and projective S-schemes). When the base is a perfect field, the results announced above are obtained from the general ones using de Jong resolution of singularities by alterations. The second reason is that mixed Weil theories defined on smooth schemes over a complete discrete valuation ring V are of interest: the analog of Theorem 2 gives a general way to compare the cohomology of the generic fiber and of the special fiber of a smooth and projective V-scheme (or, more generally, of a smooth V-scheme with good properties near infinity).

Here is a more detailed account on the contents of this paper.

These notes are split into three parts. The first one sets the basic constructions we need. That is we construct the 'effective' $\mathbf{A}^1$-derived category $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$ of a scheme S with coefficients in a ring R and recall its main geometrical and formal properties. We then introduce the Tate object $R(1)$ and define the category of Tate spectra as the category of symmetric $R(1)$-spectra. The 'non effective' $\mathbf{A}^1$-derived category $D_{\mathbf{A}^1}(S,R)$ of a scheme S with coefficients in a ring R is then the localization of the category of Tate spectra by stable $\mathbf{A}^1$-equivalences. We finish the first part by introducing the $\mathbf{A}^1$-derived category $D_{\mathbf{A}^1}(S,\mathcal{E})$ of a scheme S with coefficients in a (commutative) ring spectrum $\mathcal{E}$ (that is a (commutative) monoid object in the category of symmetric Tate spectra). The category $D_{\mathbf{A}^1}(S,\mathcal{E})$ is just defined as the localization of the category of $\mathcal{E}$-modules by the class of stable $\mathbf{A}^1$-equivalences.

The second part is properly about mixed Weil cohomologies. We also define a slightly weaker notion which we call a stable cohomology theory over a given regular scheme S. We associate canonically to any stable cohomology E a commutative ring spectra $\mathcal{E}$ and a canonical isomorphism

$$E \longrightarrow \mathbf{R}\Omega^\infty(\mathcal{E})\,.$$

In other words, E can be seen as a kind of 'Tate infinite loop space' in the category $D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})$. This means in particular that $\mathcal{E}$ represents in $D_{\mathbf{A}^1}(S,\mathbf{K})$ the cohomology theory defined by E. We get essentially by definition a 1-periodicity property for $\mathcal{E}$, that is the existence of an isomorphism $\mathcal{E}(1)\simeq\mathcal{E}$. We then study the main properties of the triangulated category $D_{\mathbf{A}^1}(S,\mathcal{E})$. In particular, we prove that Thom spaces are trivial in $D_{\mathbf{A}^1}(S,\mathcal{E})$, which imply that there is a simple theory of Chern classes and of Gysin maps in $D_{\mathbf{A}^1}(S,\mathcal{E})$. Using results of J. Riou, this allows to produce a canonical cycle class map

$$K_{2p-q}(X)^{(p)}_{\mathbf{Q}} \longrightarrow H^q(X,\mathcal{E}(p)) = H^q(X,E)(p)$$

(where $K_q(X)^{(p)}_{\mathbf{Q}}$ denotes the part of $K_q(X)$ where the k^{th} Adams operation acts by multiplication by k^p). The good functoriality properties of Gysin maps implies a Poincaré duality theorem in $D_{\mathbf{A}^1}(S,\mathcal{E})$ for smooth and projective S-schemes. In particular, for any smooth and projective S-scheme X, the object $\Sigma^\infty(\mathbf{Q}(X))\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathcal{E}$ has a strong dual in $D_{\mathbf{A}^1}(S,\mathcal{E})$ (i.e. it is a rigid object). We then prove a

[2]In fact, one could drop the regularity assumption, but then, the formulation of some of our results about the existence of a cycle class map are a little more involved: K-theory is homotopy invariant only for regular schemes. This is not a serious problem, but we decided to avoid the extra complications due to the fact algebraic K-theory is not representable in the $\mathbf{A}^1$-homotopy theory of singular schemes.

weak version of Theorem 4: if $D^{\vee}_{\mathbf{A}^1}(S,\mathcal{E})$ denotes the localizing subcategory of $D_{\mathbf{A}^1}(S,\mathcal{E})$ generated by the objects which have a strong dual, and if E is a mixed Weil theory, then the homological realization functor induces an equivalence of symmetric monoidal triangulated categories

$$D^{\vee}_{\mathbf{A}^1}(S,\mathcal{E}) \simeq D(\mathbf{K})\,.$$

Given a stable theory E', if $\mathcal{E}'$ denotes the associated ring spectrum, we associate to any morphism of presheaves of differential graded algebras $E \longrightarrow E'$ a base change functor

$$D_{\mathbf{A}^1}(S,\mathcal{E}) \longrightarrow D_{\mathbf{A}^1}(S,\mathcal{E}')$$

whose restriction to $D^{\vee}_{\mathbf{A}^1}(S,\mathcal{E})$ happens to be fully faithful whenever E is a mixed Weil theory. In particular, the cohomologies defined by E and E' have then to agree on smooth and projective S-schemes.

In the case where S is the spectrum of a perfect field, we prove that the cycle class map $\mathbf{HQ} \longrightarrow \mathcal{E}$, from Voevodsky's rational motivic cohomology spectrum to our given mixed Weil cohomology $\mathcal{E}$, is a morphism of commutative ring spectra. This is achieved by interpreting the cycle class map as an isomorphism $\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ} \simeq \mathcal{E}$ in the homotopy category of $\mathcal{E}$-modules. We then observe that the theory of de Jong alterations implies the equality:

$$D^{\vee}_{\mathbf{A}^1}(S,\mathcal{E}) = D_{\mathbf{A}^1}(S,\mathcal{E})\,.$$

Using the equivalence of categories $D_{\mathbf{A}^1}(S,\mathbf{HQ}) \simeq DM(k,\mathbf{Q})$, we deduce the expected realization functor from the triangulated category of mixed motives to the derived category of the category of $\mathbf{K}$-vector spaces. In a sequel, we also provide a shorter argument which relies on an unpublished result of F.Morel stated in [Mor06].

The last part is an elementary study of some classical mixed Weil theories We prove that, over a field of characteristic zero, algebraic de Rham cohomology is a mixed Weil theory, and we explain how Grothendieck's (resp. Kiehl's) comparison theorem between algebraic and complex analytic (resp. rigid analytic) de Rham cohomology fits in this picture.

We proceed after this to the study of Monsky-Washnnitzer cohomology as a mixed Weil theory, and revisit the Berthelot-Ogus Comparison Theorem, which relates de Rham cohomology and crystalline cohomology: given a complete discrete valuation ring V, with field of fractions of characteristic zero, and perfect residue field, for any smooth and proper V-scheme X the de Rham cohomology of the generic fiber of X and the crystalline cohomology of the special fiber of X are canonically isomorphic. Our proof of this fact also provides a simple argument to see that the triangulated category of geometrical mixed motives over V cannot be rigid: we see that, otherwise, for *any* smooth V-scheme X, the de Rham cohomology of the generic fiber of X and the Monsky-Washnitzer cohomology of the special fiber of X would agree, and this is very obviously false in general (for instance, the special fiber of X might be empty). We also explain how to define rigid cohomology from the Monsky-Washnitzer complex using the natural functorialities of $\mathbf{A}^1$-homotopy theory of schemes.

We finally explain an elementary construction of étale cohomology as a mixed Weil cohomology.

As a conclusion, let us mention that this paper deals only with the elementary part of the story: in a sequel of this paper [CD09b], we shall improve this constructions. In particular, we shall prove that any mixed Weil cohomology extends naturally to k-schemes of finite type, satisfies h-descent (in particular étale descent and proper descent), and defines a system of triangulated categories on which the six operations of Grothendieck act. By appyling this construction to rigid cohomology, this will define a convenient fundation for a good notion of p-adic coefficients.

This paper takes its origins from a seminar on p-adic regulators organized by J .Wildeshaus, D. Blottière and the second named author at university Paris 13. This is where the authors went to the problem of representing rigid cohomology as a realization functor of the triangulated category of mixed motives, and the present paper can be seen as a kind of answer (among others, see for example [Lev98, Hub00]). We would like to thank deeply Y. Henrio for all the time he spent to explain the arcanes of rigid analytic geometry and of p-adic cohomology to us. We benefited of valuable discussions with J. Ayoub, L. Breen, W. Messing and J. Riou. We feel very grateful to J. Wildeshaus for his constant warm support and enthusiasm. We also thank J. Wildeshaus and J. I. Burgos, for their joint careful reading and valuable comments.

1. Motivic homological algebra

All schemes are assumed to be noetherian and of finite Krull dimension. We will say 'S-scheme' for 'separated scheme of finite type over S'.

If $\mathcal{A}$ is an abelian category, we let $Comp(\mathcal{A})$, $K(\mathcal{A})$ and $D(\mathcal{A})$ be respectively the category of unbounded cochain complexes of $\mathcal{A}$, the same category modulo cochain homotopy equivalence, and the unbounded derived category of $\mathcal{A}$.

1.1. $\mathbf{A}^1$-invariant cohomology.

1.1.1. We suppose given a scheme S. We consider a full subcategory $\mathfrak{V}$ of the category Sm/S of smooth S-schemes satisfying the following properties[3].

(a) $\mathbf{A}^n_S$ belongs to $\mathfrak{V}$ for $n \geqslant 0$.

(b) If $X' \longrightarrow X$ is an étale morphism and if X is in $\mathfrak{V}$, then there exists a Zariski covering $Y \longrightarrow X'$ with Y in $\mathfrak{V}$.

(c) For any pullback square of S-schemes

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "u'"'] & X \arrow[d, "u"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in which u is smooth, if X,Y and Y' are in $\mathfrak{V}$, so is X'.

(d) If X and Y are in $\mathfrak{V}$, then their disjoint union $X \amalg Y$ is in $\mathfrak{V}$.

(e) For any smooth S-scheme X, there exists a Nisnevich covering $Y \longrightarrow X$ of X with Y in $\mathfrak{V}$.

[3]In practice, the category $\mathfrak{V}$ will be Sm/S itself or the full subcategory of smooth affine S-schemes.

We recall that a Nisnevich covering is a surjective and completely decomposed étale morphism. This defines the Nisnevich topology on $\mathfrak{V}$; see e.g. [Nis89, KS86, TT90, MV99].

The last property (e) ensures that the category of sheaves on $\mathfrak{V}$ is equivalent to the category of sheaves on the category of smooth S-schemes as far as we consider sheaves for the Nisnevich topology (or any stronger one).

1.1.2. A *distinguished square* is a pullback square of schemes

(1.1.2.1)
$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ U \arrow[r, "j"'] & X \end{tikzcd}$$

where j is an open immersion and f is an étale morphism such that the induced map from $f^{-1}((X-U)_{red})$ to $(X-U)_{red}$ is an isomorphism. For such a distinguished square, the map $(j,f) : U \amalg V \longrightarrow X$ is a Nisnevich covering. A very useful property of the Nisnevich topology is that any Nisnevich covering can be refined by a covering coming from a distinguished square (as far as we work with noetherian schemes). This leads to the following characterization of the Nisnevich sheaves.

A presheaf F on $\mathfrak{V}$ is a sheaf for the Nisnevich topology if and only if for any distinguished square of shape (1.1.2.1), we obtain a pullback square

(1.1.2.2)
$$\begin{tikzcd} F(X) \arrow[r, "f^*"] \arrow[d, "j^*"'] & F(V) \arrow[d, "i^*"] \\ F(U) \arrow[r, "g^*"'] & F(W) \end{tikzcd}$$

This implies that Nisnevich sheaves are stable by filtering colimits in the category of presheaves on $\mathfrak{V}$. In other words, if I is a small filtering category, and if F is a functor from I to the category of presheaves on $\mathfrak{V}$ such that F_i is a Nisnevich sheaf for all $i \in I$, then the presheaf $\varinjlim_{i\in I} F_i$ is a Nisnevich sheaf.

1.1.3. We fix a commutative ring R. Let $Sh(\mathfrak{V}, R)$ be the category of Nisnevich sheaves of R-modules on $\mathfrak{V}$. For a presheaf (of R-modules) F, we denote by F_{Nis} the Nisnevich sheaf associated to F. We can form its derived category

$$D(\mathfrak{V}, R) = D(Sh(\mathfrak{V}, R)) .$$

More precisely, the category $D(\mathfrak{V}, R)$ is obtained as the localization of the category $Comp(\mathfrak{V}, R)$ of (unbounded) complexes of the Grothendieck abelian category $Sh(\mathfrak{V}, R)$ by the class of quasi-isomorphisms. As we have an equivalence of categories

$$Sh(\mathfrak{V}, R) \simeq Sh(Sm/S, R) ,$$

we also have a canonical equivalence of categories

(1.1.3.1) $$D(\mathfrak{V}, R) \simeq D(Sm/S, R) .$$

We have a canonical functor

(1.1.3.2) $$R : \mathfrak{V} \longrightarrow Sh(\mathfrak{V}, R) \quad , \qquad X \longmapsto R(X)$$

where $R(X)$ denotes the Nisnevich sheaf associated to the presheaf

$$Y \longmapsto \text{free } R\text{-module generated by } \mathrm{Hom}_{\mathfrak{V}}(Y, X).$$

Note that according to 1.1.2, for any X in $\mathfrak{V}$, and any small filtering system $(F_i)_{i\in I}$ of $Sh(\mathfrak{V}, R)$, the canonical map

$$\varinjlim_{i\in I} \mathrm{Hom}_{Sh(\mathfrak{V},R)}(R(X), F_i) \longrightarrow \mathrm{Hom}_{Sh(\mathfrak{V},R)}(R(X), \varinjlim_{i\in I} F_i) \tag{1.1.3.3}$$

is an isomorphism (we can even take X to be any smooth S-scheme according to the previous equivalence).

For a complex K of presheaves of R-modules on $\mathfrak{V}$, we have a canonical isomorphism

$$H^n_{\mathrm{Nis}}(X, K_{\mathrm{Nis}}) = \mathrm{Hom}_{D(\mathfrak{V},R)}(R(X), K_{\mathrm{Nis}}[n]) \tag{1.1.3.4}$$

where X is an object of $\mathfrak{V}$, n is an integer, and $H^n_{\mathrm{Nis}}(X, K_{\mathrm{Nis}})$ is the Nisnevich hypercohomology with coefficients in K.

1.1.4. For a sheaf of R-modules F, and an integer n, we denote by $D^n F$ the complex concentrated in degrees n and $n+1$ whose only non trivial differential is the identity of F. We write $S^n F$ for the sheaf F seen as a complex concentrated in degree n. We have a canonical inclusion of $S^{n+1}F$ in $D^n F$. We say that a morphism of complexes of sheaves of R-modules is a $\mathfrak{V}$-*cofibration* if it is contained in the smallest class of maps stable by pushout, transfinite composition and retract that contains the maps of the form $S^{n+1}R(X) \longrightarrow D^n R(X)$ for any integer n and any X in $\mathfrak{V}$. For example, for any X in $\mathfrak{V}$, the map $0 \longrightarrow R(X)$ is a $\mathfrak{V}$-cofibration (where $R(X)$ is seen as a complex concentrated in degree 0). A complex of presheaves K is $\mathfrak{V}$-*cofibrant* if $0 \longrightarrow K$ is a $\mathfrak{V}$-cofibration.

A complex of presheaves of R-modules K on $\mathfrak{V}$ is $\mathfrak{V}_{\mathrm{Nis}}$-*local* if for any X in $\mathfrak{V}$, the canonical map

$$H^n(K(X)) \longrightarrow H^n_{\mathrm{Nis}}(X, K_{\mathrm{Nis}})$$

is an isomorphism of R-modules.

A morphism $p : K \longrightarrow L$ of complexes of presheaves of R-modules on $\mathfrak{V}$ is $\mathfrak{V}$-surjective if for any X in $\mathfrak{V}$, the map $K(X) \longrightarrow L(X)$ is surjective.

Proposition 1.1.5. *The category of complexes of Nisnevich sheaves of R-modules on $\mathfrak{V}$ is a proper Quillen closed model category structure whose weak equivalences are the quasi-isomorphisms, whose cofibrations are the $\mathfrak{V}$-cofibrations and whose fibrations are the $\mathfrak{V}$-surjective morphisms with $\mathfrak{V}_{\mathrm{Nis}}$-local kernel. In particular, for any X in $\mathfrak{V}$, $R(X)$ is $\mathfrak{V}$-cofibrant.*

Proof. If $\mathcal{X}$ is a simplicial object of $\mathfrak{V}$, we denote by $R(\mathcal{X})$ the associated complex. If $\mathcal{X}$ is a Nisnevich hypercovering of an object X of $\mathfrak{V}$, we have a canonical morphism from $R(\mathcal{X})$ to $R(X)$ and we define

$$\widetilde{R}(\mathcal{X}) = \mathrm{Cone}\big(R(\mathcal{X}) \longrightarrow R(X)\big) .$$

Let $\mathcal{G}$ be the collection of the $R(X)$'s for X in $\mathfrak{V}$, and $\mathcal{H}$ the class of the $\widetilde{R}(\mathcal{X})$'s for all the Nisnevich hypercoverings of any object X of $\mathfrak{V}$. Then $(\mathcal{G}, \mathcal{H})$ is a descent structure on $Sh(\mathfrak{V}, R)$ as defined in [CD09a, Definition 1.4], so that we can apply [CD09a, Theorem 1.7 and Corollary 4.9]. □

1.1.6. The model structure above will be called the $\mathfrak{V}$-*local model structure.*

Corollary 1.1.7. *For any complex of Nisnevich sheaves of R-modules K, there exists a quasi-isomorphism $K \longrightarrow L$ where L is $\mathfrak{V}_{\mathrm{Nis}}$-local.*

Proof. We just have to choose a factorization of $K \longrightarrow 0$ into a quasi-isomorphism $K \longrightarrow L$ followed by a fibration $L \longrightarrow 0$ for the above model structure. □

Proposition 1.1.8. *For any distinguished square*

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ U \arrow[r, "j"'] & X \end{tikzcd}$$

the induced commutative square of sheaves of R-modules

$$\begin{tikzcd} R(W) \arrow[r, "i*"] \arrow[d, "g_*"'] & R(V) \arrow[d, "f_*"] \\ R(U) \arrow[r, "j_*"'] & R(X) \end{tikzcd}$$

is exact; this means that it is cartesian and cocartesian, or equivalently that it gives rise to a short exact sequence in the category of sheaves of R-modules

$$0 \longrightarrow R(W) \xrightarrow{g_* - i_*} R(U) \oplus R(V) \xrightarrow{(j_*, f_*)} R(X) \longrightarrow 0 \ .$$

Proof. The characterization of Nisnevich sheaves given in 1.1.2 implies that the sequence

$$0 \longrightarrow R(W) \longrightarrow R(U) \oplus R(V) \longrightarrow R(X) \longrightarrow 0$$

is right exact. So the result comes from the injectivity of the map from $R(W)$ to $R(V)$ induced by i. □

1.1.9. Let K be a complex of presheaves of R-modules on $\mathfrak{V}$.

A *closed pair* will be a couple (X, Z) such that X is a scheme in $\mathfrak{V}$, $Z \subset X$ is a closed subset and $X - Z$ belongs to $\mathfrak{V}$. Let j be the immersion of $X - Z$ in X. We put

$$K_Z(X) = \mathrm{Cone}\big(K(X) \xrightarrow{j^*} K(X - Z)\big)[-1].$$

A morphism of closed pairs $f : (Y, T) \to (X, Z)$ is a morphism of schemes $f : Y \to X$ such that $f^{-1}(Z) \subset T$. The morphism of closed pairs f will be called *excisive* when the induced square

$$\begin{tikzcd} Y - T \arrow[r, "i"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ X - Z \arrow[r, "j"'] & X \end{tikzcd}$$

is distinguished.

The complex $K_Z(X)$ is obviously functorial with respect to morphisms of closed pairs. We will say that K has the *excision property on* $\mathfrak{V}$ if for any excisive morphism $f : (Y, T) \to (X, Z)$ the map $K_T(Y) \to K_Z(X)$ is a quasi-isomorphism.

We will say that K has the *Brown-Gersten property on $\mathfrak{V}$ with respect to the Nisnevich topology*, or the *B.-G.-property* for short, if for any distinguished square

$$\begin{tikzcd} W \arrow[r, "i"] \arrow[d, "g"'] & V \arrow[d, "f"] \\ U \arrow[r, "j"'] & X \end{tikzcd}$$

in $\mathfrak{V}$, the square

$$\begin{tikzcd} K(X) \arrow[r, "f^*"] \arrow[d, "j^*"'] & K(V) \arrow[d, "i^*"] \\ K(U) \arrow[r, "g^*"'] & K(W) \end{tikzcd}$$

is a homotopy pullback (or equivalently a homotopy pushout) in the category of complexes of R-modules. The latter condition means that the commutative square of complexes of R-modules obtained from the distinguished square above by applying K leads canonically to a long exact sequence "à la Mayer-Vietoris"

$$H^n(K(X)) \xrightarrow{j^*+f^*} H^n(K(U)) \oplus H^n(K(V)) \xrightarrow{g^*-i^*} H^n(K(W)) \longrightarrow H^{n+1}(K(X))$$

The complexes satisfying the B.-G.-property are in fact the fibrant objects of the model structure of Proposition 1.1.5. This is shown by the following result which is essentially due to Morel and Voevodsky.

Proposition 1.1.10. *Let K be a complex of presheaves of R-modules on $\mathfrak{V}$. Then the following conditions are equivalent.*

(i) *The complex K has the B.-G.-property.*
(i′) *The complex K has the excision property.*
(ii) *For any X in $\mathfrak{V}$, the canonical map*

$$H^n(K(X)) \longrightarrow H^n_{\mathrm{Nis}}(X, K_{\mathrm{Nis}})$$

*is an isomorphism of R-modules (*i.e. *K is $\mathfrak{V}_{\mathrm{Nis}}$-local).*

Proof. The equivalence of (i) and (i′) follows from the definition of a homotopy pullback.

As any short exact sequence of sheaves of R-modules defines canonically a distinguished triangle in $D(\mathfrak{V}, R)$, the fact that (ii) implies (i) follows easily from proposition 1.1.8. To prove that (i) implies (ii), we need a little more machinery. First, we can choose a monomorphism of complexes $K \longrightarrow L$ which induces a quasi-isomorphism between K_{Nis} and L_{Nis}, and such that L is $\mathfrak{V}_{\mathrm{Nis}}$-local. For this, we first choose a quasi-isomorphism $K_{\mathrm{Nis}} \longrightarrow M$ where M is $\mathfrak{V}_{\mathrm{Nis}}$-local (which is possible by Corollary 1.1.7). We have a natural embbeding of K into the mapping cone of its identity $\mathrm{Cone}(1_K)$. But $\mathrm{Cone}(1_K)$ is obviously $\mathfrak{V}_{\mathrm{Nis}}$-local as it is already acyclic as a complex of presheaves. This implies that the direct sum $L = \mathrm{Cone}(1_K) \oplus M$ is also $\mathfrak{V}_{\mathrm{Nis}}$-local. Moreover, as K and L both satisfy the B.-G.-property, one can check easily that the quotient presheaf L/K also has the B.-G.-property. Hence it is sufficient to prove that $H^n(L(X)/K(X)) = 0$ for any X in $\mathfrak{V}$ and any integer n.

Let us fix an object X of $\mathfrak{V}$ and an integer n. One has to consider for any $q \geqslant 0$ the presheaf T_q on the small Nisnevich site X_{Nis} of X defined by

$$T_q(Y) = H^{n-q}(L(Y)/K(Y)) \ .$$

These are B.-G.-functors as defined by Morel and Voevodsky [MV99, proof of Prop. 1.16, page 101], and for any integer $q \geqslant 0$, the Nisnevich sheaf associated to T_q is trivial (this is because $K_{\mathrm{Nis}} \longrightarrow L$ is a quasi-isomorphism of complexes of Nisnevich sheaves by construction of L). This implies by virtue of [MV99, Lemma 1.17, page 101] that $T_q = 0$ for any $q \geqslant 0$. In particular, we have $H^n(L(X)/K(X)) = 0$. Therefore L/K is an acyclic complex of presheaves over $\mathfrak{V}$, and $K \longrightarrow L$ is a quasi-isomorphism of complexes of presheaves. This proves that K is $\mathfrak{V}_{\mathrm{Nis}}$-local if and only if L is, hence that K is $\mathfrak{V}_{\mathrm{Nis}}$-local. □

Corollary 1.1.11. *Let I be a small filtering category, and K a functor from I to the category of complexes of Nisnevich sheaves of R-modules. Then for any smooth S-scheme X, the canonical maps*

$$\varinjlim_{i\in I} H^n_{\mathrm{Nis}}(X, K_i) \longrightarrow H^n_{\mathrm{Nis}}(X, \varinjlim_{i\in I} K)$$

are isomorphisms for all n.

Proof. We can suppose that K_i is $\mathfrak{V}_{\mathrm{Nis}}$-local for all $i \in I$ (we can take a termwise fibrant replacement of K with respect to the model structrure of Proposition 1.1.5). It then follows from Proposition 1.1.10 that $\varinjlim_{i\in I} K_i$ is still $\mathfrak{V}_{\mathrm{Nis}}$-local: it follows from the fact that the filtering colimits are exact that the presheaves with the B.-G.-property are stable by filtering colimits. The map

$$\varinjlim_{i\in I} H^n(K_i(X)) \longrightarrow H^n(\varinjlim_{i\in I} K_i(X))$$

is obviously an isomorphism for any X in $\mathfrak{V}$. As we are free to take $\mathfrak{V} = Sm/S$, this proves the assertion. □

1.1.12. Remember that if $\mathcal{T}$ is a triangulated category with small sums, an object X of $\mathcal{T}$ is *compact* if for any small family $(K_\lambda)_{\lambda\in\Lambda}$ of objects of $\mathcal{T}$, the canonical map

$$\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{T}}\big(X, K_\lambda\big) \longrightarrow \mathrm{Hom}_{\mathcal{T}}\big(X, \bigoplus_{\lambda\in\Lambda} K_\lambda\big)$$

is bijective (as this map is always injective, this is equivalent to say it is surjective). One denotes by $\mathcal{T}_c$ the full subcategory of $\mathcal{T}$ that consists of compact objects. It is easy to see that $\mathcal{T}_c$ is a thick subcategory of $\mathcal{T}$ (which means that $\mathcal{T}_c$ is a triangulated subcategory of $\mathcal{T}$ stable by direct factors).

Corollary 1.1.13. *For any smooth S-scheme X, $R(X)$ is a compact object of the derived category of Nisnevich sheaves of R-modules.*

Proof. As any direct sum is a filtering colimit of finite direct sums, this follows from (1.1.3.4) and Corollary 1.1.11. □

1.1.14. Let D be a triangulated category. Remember that a *localizing subcategory* of D is a full subcategory T of D with the following properties.

(i) A is in T if and only if $A[1]$ is in T.

(ii) For any distinguished triangle

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow A'[1] ,$$

if A' and A'' are in T, then A is in T.

(iii) For any (small) family $(A_i)_{i\in I}$ of objects of T, $\bigoplus_{i\in I} A_i$ is in T.

If $\mathcal{T}$ is a class of objects of D, the *localizing subcategory of D generated by* $\mathcal{T}$ is the smallest localizing subcategory of D that contains $\mathcal{T}$ (*i.e.* the intersection of all the localizing subcategories of D that contain $\mathcal{T}$).

Let $\mathcal{T}$ be the class of complexes of shape

$$\dots \longrightarrow 0 \longrightarrow R(X \times_S \mathbf{A}^1_S) \longrightarrow R(X) \longrightarrow 0 \longrightarrow \dots$$

with X in $\mathfrak{V}$ (the non trivial differential is induced by the canonical projection). Denote by $\mathsf{T}(\mathfrak{V}, \mathbf{A}^1, R)$ the localizing subcategory of $D(\mathfrak{V}, R)$ generated by $\mathcal{T}$. We define the triangulated category $D^{\mathit{eff}}_{\mathbf{A}^1}(\mathfrak{V}, R)$ as the Verdier quotient of $D(\mathfrak{V}, R)$ by $\mathsf{T}(\mathfrak{V}, \mathbf{A}^1, R)$.

$$D^{\mathit{eff}}_{\mathbf{A}^1}(\mathfrak{V}, R) = D(\mathfrak{V}, R)/\mathsf{T}(\mathfrak{V}, \mathbf{A}^1, R)$$

We know that $D(\mathfrak{V}, R) \simeq D(Sm/S, R)$, and an easy Mayer-Vietoris argument for the Zariski topology shows that the essential image of $\mathsf{T}(\mathfrak{V}, \mathbf{A}^1, R)$ in $D(Sm/S, R)$ is precisely $\mathsf{T}(Sm/S, \mathbf{A}^1, R)$. Hence we get a canonical equivalence of categories

(1.1.14.1) $$D^{\mathit{eff}}_{\mathbf{A}^1}(\mathfrak{V}, R) \simeq D^{\mathit{eff}}_{\mathbf{A}^1}(Sm/S, R) .$$

We simply put:

$$D^{\mathit{eff}}_{\mathbf{A}^1}(S, R) = D^{\mathit{eff}}_{\mathbf{A}^1}(Sm/S, R) .$$

According to F. Morel insights, the category $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$ is called the *triangulated category of effective real motives*[4] (with coefficients in R). In the sequel of this paper, we will consider the equivalence (1.1.14.1) as an equality[5]. We thus have a canonical localization functor

(1.1.14.2) $$\gamma : D(\mathfrak{V}, R) \longrightarrow D^{\mathit{eff}}_{\mathbf{A}^1}(S, R) .$$

We will say that a morphism of complexes of $Sh(\mathfrak{V}, R)$ is an $\mathbf{A}^1$-*equivalence* if its image in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$ is an isomorphism.

A complex of presheaves of R-modules K over $\mathfrak{V}$ is $\mathbf{A}^1$-*homotopy invariant* if for any X in $\mathfrak{V}$, the projection of $X \times_S \mathbf{A}^1_S$ on X induces a quasi-isomorphism

$$K(X) \longrightarrow K(X \times_S \mathbf{A}^1_S) .$$

Proposition 1.1.15. *The category of complexes of $Sh(\mathfrak{V}, R)$ is endowed with a proper Quillen model category structure whose weak equivalences are the $\mathbf{A}^1$-equivalences, whose cofibrations are the $\mathfrak{V}$-cofibrations, and whose fibrations are the $\mathfrak{V}$-surjective morphisms with $\mathbf{A}^1$-homotopy invariant and $\mathfrak{V}_{\mathrm{Nis}}$-local kernel. In particular, the fibrant objects of this model structure are exactly the $\mathbf{A}^1$-homotopy invariant and $\mathfrak{V}_{\mathrm{Nis}}$-local complexes. The corresponding homotopy category is the triangulated category of effective real motives $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$.*

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[4]This terminology comes from the fact $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$ give quadratic informations on S, which implies it is bigger than Voevodsky's triangulated category of mixed motives; see [Mor04, Mor06]. The word 'real' is meant here as opposed to 'complex'.

[5]The role of the category $\mathfrak{V}$ is only to define model category structures on the category of complexes of $Sh(\mathfrak{V}, R) \simeq Sh(Sm/S, R)$ which depend only on the local behaviour of the schemes in $\mathfrak{V}$ (e.g. the smooth affine shemes over S).

Proof. This is a direct application of [CD09a, Proposition 3.5 and Corollary 4.10]. □

1.1.16. Say that a complex K of presheaves of R-modules on $\mathfrak{V}$ is $\mathbf{A}^1$*-local* if for any X in $\mathfrak{V}$, the projection of $X \times_S \mathbf{A}^1_S$ on X induces isomorphisms in Nisnevich hypercohomology

$$H^n_{\mathrm{Nis}}(X, K_{\mathrm{Nis}}) \simeq H^n_{\mathrm{Nis}}(X \times_S \mathbf{A}^1_S, K_{\mathrm{Nis}}) \ .$$

It is easy to see that a $\mathfrak{V}_{\mathrm{Nis}}$-local complex is $\mathbf{A}^1$-local if and only if it is $\mathbf{A}^1$-homotopy invariant. In general, a complex of sheaves K is $\mathbf{A}^1$-local if and only if, for any quasi-isomorphism $K \longrightarrow L$, if L is $\mathfrak{V}_{\mathrm{Nis}}$-local, then L is $\mathbf{A}^1$-homotopy invariant. We deduce from this the following result.

Corollary 1.1.17. *The localization functor $D(\mathfrak{V}, R) \longrightarrow D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$ has a right adjoint that is fully faithful and whose essential image consists of the $\mathbf{A}^1$-local complexes. In other words, $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$ is canonically equivalent to the full subcategory of $\mathbf{A}^1$-local complexes in $D(\mathfrak{V}, R)$.*

Proof. For any complex of sheaves K, one can produce functorially a map $K \longrightarrow L_{\mathbf{A}^1}K$ which is an $\mathbf{A}^1$-equivalence with $L_{\mathbf{A}^1}K$ a $\mathfrak{V}_{\mathrm{Nis}}$-local and $\mathbf{A}^1$-homotopy invariant complex (just consider a functorial fibrant resolution of the model category of 1.1.15). Then the functor $L_{\mathbf{A}^1}$ takes $\mathbf{A}^1$-equivalences to quasi-isomorphisms of complexes of (pre)sheaves, and induces a functor

$$L_{\mathbf{A}^1} : D^{\mathit{eff}}_{\mathbf{A}^1}(S, R) \longrightarrow D(\mathfrak{V}, R)$$

which is the expected right adjoint of the localization functor. □

Example 1.1.18. The constant sheaf R is $\mathbf{A}^1$-local (if it is considered as a complex concentrated in degree 0).

1.1.19. Let R' be another commutative ring, and $R \longrightarrow R'$ a morphism of rings. The functor $K \longmapsto K \otimes_R R'$ is a symmetric monoidal left Quillen functor from $\mathit{Comp}(\mathfrak{V}, R)$ to $\mathit{Comp}(\mathfrak{V}, R')$ for the model structures of Proposition 1.1.15. Hence it has a total left derived functor

$$D^{\mathit{eff}}_{\mathbf{A}^1}(S, R) \longrightarrow D^{\mathit{eff}}_{\mathbf{A}^1}(S, R') \qquad , \quad K \longmapsto K \otimes^{\mathbf{L}}_R R'$$

whose right adjoint is the obvious forgetful functor. I.e. for a complex of sheaves of R-modules K and a complex of sheaves of R'-modules L, we have a canonical isomorphism

$$\mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}(K, L) \simeq \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R')}(K \otimes^{\mathbf{L}}_R R', L) \ .$$

Example 1.1.20. Let $\mathbf{G}_m = \mathbf{A}^1_S - \{0\}$ be the multiplicative group. It can be considered as a presheaf of groups on Sm/S, and one can check that it is a Nisnevich sheaf. Moreover, for any smooth S-scheme X, one has

$$H^i_{\mathrm{Nis}}(X, \mathbf{G}_m) = \begin{cases} \mathcal{O}^*(X) & \text{if } i = 0, \\ \mathrm{Pic}(X) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.20.1}$$

As S is assumed to be regular, $\mathbf{G}_m$ is $\mathbf{A}^1$-local as a complex concentrated in degree 0. We deduce that we have the formula

$$H^i_{\mathrm{Nis}}(X, \mathbf{G}_m) = \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(k,\mathbf{Z})}(\mathbf{Z}(X), \mathbf{G}_m[i]) \ . \tag{1.1.20.2}$$

In particular, it follows from 1.1.19 that for any smooth S-scheme X, one has a canonical morphism of abelian groups

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}(R(X), \mathbf{G}_m \otimes^{\mathbf{L}}_{\mathbf{Z}} R[1]) \ . \tag{1.1.20.3}$$

If moreover R is flat over $\mathbf{Z}$, we get the formula

$$\mathrm{Pic}(X) \otimes_{\mathbf{Z}} R \simeq \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}(R(X), \mathbf{G}_m \otimes^{\mathbf{L}}_{\mathbf{Z}} R[1]) \ . \tag{1.1.20.4}$$

Proposition 1.1.21. *Let I be a small filtering category, and K a functor from I to the category of complexes of Nisnevich sheaves of R-modules. Then for any smooth S-scheme X, the canonical map*

$$\varinjlim_{i\in I} \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}(R(X), K_i) \longrightarrow \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(k,R)}(R(X), \varinjlim_{i\in I} K)$$

is an isomorphism.

Proof. This follows from Corollary 1.1.11 once we see that $\mathbf{A}^1$-homotopy invariant complexes are stable by filtering colimits. □

Corollary 1.1.22. *For any smooth S-scheme X, $R(X)$ is a compact object of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$.*

1.2. **Derived tensor product and derived** Hom.

1.2.1. We consider a full subcategory $\mathfrak{V}$ of Sm/S as in 1.1.1 and a commutative ring R. The category of sheaves of R-modules on $\mathfrak{V}$ has a tensor product $\otimes_R$ defined in the usual way: if F and G are two sheaves of R-modules, $F \otimes_R G$ is the Nisnevich sheaf associated to the presheaf

$$X \longmapsto F(X) \otimes_R G(X) \ .$$

the unit of this tensor product is $R = R(S)$. This makes the category $Sh(\mathfrak{V}, R)$ a closed symmetric monoidal category. For two objects X and Y of $\mathfrak{V}$, we have a canonical isomorphism

$$R(X \times_S Y) \simeq R(X) \otimes_R R(Y) \ .$$

Finally, an important property of this tensor product is that for any X in $\mathfrak{V}$, the sheaf $R(X)$ is flat, by which we mean that the functor

$$F \longmapsto R(X) \otimes_R F$$

is exact. This implies that the family of the sheaves $R(X)$ for X in $\mathfrak{V}$ is *flat* in the sense of [CD09a, 2.1]. Hence we can apply Corollary 2.6 of *loc. cit.* to get that the $\mathfrak{V}_{\mathrm{Nis}}$-local model structure of 1.1.5 is compatible with the tensor product in a very (rather technical but also) gentle way: define the tensor product of two complexes of sheaves of R-modules K and L on $\mathfrak{V}$ by the formula

$$(K \otimes_R L)^n = \bigoplus_{p+q=n} K^p \otimes_R L^q$$

with differential $d(x \otimes y) = dx \otimes y + (-1)^{\deg(x)} x \otimes dy$. This defines a structure of symmetric monoidal category on $Comp(Sh(\mathfrak{V}, R))$ (the unit is just R seen as complex concentrated in degree 0, and the symmetry rule is given by the usual formula $x \otimes y \longmapsto (-1)^{\deg(x)\deg(y)} y \otimes x$). A consequence of *loc. cit.* Corollary 2.6 is that the functor $(K, L) \longmapsto K \otimes_R L$ is a left Quillen bifunctor, which implies in particular that it has a well behaved total left derived functor

$$D(\mathfrak{V}, R) \times D(\mathfrak{V}, R) \longrightarrow D(\mathfrak{V}, R) \quad , \qquad (K, L) \longmapsto K \otimes^{\mathbf{L}}_R L \ .$$

Moreover, for a given complex L, the functor

$$D(\mathfrak{V},R) \longrightarrow D(\mathfrak{V},R) \quad , \qquad K \longmapsto K \otimes^{\mathbf{L}}_R L$$

is the total left derived functor of the functor $K \longmapsto K \otimes_R L$ (see Remark 2.9 of *loc. cit.*). This means that for any $\mathfrak{V}$-cofibrant complex K (that is, a complex such that the map $0 \longrightarrow K$ is a $\mathfrak{V}$-cofibration), the canonical map

$$K \otimes^{\mathbf{L}}_R L \longrightarrow K \otimes_R L$$

is an isomorphism in $D(\mathfrak{V},R)$ for any complex L. In particular, if F is a direct factor of some $R(X)$ (with X in $\mathfrak{V}$), then for any complex of sheaves L, the map

$$F \otimes^{\mathbf{L}}_R L \longrightarrow F \otimes_R L$$

is an isomorphism in $D(\mathfrak{V},R)$. This derived tensor product makes $D(\mathfrak{V},R)$ a closed symmetric monoidal triangulated category. This means that for two objects L and M of $D(\mathfrak{V},R)$, there is an object $\mathbf{RHom}(L,M)$ of $D(\mathfrak{V},R)$ that is defined by the universal property

$$\forall K \in D(\mathfrak{V},R) \ , \quad \mathrm{Hom}_{D(\mathfrak{V},R)}(K \otimes^{\mathbf{L}}_R L, M) \simeq \mathrm{Hom}_{D(\mathfrak{V},R)}(K, \mathbf{RHom}(L,M)).$$

The functor **RHom** can also be characterized as the total right derived functor of the internal Hom of the category of complexes of Nisnevich sheaves of R-moddules on $\mathfrak{V}$. If L is $\mathfrak{V}$-cofibrant and if M if $\mathfrak{V}_{\mathrm{Nis}}$-local, then $\mathbf{RHom}(L,M)$ can be represented by the complex of sheaves

$$X \longmapsto \mathrm{Tot}\big[\mathrm{Hom}_{Sh(\mathfrak{V},R)}(R(X) \otimes_R L, M)\big].$$

The derived tensor product on $D(\mathfrak{V},R)$ induces a derived tensor product on $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$ as follows.

Proposition 1.2.2. *The tensor product of complexes $\otimes_R$ has a total left derived functor*

$$D^{\mathit{eff}}_{\mathbf{A}^1}(S,R) \times D^{\mathit{eff}}_{\mathbf{A}^1}(S,R) \longrightarrow D^{\mathit{eff}}_{\mathbf{A}^1}(S,R) \quad , \qquad (K,L) \longmapsto K \otimes^{\mathbf{L}}_R L$$

that makes $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$ a closed symmetric tensor triangulated category. Moreover, the localization functor $D(\mathfrak{V},R) \longrightarrow D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$ is a triangulated symmetric monoidal functor.

Proof. This follows easily from [CD09a, Corollary 3.14] applied to the classes $\mathcal{G}$ and $\mathcal{H}$ defined in the proof of 1.1.5 and to the class $\mathcal{T}$ of complexes of shape

$$\cdots \longrightarrow 0 \longrightarrow R(X \times_S \mathbf{A}^1_S) \longrightarrow R(X) \longrightarrow 0 \longrightarrow \cdots$$

with X in $\mathfrak{V}$. □

1.2.3. It follows from Proposition 1.2.2 that the category $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$ has an internal Hom that we still denote by **RHom**. Hence for three objects K, L and M in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$, we have a canonical isomorphism

$$\mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}(K \otimes^{\mathbf{L}}_R L, M) \simeq \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}(K, \mathbf{RHom}(L,M)). \tag{1.2.3.1}$$

If L is $\mathfrak{V}$-cofibrant and M is $\mathfrak{V}_{\mathrm{Nis}}$-local and $\mathbf{A}^1$-local, then

$$\mathbf{RHom}(L,M) = \mathrm{Tot}\big[\mathrm{Hom}_{Sh(\mathfrak{V},R)}(R(-) \otimes_R L, M)\big]. \tag{1.2.3.2}$$

Proposition 1.2.4. *If L is a compact object of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$, then for any small family $(K_\lambda)_{\lambda\in\Lambda}$ of objects of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$, the canonical map*

$$\bigoplus_{\lambda\in\Lambda}\mathbf{RHom}\big(L,K_\lambda\big)\longrightarrow\mathbf{RHom}\big(L,\bigoplus_{\lambda\in\Lambda}K_\lambda\big)$$

is an isomorphism in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$.

Proof. Once the family $(K_\lambda)_{\lambda\in\Lambda}$ is fixed, this map defines a morphism of triangulated functors from the triangulated category of compact objects of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$ to $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$. Therefore, it is sufficient to check this property when $L=R(Y)$ with Y in $\mathfrak{V}$. This is equivalent to say that for any X in $\mathfrak{V}$, the map

$$\operatorname{Hom}\big(R(X),\bigoplus_{\lambda\in\Lambda}\mathbf{RHom}\big(R(Y),K_\lambda\big)\big)\longrightarrow\operatorname{Hom}\big(R(X),\mathbf{RHom}\big(R(Y),\bigoplus_{\lambda\in\Lambda}K_\lambda\big)\big)$$

is bijective. As $R(X)$ is compact (1.1.22), we have

$$\operatorname{Hom}\big(R(X),\bigoplus_{\lambda\in\Lambda}\mathbf{RHom}\big(R(Y),K_\lambda\big)\big)\simeq\bigoplus_{\lambda\in\Lambda}\operatorname{Hom}\big(R(X),\mathbf{RHom}\big(R(Y),K_\lambda\big)\big),$$

and as $R(X\times_S Y)\simeq R(X)\otimes^{\mathbf{L}}_R R(Y)$ is compact as well, we have

$$\begin{aligned}\operatorname{Hom}\big(R(X),\mathbf{RHom}\big(R(Y),\bigoplus_{\lambda\in\Lambda}K_\lambda\big)\big)&\simeq\operatorname{Hom}\big(R(X)\otimes^{\mathbf{L}}_R R(Y),\bigoplus_{\lambda\in\Lambda}K_\lambda\big)\\&\simeq\bigoplus_{\lambda\in\Lambda}\operatorname{Hom}\big(R(X)\otimes^{\mathbf{L}}_R R(Y),K_\lambda\big)\\&\simeq\bigoplus_{\lambda\in\Lambda}\operatorname{Hom}\big(R(X),\mathbf{RHom}(R(Y),K_\lambda)\big).\end{aligned}$$

This implies our claim immediately. □

1.2.5. Let R-Mod be the category of R-modules. If M is an R-module, we still denote by M the constant Nisnevich sheaf of R-modules on $\mathfrak{V}$ generated by M. This defines a symmetric monoidal functor from the category of (unbounded) complexes of R-modules $\mathit{Comp}(R)$ to the category $\mathit{Comp}(\mathfrak{V},R)$

$$\mathit{Comp}(R)\longrightarrow\mathit{Comp}(\mathfrak{V},R)\quad,\qquad M\longmapsto M. \tag{1.2.5.1}$$

This functor is a left adjoint to the global sections functor

$$\Gamma:\mathit{Comp}(\mathfrak{V},R)\longrightarrow\mathit{Comp}(R)\quad,\qquad M\longmapsto\Gamma(M)=\Gamma(S,M). \tag{1.2.5.2}$$

The category $\mathit{Comp}(R)$ is a Quillen model category with the quasi-isomorphims as weak equivalences and the degreewise surjective maps as fibrations (see *e.g.* [Hov99, Theorem 2.3.11]). We call this model structure the *projective model structure.* This implies that the constant sheaf functor (1.2.5.1) is a left Quillen functor for the model structures of Propositions 1.1.5 and 1.1.15 on $\mathit{Comp}(\mathfrak{V},R)$. Therefore, the global sections functor (1.2.5.2) is a right Quillen functor and has total right derived functor

$$\mathbf{R}\Gamma:D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)\longrightarrow D(R) \tag{1.2.5.3}$$

where $D(R)$ denotes the derived category of R. For two objects M and N of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$, we define

$$\mathbf{R}\operatorname{Hom}(M,N)=\mathbf{R}\Gamma\big(\mathbf{RHom}(M,N)\big). \tag{1.2.5.4}$$

We invite the reader to check that $\mathbf{R}\mathrm{Hom}$ is the derived Hom of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$. In particular, for any integer n, we have a canonical isomorphism

$$H^n\big(\mathbf{R}\mathrm{Hom}(M,N)\big) \simeq \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}(M,N[n]). \tag{1.2.5.5}$$

1.3. **Tate object and purity.**

1.3.1. Let $\mathbf{G}_m = \mathbf{A}^1_S - \{0\}$ be the multiplicative group scheme over S. The unit of $\mathbf{G}_m$ defines a morphism $R = R(S) \longrightarrow R(\mathbf{G}_m)$, and we define the *Tate object* $R(1)$ as the cokernel

$$R(1) = \mathrm{coker}\big(R \longrightarrow R(\mathbf{G}_m)\big)[-1]$$

(this definition makes sense in the category of complexes of $\mathit{Sh}(\mathfrak{V},R)$ as well as in $D(\mathfrak{V},R)$ or in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$ as we take the cokernel of a split monomorphism). By definition, $R(1)[1]$ is a direct factor of $R(\mathbf{G}_m)$, so that $R(1)$ is $\mathfrak{V}$-cofibrant. Hence for any integer $n \geqslant 0$, $R(n) = R(1)^{\otimes n}$ is also $\mathfrak{V}$-cofibrant. For a complex K of $\mathit{Sh}(\mathfrak{V},R)$, we define $K(n) = K \otimes_R R(n)$. As $R(n)$ is $\mathfrak{V}$-cofibrant, the map

$$K \otimes^{\mathbf{L}}_R R(n) \longrightarrow K \otimes_R R(n) = K(n)$$

is an isomorphism in $D^{\mathit{eff}}_{\mathbf{A}^1}(k,R)$. Another description of $R(1)$ is the following.

Proposition 1.3.2. *The inclusion of $\mathbf{G}_m$ in $\mathbf{A}^1$ induces a canonical split distinguished triangle in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$*

$$R(\mathbf{G}_m) \longrightarrow R(\mathbf{A}^1_S) \xrightarrow{0} R(1)[2] \longrightarrow R(\mathbf{G}_m)[1]$$

that gives the canonical decomposition $R(\mathbf{G}_m) = R \oplus R(1)[1]$.

Proof. This follows formally from the definition of $R(1)$ and from the fact that $R(\mathbf{A}^1) = R$ in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$. $\square$

1.3.3. Let $\Delta^{op}\mathit{Sh}(\mathit{Sm}/S)$ be the category of Nisnevich sheaves of simplicial sets on Sm/S. Morel and Voevodsky defined in [MV99] the $\mathbf{A}^1$-homotopy theory in $\Delta^{op}\mathit{Sh}(\mathit{Sm}/S)$. In particular, we have a notion of $\mathbf{A}^1$-weak equivalences of simplicial sheaves that defines a proper model category structure (with the monomorphisms as cofibrations). Furthermore, we have a canonical functor

$$\Delta^{op}\mathit{Sh}(\mathit{Sm}/S) \longrightarrow \mathit{Comp}(\mathit{Sm}/S,R) \quad , \qquad \mathcal{X} \longmapsto R(\mathcal{X})$$

which has the following properties; see e.g. [Mor03, Mor05].

(1) The functor R above preserves colimits.
(2) The functor R preserves monomorphisms.
(3) The functor R sends $\mathbf{A}^1$-weak equivalences to $\mathbf{A}^1$-equivalences.

We deduce from these properties that the functor R sends homotopy pushout squares of $\Delta^{op}\mathit{Sh}(\mathit{Sm}/S)$ to homotopy pushout squares of $\mathit{Comp}(\mathit{Sm}/S,R)$ and induces a functor

$$R : \mathcal{H}(S) \longrightarrow D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$$

where $\mathcal{H}(S)$ denotes the localization of $\Delta^{op}\mathit{Sh}(\mathit{Sm}/S)$ by the $\mathbf{A}^1$-weak equivalences.

This implies that all the results of [MV99] that are formulated in terms of $\mathbf{A}^1$-weak equivalences (or isomorphisms in $\mathcal{H}(S)$) and in terms of homotopy pushout have their counterpart in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$. We give below the results we will need that come from this principle.

1.3.4. Let X be a smooth S-scheme and $\mathcal{V}$ a vector bundle over X. Consider the open immersion $j : \mathcal{V}^\times \to \mathcal{V}$ of the complement of the zero section of $\mathcal{V}/X$. We define the *Thom space of* $\mathcal{V}$ as the quotient

$$R(\mathit{Th}\,\mathcal{V}) = \operatorname{coker}\big(R(\mathcal{V}^\times) \xrightarrow{j_*} R(\mathcal{V})\big) \ . \tag{1.3.4.1}$$

We thus have a short exact sequence of sheaves of R-modules

$$0 \longrightarrow R(\mathcal{V}^\times) \longrightarrow R(\mathcal{V}) \longrightarrow R(\mathit{Th}\,\mathcal{V}) \longrightarrow 0 \ . \tag{1.3.4.2}$$

Proposition 1.3.5. *Let $\mathcal{O}^n$ be the trivial vector bundle of dimension n on a smooth S-scheme X. Then we have a canonical isomorphism in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$:*

$$R(\mathit{Th}\,\mathcal{O}^n) \simeq R(X)(n)[2n] \ .$$

Proof. This follows from Proposition 1.3.2 and from the second statement of [MV99, Proposition 2.17, page 112]. □

For a given vector bundle $\mathcal{V}$, over a S-scheme X, we will denote by $\mathbf{P}(\mathcal{V}) \longrightarrow X$ the corresponding projective bundle.

Proposition 1.3.6. *Let $\mathcal{V}$ be a vector bundle on a smooth S-scheme X. Then we have a canonical distinguished triangle in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$*

$$R(\mathbf{P}(\mathcal{V})) \longrightarrow R(\mathbf{P}(\mathcal{V}\oplus\mathcal{O})) \longrightarrow R(\mathit{Th}\,\mathcal{V}) \longrightarrow R(\mathbf{P}(\mathcal{V}))[1] \ .$$

Proof. This follows from Proposition 1.3.2 and from the third statement of [MV99, Proposition 2.17, page 112]. □

Corollary 1.3.7. *We have a canonical distinguished triangle in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$*

$$R(\mathbf{P}^n_S) \longrightarrow R(\mathbf{P}^{n+1}_S) \longrightarrow R(n+1)[2n+2] \longrightarrow R(\mathbf{P}^n_S)[1] \ .$$

Moreover, this triangle splits canonically for $n=0$ and gives the decomposition

$$R(\mathbf{P}^1_S) = R \oplus R(1)[2] \ .$$

Proof. This is a direct consequence of Propositions 1.3.5 and 1.3.6. The splitting of the case $n=0$ comes obviously from the canonical map from $\mathbf{P}^1_S$ to S. □

1.3.8. The inclusions $\mathbf{P}^n_S \subset \mathbf{P}^{n+1}_S$ allow us to define the Nisnevich sheaf of sets

$$\mathbf{P}^\infty_S = \varinjlim_{n\geqslant 0} \mathbf{P}^n_S \tag{1.3.8.1}$$

We get a Nisnevich sheaf of R-modules

$$R(\mathbf{P}^\infty_S) = \varinjlim_{n\geqslant 0} R(\mathbf{P}^n_S) \ . \tag{1.3.8.2}$$

For a complex K of sheaves of R-modules, we define the hypercohomology of $\mathbf{P}^\infty_S$ with coefficients in K to be

$$H^i_{\mathrm{Nis}}(\mathbf{P}^\infty_S, K) = \operatorname{Hom}_{D(Sm/S,R)}(R(\mathbf{P}^\infty_S), K[i]) \ . \tag{1.3.8.3}$$

Proposition 1.3.9. *There is a short exact sequence*

$$0 \longrightarrow \varprojlim^1_{n\geqslant 0} H^{i-1}_{\mathrm{Nis}}(\mathbf{P}^n_S, K) \longrightarrow H^i_{\mathrm{Nis}}(\mathbf{P}^\infty_S, K) \longrightarrow \varprojlim_{n\geqslant 0} H^i_{\mathrm{Nis}}(\mathbf{P}^n_S, K) \longrightarrow 0 \ .$$

Proof. As the filtering colimits are exact in $\mathit{Sh}(Sm/S,R)$ we have an isomorphism $\underrightarrow{\operatorname{holim}}\, R(\mathbf{P}^n_S) \simeq R(\mathbf{P}^\infty_S)$ in $D(Sm/S,R)$. This result is thus a direct application of the Milnor short exact sequence applied to this homotopy colimit (see e.g. [Hov99, Proposition 7.3.2]). □

Proposition 1.3.10 (Purity Theorem)**.** *Let $i : Z \longrightarrow X$ a closed immersion of smooth S-schemes, and $U = X - i(Z)$. Denote by $\mathcal{N}_{X,Z}$ the normal vector bundle of i. Then there is a canonical distinguished triangle in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$*

$$R(U) \longrightarrow R(X) \longrightarrow R(\mathit{Th}\,\mathcal{N}_{X,Z}) \longrightarrow R(U)[1] \ .$$

Proof. This follows from [MV99, Theorem 2.23, page 115]. □

Corollary 1.3.11. *There is a canonical decomposition $R(\mathbf{A}^n_S - \{0\}) = R \oplus R(n)[2n-1]$ in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$.*

Proof. The Purity Theorem and Proposition 1.3.5 give a distinguished triangle

$$R(\mathbf{A}^n_S - \{0\}) \longrightarrow R(\mathbf{A}^n_S) \longrightarrow R(n)[2n] \longrightarrow R(\mathbf{A}^n_S - \{0\})[1] \ .$$

But this triangle is isomorphic to the distinguished triangle

$$R(\mathbf{A}^n_S - \{0\}) \longrightarrow R \longrightarrow Q[1] \longrightarrow R(\mathbf{A}^n_S - \{0\})[1]$$

where Q is the kernel of the obvious map $R(\mathbf{A}^n_S - \{0\}) \longrightarrow R$, which shows that these triangle split. □

1.4. **Tate spectra.**

1.4.1. We want the derived tensor product by $R(1)$ to be an equivalence of categories. As this is not the case in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$, we will modify the category $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$ and construct the triangulated category of real motives $D_{\mathbf{A}^1}(S, R)$ in which this will occur by definition. For this purpose, we will define the model category of symmetric Tate spectra. We will give only the minimal definitions we will need to work with. We invite the interested reader to have look at [CD09a, Section 6] for a more complete account. The main properties of $D_{\mathbf{A}^1}(S, R)$ are listed in 1.4.4.

We consider given a category of smooth S-schemes $\mathfrak{V}$ as in 1.1.1.

1.4.2. A *symmetric Tate spectrum* (in $\mathit{Sh}(\mathfrak{V}, R)$) is a collection $E = (E_n, \sigma_n)_{n \geqslant 0}$, where for each integer $n \geqslant 0$, E_n is a complex of Nisnevich sheaves on $\mathfrak{V}$ endowed with an action of the symmetric group $\mathfrak{S}_n$, and $\sigma_n : R(1) \otimes_R E_n \longrightarrow E_{n+1}$ is a morphism of complexes, such that the induced maps obtained by composition

$$R(1)^{\otimes m} \otimes_R E_n \longrightarrow R(1)^{\otimes m-1} \otimes_R E_{n+1} \longrightarrow \cdots \longrightarrow R(1) \otimes_R E_{m+n-1} \longrightarrow E_{m+n}$$

are $\mathfrak{S}_m \times \mathfrak{S}_n$-equivariant. We have to define the actions to be precise: $\mathfrak{S}_m$ acts on $R(m) = R(1)^{\otimes m}$ by permutation, and the action on E_{m+n} is induced by the diagonal inclusion $\mathfrak{S}_m \times \mathfrak{S}_n \subset \mathfrak{S}_{m+n}$. A morphism of symmetric Tate spectra $u : (E_n, \sigma_n) \longrightarrow (F_n, \tau_n)$ is a collection of $\mathfrak{S}_n$-equivariant maps $u_n : E_n \longrightarrow F_n$ such that the squares

$$\begin{tikzcd}
R(1) \otimes_R E_n \arrow[r, "\sigma_n"] \arrow[d, "R(1)\otimes u_n"'] & E_{n+1} \arrow[d, "u_{n+1}"] \\
R(1) \otimes_R F_n \arrow[r, "\tau_n"'] & F_{n+1}
\end{tikzcd}$$

commute. We denote by $\mathrm{Sp}_{\mathit{Tate}}(\mathfrak{V}, R)$ the category of symmetric Tate spectra. If A is a complex of sheaves of R-modules on $\mathfrak{V}$, we define its *infinite suspension* $\Sigma^\infty(A)$ as the symmetric Tate spectrum that consists of the collection $(A(n), 1_{A(n+1)})_{n \geqslant 0}$

where $\mathfrak{S}_n$ acts on $A(n) = R(1)^{\otimes n} \otimes_R A$ by permutation on $R(n) = R(1)^{\otimes n}$. This defines the infinite suspension functor

$$\Sigma^\infty : \mathit{Comp}(\mathfrak{V}, R) \longrightarrow \mathrm{Sp}_{\mathit{Tate}}(\mathfrak{V}, R) \tag{1.4.2.1}$$

This functor has a right adjoint

$$\Omega^\infty : \mathrm{Sp}_{\mathit{Tate}}(\mathfrak{V}, R) \longrightarrow \mathit{Comp}(\mathfrak{V}, R) \tag{1.4.2.2}$$

defined by $\Omega^\infty(E_n, \sigma_n)_{n \geqslant 0} = E_0$. According to [CD09a, 6.14 and 6.20] we can define a (R-linear) tensor product of symmetric spectra $E \otimes_R F$ satisfying the following properties (and these properties determine this tensor product up to a canonical isomorphism).

(1) This tensor product makes the category of symmetric Tate spectra a closed symmetric monoidal category with $\Sigma^\infty(R(S))$ as unit.
(2) The infinite suspension functor (1.4.2.1) is a symmetric monoidal functor.

Say that a map of symmetric Tate spectra $u : (E_n, \sigma_n) \longrightarrow (F_n, \tau_n)$ is a *quasi-isomorphism* if the map $u_n : E_n \longrightarrow F_n$ is a quasi-isomorphism of complexes of Nisnevich sheaves of R-modules for any $n \geqslant 0$. We define the *Tate derived category of* $\mathit{Sh}(\mathfrak{V}, R)$ as the localization of $\mathrm{Sp}_{\mathit{Tate}}(\mathfrak{V}, R)$ by the class of quasi-isomorphisms. We will write $D_{\mathit{Tate}}(\mathfrak{V}, R)$ for this "derived category". One can check that $D_{\mathit{Tate}}(\mathfrak{V}, R)$ is a triangulated category (according to [CD09a, Remark 6.19], this is the homotopy category of a stable model category) and that the functor induced by Σ^∞ is a triangulated functor (because this is a left Quillen functor between stable model categories).

A symmetric Tate spectrum $E = (E_n, \sigma_n)_{n \geqslant 0}$ is a *weak* Ω^∞*-spectrum* if for any integer $n \geqslant 0$, the map σ_n induces an isomorphism $E_n \simeq \mathbf{RHom}(R(1), E_{n+1})$ in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, R)$. A symmetric Tate spectrum $E = (E_n, \sigma_n)_{n \geqslant 0}$ is a Ω^∞*-spectrum* if it is a weak Ω^∞-spectrum and if, for any integer $n \geqslant 0$, the complex E_n is $\mathfrak{V}_{\mathrm{Nis}}$-local and $\mathbf{A}^1$-homotopy invariant.

A morphism of symmetric Tate spectra $u : A \longrightarrow B$ is a *stable* $\mathbf{A}^1$*-equivalence* if for any weak Ω^∞-spectrum E, the map

$$u^* : \mathrm{Hom}_{D_{\mathit{Tate}}(\mathfrak{V}, R)}(B, E) \longrightarrow \mathrm{Hom}_{D_{\mathit{Tate}}(\mathfrak{V}, R)}(A, E)$$

is an isomorphism of R-modules.

A morphism of Tate spectra is a *stable* $\mathbf{A}^1$*-fibration* if it is termwise $\mathfrak{V}_{\mathrm{Nis}}$-surjective and if its kernel is a Ω^∞-spectrum.

A morphism of Tate spectra is a *stable* $\mathfrak{V}$*-cofibration* if it has the left lifting property with respect to the stable $\mathbf{A}^1$-fibrations which are also stable $\mathbf{A}^1$-equivalences. A symmetric Tate spectrum E is *stably* $\mathfrak{V}$*-cofibrant* if the map $0 \longrightarrow E$ is a stable $\mathfrak{V}$-cofibration.

Proposition 1.4.3. *The category of symmetric Tate spectra is a stable proper symmetric monoidal model category with the stable* $\mathbf{A}^1$*-equivalences as the weak equivalences, the stable* $\mathbf{A}^1$*-fibrations as fibrations and the stable* $\mathfrak{V}$*-cofibrations as cofibrations. The infinite suspension functor is a symmetric left Quillen functor that sends the* $\mathbf{A}^1$*-equivalences to the stable* $\mathbf{A}^1$*-equivalences. Moreover, the tensor product by any stably* $\mathfrak{V}$*-cofibrant symmetric Tate spectrum preserves the stable* $\mathbf{A}^1$*-equivalences.*

Proof. The first assertion is an application of [CD09a, Proposition 6.15]. The fact that the functor Σ^∞ preserves weak equivalences comes from *loc. cit.*, Proposition 6.18. The last assertion follows from *loc. cit.*, Proposition 6.35. □

1.4.4. The proposition above means the following.

Define the *triangulated category of real mixed motives* $D_{\mathbf{A}^1}(S,R)$ as the localization of the category $\mathrm{Sp}_{Tate}(\mathfrak{V},R)$ by the class of stable $\mathbf{A}^1$-equivalences. Then $D_{\mathbf{A}^1}(S,R)$ is a triangulated category with infinite direct sums and products. To be more precise, any short exact sequence in $\mathrm{Sp}_{Tate}(\mathfrak{V},R)$ gives rise canonically to an exact triangle in $D_{\mathbf{A}^1}(S,R)$, and any distinguished triangle is isomorphic to an exact triangle that comes from a short exact sequence. Furthermore, this triangulated category does not depend on the category $\mathfrak{V}$: the category $\mathfrak{V}$ is only a technical tool to define a model category structure that is well behaved with the tensor product and Nisnevich descent in $\mathfrak{V}$.

The infinite suspension functor sends $\mathbf{A}^1$-equivalences to stable $\mathbf{A}^1$-equivalences and thus induces a functor

$$\Sigma^\infty : D^{\mathit{eff}}_{\mathbf{A}^1}(S,R) \longrightarrow D_{\mathbf{A}^1}(S,R)\ . \tag{1.4.4.1}$$

The right adjoint of the infinite suspension functor has a total right derived functor

$$\mathbf{R}\Omega^\infty : D_{\mathbf{A}^1}(S,R) \longrightarrow D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)\ . \tag{1.4.4.2}$$

For a (weak) Ω^∞-spectrum E, one has

$$\mathbf{R}\Omega^\infty(E) = E_0\ . \tag{1.4.4.3}$$

The tensor product on $\mathrm{Sp}_{Tate}(\mathfrak{V},R)$ has a total left derived functor

$$D_{\mathbf{A}^1}(S,R) \times D_{\mathbf{A}^1}(S,R) \longrightarrow D_{\mathbf{A}^1}(S,R) \quad , \qquad (E,F) \longmapsto E \otimes^{\mathbf{L}}_R F\ . \tag{1.4.4.4}$$

If E is stably $\mathfrak{V}$-cofibrant, then the canonical map $E \otimes^{\mathbf{L}}_R F \longrightarrow E \otimes_R F$ is an isomorphism in $D_{\mathbf{A}^1}(S,R)$. Moreover, the functor (1.4.4.1) is symmetric monoidal. In particular, for two complexes of Nisnevich sheaves of R-modules A and B we have a canonical isomorphism

$$\Sigma^\infty(A \otimes^{\mathbf{L}}_R B) \simeq \Sigma^\infty(A) \otimes^{\mathbf{L}}_R \Sigma^\infty(B)\ . \tag{1.4.4.5}$$

The category $D_{\mathbf{A}^1}(S,R)$ has also an internal Hom that we denote by $\mathbf{RHom}(E,F)$.

We will write $\underline{R} = \Sigma^\infty(R(S))$, and for a smooth S-scheme X, we define $\underline{R}(X)$ to be $\Sigma^\infty(R(X))$. We also define $\underline{R}(n) = \Sigma^\infty(R(n))$ for $n \geqslant 0$. Note that $\underline{R}$ is the unit of the (derived) tensor product. We define the symmetric Tate spectrum $\underline{R}(-1)$ by the formula $\underline{R}(-1)_n = R(n+1)$ with the action of $\mathfrak{S}_n$ defined as the action by permutations on the first n factors of $R(n+1) = R(1)^{\otimes n} \otimes_R R(1)$. The maps $\underline{R}(-1)_n \otimes R(1) \longrightarrow \underline{R}(-1)_{n+1}$ are just the identities. One can check that $\underline{R}(-1)$ is $\mathfrak{V}$-cofibrant.

Proposition 1.4.5. *The object $\underline{R}(1)$ is invertible in $D_{\mathbf{A}^1}(S,R)$ and we have an isomorphism $\underline{R}(-1) \simeq \underline{R}(1)^{-1}$. In other words, there are isomorphisms*

$$\underline{R}(1) \otimes^{\mathbf{L}}_R \underline{R}(-1) \simeq \underline{R} \quad \textit{and} \quad \underline{R}(-1) \otimes^{\mathbf{L}}_R \underline{R}(1) \simeq \underline{R}\ .$$

Proof. This follows from [CD09a, Proposition 6.24]. □

1.4.6. For an integer $n \geqslant 0$, we define $\underline{R}(-n) = \underline{R}(-1)^{\otimes n}$. For an integer n, and a symmetric Tate spectrum E, we define

$$E(n) = E \otimes \underline{R}(n)\ .$$

As $\underline{R}(n)$ is $\mathfrak{V}$-cofibrant, the canonical maps $E\otimes^{\mathbf{L}}_R \underline{R}(n) \longrightarrow E\otimes_R \underline{R}(n) = E(n)$ are isomorphisms in $D_{\mathbf{A}^1}(S,R)$.

We will say that a symmetric Tate spectrum $E=(E_n,\sigma_n)_{n\geqslant 0}$ is a *weak Ω^∞-spectrum* if for any integer $n\geqslant 0$, the map σ_n induces by adjunction an isomorphism

$$E_n \simeq \mathbf{RHom}(R(1),E_{n+1})$$

in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$.

Proposition 1.4.7. *Let E be a weak Ω^∞-spectrum. Then for any integer $n\geqslant 0$ and any complex of Nisnevich sheaves of R-modules A, there is a canonical isomorphism of R-modules*

$$\operatorname{Hom}_{D_{\mathbf{A}^1}(S,R)}(\Sigma^\infty(A),E(n)) \simeq \operatorname{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}(A,E_n)\ .$$

In particular, for any smooth S-scheme X, one has isomorphisms

$$H^i_{\mathrm{Nis}}(X,E_n) \simeq \operatorname{Hom}_{D_{\mathbf{A}^1}(S,R)}(\underline{R}(X),E(n)[i])\ .$$

Proof. This is an application of [CD09a, Proposition 6.28]. □

Corollary 1.4.8. *A morphism of weak Ω^∞-spectra $E\longrightarrow F$ is a stable $\mathbf{A}^1$-equivalence if and only if the map $E_n\longrightarrow F_n$ is a $\mathbf{A}^1$-equivalence for all $n\geqslant 0$.*

Proposition 1.4.9. *For any smooth S-scheme X and any integer n, $\underline{R}(X)(n)$ is a compact object of $D_{\mathbf{A}^1}(S,R)$.*

Proof. Let $(E_\lambda)_{\lambda\in\Lambda}$ be a small family of Tate spectra. We want to show that the map

$$\bigoplus_{\lambda\in\Lambda}\operatorname{Hom}_{D_{\mathbf{A}^1}(S,R)}\big(\underline{R}(X)(n),E_\lambda\big) \longrightarrow \operatorname{Hom}_{D_{\mathbf{A}^1}(S,R)}\big(\underline{R}(X)(n),\bigoplus_{\lambda\in\Lambda}E_\lambda\big)$$

is bijective. Replacing the spectra E_λ by the spectra $E_\lambda(-n)$, we can suppose that $n=0$. Furthermore, we can assume that the spectra E_λ are weak Ω^∞-spectra. As $R(1)$ is a compact object of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)$ (this is by definition a direct factor of the object $R(\mathbf{G}_m)$ which is compact by 1.1.22), it follows from Proposition 1.2.4 that $\bigoplus_{\lambda\in\Lambda}E_\lambda$ is a weak Ω^∞-spectrum as well. Therefore, by Proposition 1.4.7, we have

$$\begin{aligned}\bigoplus_{\lambda\in\Lambda}\operatorname{Hom}_{D_{\mathbf{A}^1}(S,R)}\big(\underline{R}(X),E_\lambda\big) &\simeq \bigoplus_{\lambda\in\Lambda}\operatorname{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}\big(R(X),E_{0,\lambda}\big)\\ &\simeq \operatorname{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}\big(R(X),\bigoplus_{\lambda\in\Lambda}E_{0,\lambda}\big)\\ &\simeq \operatorname{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}\big(R(X),\big(\bigoplus_{\lambda\in\Lambda}E_\lambda\big)_0\big)\\ &\simeq \operatorname{Hom}_{D_{\mathbf{A}^1}(S,R)}\big(\underline{R}(X),\bigoplus_{\lambda\in\Lambda}E_\lambda\big).\end{aligned}$$

This proves the result. □

1.4.10. The functor

$$\mathit{Comp}(R) \longrightarrow \mathrm{Sp}_{\mathit{Tate}}(\mathfrak{V},R)\quad,\qquad M\longmapsto \Sigma^\infty(M). \tag{1.4.10.1}$$

is a left Quillen functor from the projective model structure on $\mathit{Comp}(R)$ (see 1.2.5) to the model structure of Proposition 1.4.3. The right adjoint of (1.4.10.1)

$$\mathrm{Sp}_{\mathit{Tate}}(\mathfrak{V},R) \longrightarrow \mathit{Comp}(R)\quad,\qquad M\longmapsto \Gamma(\Omega^\infty(M)) \tag{1.4.10.2}$$

is a right Quillen functor. The corresponding total right derived functor is canonically isomorphic to the composed functor $\mathbf{R}\Gamma \circ \mathbf{R}\Omega^\infty$.

For two objects E and F of $D_{\mathbf{A}^1}(S,R)$, we define

$$\mathbf{R}\mathrm{Hom}(E,F) = \mathbf{R}\Gamma\Big(\mathbf{R}\Omega^\infty\big(\mathbf{RHom}(E,F)\big)\Big). \tag{1.4.10.3}$$

For any integer n, we have a canonical isomorphism

$$H^n\big(\mathbf{R}\mathrm{Hom}(E,F)\big) \simeq \mathrm{Hom}_{D_{\mathbf{A}^1}(S,R)}(E,F[n]). \tag{1.4.10.4}$$

1.5. **Ring spectra.**

1.5.1. A *ring spectrum* is a monoid object in the category of symmetric Tate spectra. A ring spectrum is *commutative* if it is commutative as a monoid object of $\mathrm{Sp}_{Tate}(\mathfrak{V},R)$.

Given a ring spectrum $\mathcal{E}$, one can form the category of left $\mathcal{E}$-modules. These are the symmetric Tate spectra M endowed with a left action of $\mathcal{E}$

$$\mathcal{E} \otimes_R M \longrightarrow M$$

satisfying the usual associativity and unit properties.

We denote by $\mathrm{Sp}_{Tate}(\mathfrak{V},\mathcal{E})$ the category of left $\mathcal{E}$-modules. There is a *base change functor*

$$\mathrm{Sp}_{Tate}(\mathfrak{V},R) \longrightarrow \mathrm{Sp}_{Tate}(\mathfrak{V},\mathcal{E}) \quad , \qquad F \longmapsto \mathcal{E} \otimes_R F \tag{1.5.1.1}$$

which is a left adjoint of the forgetful functor

$$\mathrm{Sp}_{Tate}(\mathfrak{V},\mathcal{E}) \longrightarrow \mathrm{Sp}_{Tate}(\mathfrak{V},R) \quad , \qquad M \longmapsto M\ . \tag{1.5.1.2}$$

If $\mathcal{E}$ is commutative, the category $\mathrm{Sp}_{Tate}(\mathfrak{V},\mathcal{E})$ is canonically endowed with a closed symmetric monoidal category structure such that the functor (1.5.1.1) is a symmetric monoidal functor. We denote by $\otimes_{\mathcal{E}}$ the corresponding tensor product. The unit of this monoidal structure is $\mathcal{E}$ seen as an $\mathcal{E}$-module.

A morphism of $\mathcal{E}$-modules is a *stable* $\mathbf{A}^1$*-equivalence* (resp. a *stable* $\mathbf{A}^1$*-fibration*) if it is so as a morphism of symmetric Tate spectra. A morphism of $\mathcal{E}$-modules is a stable $\mathfrak{V}$-cofibration if it is has the left lifting property with respect to the stable $\mathbf{A}^1$-fibrations which are also stable $\mathbf{A}^1$-equivalences.

Proposition 1.5.2. *For a given ring spectrum $\mathcal{E}$, the category $\mathrm{Sp}_{Tate}(\mathfrak{V},\mathcal{E})$ is endowed with a stable proper model category structure with the stable $\mathbf{A}^1$-equivalences as weak equivalences, the stable $\mathbf{A}^1$-fibrations as fibrations, and the stable $\mathfrak{V}$-cofibrations as cofibrations. The base change functor* (1.5.1.1) *is a left Quillen functor. Moreover, if $\mathcal{E}$ is commutative, then this model structure is symmetric monoidal.*

Proof. See [CD09a, Corollary 6.39]. □

1.5.3. Let $\mathcal{E}$ be a commutative ring spectrum. We define $D_{\mathbf{A}^1}(S,\mathcal{E})$ to be the localization of the category $\mathrm{Sp}_{Tate}(\mathfrak{V},\mathcal{E})$ by the class of stable $\mathbf{A}^1$-equivalences. It follows from the proposition above that this category is canonically endowed with a triangulated category structure. The base change functor has a total left derived functor

$$D_{\mathbf{A}^1}(S,R) \longrightarrow D_{\mathbf{A}^1}(S,\mathcal{E}) \quad , \qquad F \longmapsto \mathcal{E} \otimes^{\mathbf{L}}_R F \tag{1.5.3.1}$$

which is a left adjoint of the forgetful functor

$$D_{\mathbf{A}^1}(S,\mathcal{E}) \longrightarrow D_{\mathbf{A}^1}(S,R) \quad , \qquad M \longmapsto M\ . \tag{1.5.3.2}$$

The forgetful functor (1.5.3.2) is conservative (which means that an object of $D_{\mathbf{A}^1}(S,\mathcal{E})$ is null if and only if it is null in $D_{\mathbf{A}^1}(S,R)$. There is a derived tensor product

$$D_{\mathbf{A}^1}(S,\mathcal{E}) \times D_{\mathbf{A}^1}(S,\mathcal{E}) \longrightarrow D_{\mathbf{A}^1}(S,\mathcal{E}) \quad , \qquad (M,N) \longmapsto M \otimes^{\mathbf{L}}_{\mathcal{E}} N \tag{1.5.3.3}$$

that turns $D_{\mathbf{A}^1}(S,\mathcal{E})$ into a symmetric monoidal triangulated category (by applying [Hov99, Theorem 4.3.2] to the model structure of Proposition 1.5.2). The derived base change functor (1.5.3.1) is of course a symmetric monoidal functor. The category $D_{\mathbf{A}^1}(S,\mathcal{E})$ also has an internal Hom that we denote by $\mathbf{RHom}_{\mathcal{E}}(M,N)$. We thus have the formula

$$\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(L \otimes^{\mathbf{L}}_{\mathcal{E}} M, N) \simeq \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(L, \mathbf{RHom}_{\mathcal{E}}(M,N)). \tag{1.5.3.4}$$

It follows from 1.4.10 that the functor

$$\mathit{Comp}(R) \longrightarrow \mathrm{Sp}_{\mathit{Tate}}(\mathfrak{V},\mathcal{E}) \quad , \qquad M \longmapsto \mathcal{E} \otimes_R \Sigma^\infty(M). \tag{1.5.3.5}$$

is a left Quillen functor. The right derived functor of its right adjoint is the composition of the forgetful functor (1.5.3.2) with the functor $\mathbf{R}\Gamma \circ \mathbf{R}\Omega^\infty$. For two objects M and N of $D_{\mathbf{A}^1}(S,\mathcal{E})$, we define

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M,N) = \mathbf{R}\Gamma\Big(\mathbf{R}\Omega^\infty\big(\mathbf{RHom}_{\mathcal{E}}(M,N)\big)\Big). \tag{1.5.3.6}$$

For any integer n, we have a canonical isomorphism

$$H^n\big(\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M,N)\big) \simeq \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(M, N[n]). \tag{1.5.3.7}$$

For a smooth S-scheme X, we define the free $\mathcal{E}$-module generated by X as

$$\mathcal{E}(X) = \mathcal{E} \otimes_R \underline{R}(X) = \mathcal{E} \otimes_R \Sigma^\infty(R(X)) \ .$$

As $\underline{R}(X)$ is stably $\mathfrak{V}$-cofibrant, the canonical map

$$\mathcal{E} \otimes^{\mathbf{L}}_R \underline{R}(X) \longrightarrow \mathcal{E} \otimes_R \underline{R}(X) = \mathcal{E}(X)$$

is an isomorphism in $D_{\mathbf{A}^1}(S,R)$ (hence in $D_{\mathbf{A}^1}(S,\mathcal{E})$ as well). This implies that for any $\mathcal{E}$-module M, we have canonical isomorphisms

$$\mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,R)}(R(X), \mathbf{R}\Omega^\infty(M)) \simeq \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(\mathcal{E}(X), M) \ . \tag{1.5.3.8}$$

Note that as the forgetful functor (1.5.3.2) preserves direct sums, Proposition 1.4.9 implies that $\mathcal{E}(X)(n)$ is a compact object of $D_{\mathbf{A}^1}(S,\mathcal{E})$ for all smooth S-scheme X and integer n.

2. Modules over a Weil spectrum

From now on, we assume the given scheme S is regular.

Let $\mathfrak{V}$ be a full subcategory of smooth S-schemes satisfying the hypothesis of 1.1.1. We also fix a field of characteristic zero $\mathbf{K}$ called the *field of coefficients.*

2.1. **Mixed Weil theory.**

2.1.1. Let E be a complex of presheaves of $\mathbf{K}$-module on $\mathfrak{V}$ which has the Brown-Gersten property and is $\mathbf{A}^1$-homotopy invariant. From Proposition 1.1.10, the first property means that $H^i(E(X)) = H^i_{\mathrm{Nis}}(X, E_{\mathrm{Nis}})$ for any scheme X in $\mathfrak{V}$, and any integer i. The second property implies the complex of sheaves E_{Nis} is quasi-isomorphic to a $\mathbf{A}^1$-local complex. In the sequel we will write E for the corresponding object of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})$. Whence we obtain, for any smooth S-scheme X in $\mathfrak{V}$, a canonical isomorphism

$$H^i(E(X)) = \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})}(\mathbf{K}(X), E[i])\,.$$

Suppose moreover that E has a structure of a presheaf of commutative differential graded K-algebras. This structure corresponds to morphisms of presheaves

$$E \otimes_K E \xrightarrow{\mu} E, \qquad \mathbf{K} \xrightarrow{\eta} E$$

satisfying the usual identities (corresponding to the associativity and commutativity properties of the multiplication μ and to the the fact η is a unit). Applying the associated Nisnevich sheaf functor, we obtain in $\mathit{Comp}(\mathit{Sh}(S,\mathbf{K}))$ the following morphisms

$$E_{\mathrm{Nis}} \otimes_K E_{\mathrm{Nis}} \xrightarrow{\mu} E_{\mathrm{Nis}}, \qquad \mathbf{K} \xrightarrow{\eta} E_{\mathrm{Nis}}.$$

As the sheafifying functor and the tensor product over $\mathbf{K}$ are exact, these morphisms indeed induce a commutative monoid structure on E, as an object of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})$.

2.1.2. Consider now a merely commutative monoid object E of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})$.

Let us denote by $\mu : E \otimes^{\mathbf{L}}_{\mathbf{K}} E \longrightarrow E$ and $\eta : \mathbf{K} \longrightarrow E$ respectively the multiplication and the unit maps.

If M is an object of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})$, we set $H^i(M,E) = \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})}(M, E[i])$. For two objects M and N of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})$, we define the external cup product

$$H^p(M,E) \otimes_{\mathbf{K}} H^q(N,E) \longrightarrow H^{p+q}(M \otimes^{\mathbf{L}}_{\mathbf{K}} N, E)$$

as follows. Considering two morphisms $\alpha : M \longrightarrow E[p]$ and $\beta : N \longrightarrow E[q]$ in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})$, we define a map $\alpha \otimes_\mu \beta$ as the composite

$$M \otimes^{\mathbf{L}}_{\mathbf{K}} N \xrightarrow{\alpha \otimes^{\mathbf{L}}_{\mathbf{K}} \beta} E[p] \otimes^{\mathbf{L}}_{\mathbf{K}} E[q] \xrightarrow{\mu[p+q]} E[p+q]$$

that is the expected product of α and β.

For a smooth S-scheme X, we simply write $H^i(X,E) = H^i(\mathbf{K}(X),E)$. We can consider the diagonal embedding $X \longrightarrow X \times_S X$ which induces a comultiplication $\delta_* : \mathbf{K}(X) \longrightarrow \mathbf{K}(X) \otimes^{\mathbf{L}}_{\mathbf{K}} \mathbf{K}(X)$. This allows to define as usual a 'cup product' on $H^*(X,E)$ by the formula

$$\alpha\,.\,\beta = (\alpha \otimes_\mu \beta) \circ \delta_*\ .$$

We will always consider $H^*(X,E)$ as a graded $\mathbf{K}$-algebra with this cup product.

We introduce the following axioms :

W1 *Dimension.*— $H^i(S,E) \simeq \begin{cases} \mathbf{K} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$

W2 *Stability.*— $\dim_{\mathbf{K}} H^i(\mathbf{G}_m, E) = \begin{cases} 1 & \text{if } i = 0 \text{ or } i = 1, \\ 0 & \text{otherwise.} \end{cases}$

W3 *Künneth formula.*— For any smooth S-schemes X and Y, the exterior cup product induces an isomorphism

$$\bigoplus_{p+q=n} H^p(X,E)\otimes_{\mathbf{K}} H^q(Y,E) \xrightarrow{\sim} H^n(X\times_S Y,E)\ .$$

W3′ *Weak Künneth formula.*— For any smooth S-scheme X, the exterior cup product induces an isomorphism

$$\bigoplus_{p+q=n} H^p(X,E)\otimes_{\mathbf{K}} H^q(\mathbf{G}_m,E) \xrightarrow{\sim} H^n(X\times_S \mathbf{G}_m,E)\ .$$

2.1.3. Under assumptions W1 and W2, we will call any non zero element $c\in H^1(\mathbf{G}_m,E)$ a *stability class*. Note that such a class corresponds to a non trivial map

$$c:\mathbf{K}(1)\longrightarrow E$$

in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})$. In particular, if E is a presheaf of commutative diffential graded $\mathbf{K}$-algebras which has the B.-G.-property and is $\mathbf{A}^1$-homotopy invariant, then such a stability class can be lifted to an actual map of complexes of presheaves. Such a lift will be called a *stability structure* on E.

Remark that, in the formulation of axiom W3 (resp. W3′) we might require the Künneth formula to hold only for X and Y (resp. X) in $\mathfrak{V}$: as any smooth S-scheme is locally in $\mathfrak{V}$ for the Nisnevich topology, this apparently weaker condition implies the general one by a Mayer-Vietoris argument.

Definition 2.1.4. A *mixed Weil theory*[6] is a presheaf E of commutative differential graded $\mathbf{K}$-algebras on $\mathfrak{V}$ which has the Brown-Gersten property (or equivalently the excision property, see 1.1.10), is $\mathbf{A}^1$-homotopy invariant, and satisfies the properties W1, W2 and W3 stated above.

A *stable theory* is a presheaf E of commutative differential graded $\mathbf{K}$-algebras on $\mathfrak{V}$ which has the Brown-Gersten property, is $\mathbf{A}^1$-homotopy invariant, and satisfies the properties W1, W2 and W3′.

2.1.5. Any stable theory E gives rise canonically to a commutative ring spectrum $\mathcal{E}$, as explained below. The idea to define the spectrum $\mathcal{E}$ consists essentially to consider a weighted version of E (this should be clearer considering the comments given in 2.1.7).

Let $\mathrm{Hom}^*(\mathbf{K}(1),E)$ be the complex of maps of complexes of sheaves from $\mathbf{K}(1)$ to E (the category $\mathit{Comp}(\mathfrak{V},\mathbf{K})$ is naturally enriched in complexes of $\mathbf{K}$-vector spaces). As $\mathbf{K}(1)$ is $\mathfrak{V}$-cofibrant and as E is fibrant with respect to the model category structure of Proposition 1.1.15, we have for any integer i

$$H^i(\mathrm{Hom}^*(\mathbf{K}(1),E)) = \begin{cases} H^1(\mathbf{G}_m,E) & \text{if } i=0,\\ 0 & \text{otherwise.}\end{cases} \tag{2.1.5.1}$$

Consider the constant sheaf of complexes on $\mathfrak{V}$

$$L=\mathrm{Hom}^*(\mathbf{K}(1),E)_S \tag{2.1.5.2}$$

associated to the complex $\mathrm{Hom}^*(\mathbf{K}(1),E)$. We can now define a symmetric Tate spectrum $\mathcal{E}=(\mathcal{E}_n,\sigma_n)_{n\geqslant 0}$ as follows. Put first $\mathcal{E}_n=\mathbf{Hom}(L^{\otimes n},E)$ (here $\mathbf{Hom}$

[6]In what follows, we will prove this terminology is not usurpated: a consequence of the main results of this paper is that, when S is the spectrum of a field k, the restriction of the functor $H^*(\,.\,,E)$ to smooth and projective k-schemes is a Weil cohomology in the sense defined in [And04].

stands for the internal Hom in the category $Comp(\mathfrak{V}, \mathbf{K})$). We have a canonical map

$$L = \mathrm{Hom}^*(\mathbf{K}(1), E)_S \longrightarrow \mathbf{Hom}(\mathbf{K}(1), E)$$

which gives a map

$$\mathbf{K}(1) \otimes_{\mathbf{K}} L \longrightarrow E\,, \tag{2.1.5.3}$$

and tensoring with $\mathbf{Hom}(L^{\otimes n}, E)$ gives a map

$$\mathbf{K}(1) \otimes_{\mathbf{K}} L \otimes_{\mathbf{K}} \mathbf{Hom}(L^{\otimes n}, E) \longrightarrow E \otimes_{\mathbf{K}} \mathbf{Hom}(L^{\otimes n}, E)\,. \tag{2.1.5.4}$$

The product on E induces a canonical action of E on $\mathbf{Hom}(L^{\otimes n}, E)$:

$$E \otimes_{\mathbf{K}} \mathbf{Hom}(L^{\otimes n}, E) \longrightarrow \mathbf{Hom}(L^{\otimes n}, E)\,. \tag{2.1.5.5}$$

The composition of (2.1.5.4) and (2.1.5.5) finally leads to a morphism

$$\mathbf{K}(1) \otimes_{\mathbf{K}} L \otimes_{\mathbf{K}} \mathbf{Hom}(L^{\otimes n}, E) \longrightarrow \mathbf{Hom}(L^{\otimes n}, E)\,. \tag{2.1.5.6}$$

The map $\sigma_n : \mathcal{E}_n(1) \longrightarrow \mathcal{E}_{n+1}$ is defined at last from (2.1.5.6) by transposition, using the isomorphism $\mathbf{Hom}(L, \mathbf{Hom}(L^{\otimes n}, E) \simeq \mathbf{Hom}(L^{\otimes(n+1)}, E)$. The action of $\mathfrak{S}_n$ on $\mathcal{E}_n$ is by permutation of factors in $L^{\otimes n}$. Note that the fact $\mathcal{E}$ is well defined relies heavily on the fact E is commutative as a differential graded algebra. We define in the same spirit a commutative ring spectrum structure on $\mathcal{E}$. The unit map $\underline{\mathbf{K}} \longrightarrow \mathcal{E}$ is determined by a sequence of maps $\eta_n : \mathbf{K}(n) \longrightarrow \mathcal{E}_n$. The map η_0 is of course the unit of E, and the rest of the sequence is then obtained easily by induction: if η_{n-1} is defined, then η_n is obtained as the composition

$$\mathbf{K}(n) \xrightarrow{\eta_{n-1}(1)} \mathcal{E}_{n-1}(1) \xrightarrow{\sigma_{n-1}} \mathcal{E}_n\,.$$

The multiplication of $\mathcal{E}$ is determined by maps $\mu_{m,n} : \mathcal{E}_m \otimes_{\mathbf{K}} \mathcal{E}_n \longrightarrow \mathcal{E}_{m+n}$ which are defined by composition of the obvious maps below.

$$\mathbf{Hom}(L^{\otimes m}, E) \otimes_{\mathbf{K}} \mathbf{Hom}(L^{\otimes m}, E) \to \mathbf{Hom}(L^{\otimes(m+n)}, E \otimes_{\mathbf{K}} E) \to \mathbf{Hom}(L^{\otimes(m+n)}, E)$$

Proposition 2.1.6. *Let E be a stable theory. The associated commutative ring spectrum $\mathcal{E}$ is a weak Ω^∞-spectrum, and there is a canonical isomorphism $E \simeq \mathbf{R}\Omega^\infty(\mathcal{E})$ in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{K})$. In other words, for any sheaf of complexes M, we have canonical isomorphisms*

$$\mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})}(M, E) \simeq \mathrm{Hom}_{D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})}(M, \mathbf{R}\Omega^\infty(\mathcal{E})) \simeq \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{K})}(\mathbf{L}\Sigma^\infty(M), \mathcal{E}).$$

Furthermore, any stability structure on E defines an isomorphism $\mathcal{E}(1) \simeq \mathcal{E}$ in $D_{\mathbf{A}^1}(S, \mathbf{K})$.

Proof. It follows from (2.1.5.1) that the complex $\mathrm{Hom}^*(\mathbf{K}(1), E)$ is quasi-isomorphic to the constant sheaf associated to the vector space $H^1(\mathbf{G}_m, E)$. As a consequence, the constant sheaf L is a $\mathfrak{V}$-cofibrant[7] complex which is (non canonically) isomorphic to $\mathbf{K}$ in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{K})$. Taking into account that E is quasi-isomorphic, as a presheaf, to its fibrant replacement in the model structure of Proposition 1.1.15, we also have a canonical isomorphism in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{K})$

$$\mathcal{E}_n = \mathbf{Hom}(L^{\otimes n}, E) \simeq \mathbf{RHom}(L^{\otimes n}, E)\,.$$

[7] As we work with a field of coefficients $\mathbf{K}$, any constant sheaf of complexes of vector spaces is $\mathfrak{V}$-cofibrant.

Hence we can get a non canonical isomorphism $\mathcal{E}_n \simeq E$ which corresponds to the choice of a generator c of $H^1(\mathbf{G}_m, E)$. Under such an identification, the structural maps

$$\mathcal{E}_n \longrightarrow \mathbf{RHom}(\mathbf{K}(1), \mathcal{E}_{n+1})$$

all correspond in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{K})$ to the map

$$\tau_c : E \longrightarrow \mathbf{RHom}(\mathbf{K}(1), E)$$

induced by transposition of the map $E(1) \longrightarrow E$, obtained as the cup product of the identity of E and of the map $\mathbf{K}(1) \longrightarrow E$ coming from the chosen generator c. The weak Künneth formula and the stability axiom thus imply that the map τ_c above is an isomorphism in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{K})$. This proves that $\mathcal{E}$ is indeed a weak Ω^∞-spectrum. The reformulation of this assertion comes directly from Proposition 1.4.7.

Consider now a stability structure $c : \mathbf{K}(1) \longrightarrow E$ on E. We have to define a morphism of symmetric Tate spectra $u : \mathcal{E}(1) \longrightarrow \mathcal{E}$, which corresponds to $\mathfrak{S}_n$-equivariant maps commuting with the σ_n's

$$u_n : \mathbf{Hom}(L^{\otimes n}, E)(1) \longrightarrow \mathbf{Hom}(L^{\otimes n}, E)\,.$$

Such a map u_n is determined by a map

$$v_n : L^{\otimes n} \otimes_{\mathbf{K}} \mathbf{Hom}(L^{\otimes n}, E)(1) \longrightarrow E\,.$$

We already have an evaluation map twisted by $\mathbf{K}(1)$

$$L^{\otimes n} \otimes_{\mathbf{K}} \mathbf{Hom}(L^{\otimes n}, E)(1) \longrightarrow E(1)\,,$$

so that to define v_n, we are reduced to define a map

$$E(1) \longrightarrow E\,;$$

this is obtained as the cup product of the identity of E with the given map c. The fact that u is an isomorphism in $D^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{K})$ comes again from the stability axiom and from the weak Künneth formula. $\square$

2.1.7. Given a stable theory and its associated commutative ring spectrum $\mathcal{E}$, for a smooth S-scheme X and two integers p and q, we define the q^{th} *group of cohomology of X of twist p with coefficients in* $\mathcal{E}$ to be

$$H^q(X, \mathcal{E}(p)) = \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{K})}(\underline{\mathbf{K}}(X), \mathcal{E}(p)[q])\,. \tag{2.1.7.1}$$

We obviously have

$$H^q(X, E) = H^q(X, \mathcal{E})\,, \tag{2.1.7.2}$$

and more generally, if $p \geqslant 0$, $H^*(X, \mathcal{E}(p))$ is just the Nisnevich hypercohomology of X with coefficients in the sheaf of complexes $\mathbf{Hom}(L^{\otimes p}, E)$. Hence for any integer p, any choice of a generator of $H^1(\mathbf{G}_m, E)$ determines a non canonical (but still functorial) isomorphism $H^q(X, E) \simeq H^q(X, \mathcal{E}(p))$.

We also define complexes

$$\mathbf{R}\Gamma(X, \mathcal{E}(p)) = \mathbf{R}\mathrm{Hom}_{\mathbf{K}}(\underline{\mathbf{K}}(X), \mathcal{E}(p)) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}(X), \mathcal{E}(p)) \tag{2.1.7.3}$$

and we get by definition

$$H^q(\mathbf{R}\Gamma(X, \mathcal{E}(p))) = H^q(X, \mathcal{E}(p))\,. \tag{2.1.7.4}$$

2.1.8. Let X be a smooth S-scheme and $\alpha : \mathcal{E}(X) \longrightarrow \mathcal{E}(p)[i]$ and $\beta : \mathcal{E}(X) \longrightarrow \mathcal{E}(q)[j]$ be morphisms of $\mathcal{E}$-modules, corresponding to cohomological classes. The cup product of α and β over X then corresponds to a map of $\mathcal{E}$-modules

$$\alpha\,.\,\beta : \mathcal{E}(X) \longrightarrow \mathcal{E}(p+q)[i+j]$$

defined as the composition

$$\mathcal{E}(X) \xrightarrow{\delta_*} \mathcal{E}(X \times_S X) \simeq \mathcal{E}(X) \otimes^{\mathbf{L}}_{\mathcal{E}} \mathcal{E}(X) \xrightarrow{\alpha \otimes^{\mathbf{L}}_{\mathcal{E}} \beta} \mathcal{E}(p)[i] \otimes^{\mathbf{L}}_{\mathcal{E}} \mathcal{E}(q)[j] \simeq \mathcal{E}(p+q)[i+j].$$

2.2. **First Chern classes.** We assume a stable theory E is given. We will consider its associated commutative ring spectrum $\mathcal{E}$ (2.1.5), and the corresponding cohomology groups (2.1.7).

2.2.1. Recall we have a canonical decomposition $\mathbf{K}(\mathbf{G}_m) = \mathbf{K} \oplus \mathbf{K}(1)[1]$ in the category $D^{eff}_{\mathbf{A}^1}(S, \mathbf{K})$. The unit map $\mathbf{K} \longrightarrow \mathcal{E}$ determines by twisting and shifting a map

$$c : \mathbf{K}(1)[1] \longrightarrow \mathcal{E}(1)[1]\,. \tag{2.2.1.1}$$

The morphism (2.2.1.1), seen in $D^{eff}_{\mathbf{A}^1}(S, \mathbf{K})$ corresponds to a non trivial cohomology class in $\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{K})}(\mathbf{K}(1)[1], \mathcal{E}(1)[1]) = H^1(\mathbf{G}_m, \mathcal{E}(1))$.

We also have a decomposition $\mathbf{K}(\mathbf{P}^1_S) = \mathbf{K} \oplus \mathbf{K}(1)[2]$, so that (2.2.1.1) also corresponds to a cohomology class c in $H^2(\mathbf{P}^1_S, \mathcal{E}(1))$ that will be called the *canonical orientation of* $\mathcal{E}$.

Note also that the decomposition $\mathbf{K}(\mathbf{P}^1_S) = \mathbf{K} \oplus \mathbf{K}(1)[2]$ and the weak Künneth formula implies the Künneth formula holds with respect to products of type $\mathbf{P}^1_S \times_S X$ for Nisnevich cohomology with coefficients in E. We will still refer to this as the 'weak Künneth formula'.

Lemma 2.2.2. *For any integer $n \geqslant 0$, the graded vector space $H^*(\mathbf{K}(n), E)$ is (non canonically) isomorphic to $\mathbf{K}$ concentrated in degree zero.*

Proof. The case $n = 0$ is precisely W1. Assume $n \geqslant 1$. We can begin by a choice of a stability structure on E, which defines, using W2 and the weak Künneth formula, an isomorphism in $D^{eff}_{\mathbf{A}^1}(S, \mathbf{K})$:

$$E \simeq \mathbf{RHom}(\mathbf{K}(1), E)\,.$$

This gives

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{D^{eff}_{\mathbf{A}^1}(S,\mathbf{K})}(\mathbf{K}(n), E) &\simeq \mathbf{R}\mathrm{Hom}_{D^{eff}_{\mathbf{A}^1}(S,\mathbf{K})}(\mathbf{K}(n-1), \mathbf{RHom}(\mathbf{K}(1), E)) \\ &\simeq \mathbf{R}\mathrm{Hom}_{D^{eff}_{\mathbf{A}^1}(S,\mathbf{K})}(\mathbf{K}(n-1), E)\,.\end{aligned}$$

We conclude by induction on n. □

For any integer $1 \leqslant n \leqslant m$, we let $\iota_{n,m} : \mathbf{P}^n_S \longrightarrow \mathbf{P}^m_S$ be the embedding given by $(x_0 : \ldots : x_n) \longmapsto (x_0 : \ldots : x_n : 0 : \ldots : 0)$.

Lemma 2.2.3. *For any integer $n > 0$, the cohomology group $H^*(\mathbf{P}^n_S, E)$ is concentrated in degrees i such that i is even and $i \in [0, 2n]$.*

For any integer $0 \leqslant n \leqslant m$, $\iota^*_{n,m} : H^*(\mathbf{P}^m_S, E) \longrightarrow H^*(\mathbf{P}^n_S, E)$ is an isomorphism in degrees $i \in [0, 2n]$.

Proof. The case where $m = 1$ is already known (2.2.1). The remaining assertions follow then by induction from the canonical distinguished triangle

$$\mathbf{K}(\mathbf{P}^{n-1}_S) \xrightarrow{\iota_{n-1,n}} \mathbf{K}(\mathbf{P}^n_S) \longrightarrow \mathbf{K}(n)[2n] \longrightarrow \mathbf{K}(\mathbf{P}^{n-1}_S)[1]$$

in $D^{\mathit{eff}}_{\mathbf{A}^1}(S,\mathbf{K})$ (see Corollary 1.3.7), and using Lemma 2.2.2. □

Lemma 2.2.4. *Let $n \geqslant 2$ and σ be a permutation of the set $\{0,\dots,n\}$.*
Consider the morphism $\sigma : \mathbf{P}^n_S \longrightarrow \mathbf{P}^n_S, (x_0 : \dots : x_n) \longmapsto (x_{\sigma(0)} : \dots : x_{\sigma(n)})$.
Then $\sigma^* : H^2(\mathbf{P}^n_S, E) \longrightarrow H^2(\mathbf{P}^n_S, E)$ is the identity.

Proof. We consider first the case $n \geqslant 3$. We can assume σ is the transposition $(n-1,n)$. Then $\sigma\iota_{1,n} = \iota_{1,n}$ and the claim follows from the preceeding lemma.

It remains to prove the case $n = 2$. Let σ a transposition of $\{0,1,2\}$. There is then a transposition τ of $\{0,1,2,3\}$ such that $\iota_{2,3}\sigma = \tau\iota_{2,3}$. As we already know that τ induces the identity in degree 2 cohomology. By applying Lemma 2.2.3, we see that the map $\iota_{2,3}$ induces an isomorphism in degree 2 cohomology as well. We thus get, by functoriality, $\sigma^*\iota^*_{2,3} = \iota^*_{2,3}\tau^* = \iota^*_{2,3}$, with $\iota^*_{2,3}$ invertible, which ends the proof. □

2.2.5. For any integer $n, m \geqslant 0$, we will consider the Segre embedding

$$\sigma_{n,m} : \mathbf{P}^n_S \times \mathbf{P}^m_S \longrightarrow \mathbf{P}^{nm+n+m}_S \tag{2.2.5.1}$$

and the n-fold Segre embedding

$$\sigma^{(n)} : (\mathbf{P}^1_S)^n \longrightarrow \mathbf{P}^{2^n-1}_S\,. \tag{2.2.5.2}$$

Proposition 2.2.6. *There exists a unique sequence $(c_{1,n})_{n>0}$ of cohomology classes, with $c_{1,n} \in H^2(\mathbf{P}^n_S, \mathcal{E}(1))$, such that :*

(i) *$c_{1,1} = c$ is the canonical orientation of $\mathcal{E}$;*
(ii) *for any integers $1 \leqslant n \leqslant m$, $\iota^*_{n,m}(c_{1,m}) = c_{1,n}$.*

Moreover, the following formulas hold :

(iii) *for any integer $n > 0$, $c^n_{1,n} \neq 0$ and $c^{n+1}_{1,n} = 0$;*
(iv) *for any integers $n, m > 0$, $\sigma^*_{n,m}(c_{1,nm+n+m}) = \pi^*_n(c_{1,n}) + \pi^*_m(c_{1,m})$, where π_n and π_m denote the projections from $\mathbf{P}^n_S \times_S \mathbf{P}^m_S$ to $\mathbf{P}^n_S$ and $\mathbf{P}^m_S$ respectively.*

Proof. The unicity statement is clear from 2.2.3.

Let $n \geqslant 2$ be an integer and consider the embedding

$$p : \mathbf{P}^1_S \longrightarrow (\mathbf{P}^1_S)^n\ , \quad (x:y) \longmapsto ((x:y),(0:1),\dots,(0:1))\,.$$

The morphism $\iota_{1,2^n-1}$ factors[8] as

$$\mathbf{P}^1_S \xrightarrow{p} (\mathbf{P}^1_S)^n \xrightarrow{\sigma^{(n)}} \mathbf{P}^{2^n-1}_S$$

which induces in cohomology

$$H^2(\mathbf{P}^{2^n-1}_S, \mathcal{E}(1)) \xrightarrow{\sigma^{(n)*}} H^2((\mathbf{P}^1_S)^n, \mathcal{E}(1)) \xrightarrow{p^*} H^2(\mathbf{P}^1_S, \mathcal{E}(1)).$$

Let t be the unique class in $H^2(\mathbf{P}^{2^n-1}_S, \mathcal{E}(1))$ which is sent to c by the isomorphism $\iota^*_{1,2^n-1}$ (in degree 2 cohomology). Using the weak Künneth formula and Lemma 2.2.2, we obtain a decomposition

$$H^2((\mathbf{P}^1_S)^n, \mathcal{E}(1)) = \mathbf{K}.u_1 \oplus \dots \oplus \mathbf{K}.u_n$$

[8]This factorization might hold eventually only up to a permutation of coordinates in $\mathbf{P}^{2^n-1}_S$ (depending on the choices made to define the Segre embeddings), but this will be harmless by Lemma 2.2.4.

(where u_i stands for the the generator corresponding to c in the ith copy of $H^2(\mathbf{P}^1_S, \mathcal{E}(1))$ in $H^2((\mathbf{P}^1_S)^n, \mathcal{E}(1)))$. The map p^* is then the $\mathbf{K}$-linear map sending u_1 to c and u_i to 0 for $i > 1$. This implies $\sigma^{(n)*}(t) = u_1 + \lambda_2.u_2 + \cdots + \lambda_n.u_n$ for $\lambda_i \in \mathbf{K}$.

But from Lemma 2.2.4, the class t is invariant under permutations of the coordinates of $\mathbf{P}^{2^n-1}_S$. This implies $\sigma^{(n)*}(t)$ is invariant under permutations of the factors of $(\mathbf{P}^1_S)^n$ which gives $\lambda_2 = \cdots = \lambda_n = 1$.

Then $\sigma^{(n)*}(t^n) = n!\, u_1 \cdots u_n$ which is non zero by the weak Künneth formula and Lemma 2.2.2 (remember $\mathbf{K}(\mathbf{P}^1_S) = \mathbf{K} \oplus \mathbf{K}(1)[2]$). This implies $t^n \neq 0$.

We put $c_{1,n} = \iota^*_{n,2^n-1}(t)$ so that we have $\iota^*_{1,n}(c_{1,n}) = c_{1,1}$.

As $\iota_{n,2^n-1}$ induces an isomorphism in degree less than $2n$, we see that $c_{1,n}^n \neq 0$ and $c_{1,n}^{n+1} = 0$ (indeed, $u_i^2 = 0$ for any $i = 1, \ldots, n$).

The existence of the sequence and property (iii) are then proved. Moreover, by the unicity statement, we see that the class t introduced in the preceeding reasoning is $t = c_{1,2^n-1}$.

Property (iv) follows from the relation $\sigma^{(n)*}(c_{1,2^n-1}) = u_1 + \ldots + u_n$ and from property (ii). □

2.2.7. Remember from 1.3.8 the ind-scheme $\mathbf{P}^\infty_S$ defined by the tower of inclusions

$$\mathbf{P}^1_S \longrightarrow \cdots \longrightarrow \mathbf{P}^n_S \xrightarrow{\iota_n} \mathbf{P}^{n+1}_S \longrightarrow \cdots$$

We set $H^q(\mathbf{P}^\infty_S, \mathcal{E}(p)) = \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{K})}(\mathbf{L}\Sigma^\infty(\mathbf{K}(\mathbf{P}^\infty_k)), \mathcal{E}(p)[q])$.

Corollary 2.2.8. *The sequence $(c_{1,n})_{n>0}$ of the previous proposition gives an element $\mathfrak{c}$ of $H^2(\mathbf{P}^\infty_S, \mathcal{E}(1))$ which induces an isomorphism of graded $\mathbf{K}$-algebras*

$$\mathbf{K}[[\mathfrak{c}]] = \prod_{n \geqslant 0} H^{2n}(\mathbf{P}^\infty_S, \mathcal{E}(n)).$$

Proof. Using Proposition 1.4.7, we have the Milnor short exact sequences (1.3.9)

$$0 \longrightarrow \varprojlim\nolimits^1_{n\geqslant 0} H^{i-1}(\mathbf{P}^n_S, \mathcal{E}(p)) \longrightarrow H^i(\mathbf{P}^\infty_S, \mathcal{E}(p)) \xrightarrow{(*)} \varprojlim\nolimits_{n\geqslant 0} H^i(\mathbf{P}^n_S, \mathcal{E}(p)) \longrightarrow 0\ .$$

The $\varprojlim^1$ of a constant functor is null, and thus Lemma 2.2.3 implies $(*)$ is an isomorphism. The corollary then follows directly from the previous proposition. □

2.2.9. The sequence $(c_{1,n})_{n>0}$ induces a morphism in $D_{\mathbf{A}^1}(S, \mathbf{K})$

$$\mathfrak{c} : \Sigma^\infty \mathbf{K}(\mathbf{P}^\infty_S) \longrightarrow \mathcal{E}(1)[2]\,. \tag{2.2.9.1}$$

As a consequence, using the functor $\mathbf{K} : \mathcal{H}(S) \longrightarrow D^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{K})$ introduced in 1.3.3, for any smooth S-scheme X, we obtain a canonical map

$$[X, \mathbf{P}^\infty_S] \longrightarrow \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{K})}(\Sigma^\infty \mathbf{K}(X), \Sigma^\infty \mathbf{K}(\mathbf{P}^\infty_S))\,.$$

The map (2.2.9.1) induces a map

$$\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{K})}(\Sigma^\infty \mathbf{K}(X), \Sigma^\infty \mathbf{K}(\mathbf{P}^\infty_S)) \longrightarrow \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{K})}(\Sigma^\infty \mathbf{K}(X), \mathcal{E}(1)[2])\,.$$

As the base scheme S is regular, it follows from [MV99, Proposition 3.8, p. 138] that we have a natural bijection

$$[X, \mathbf{P}^\infty_S] \simeq \mathrm{Pic}(X)\,. \tag{2.2.9.2}$$

We have thus associated to (2.2.9.1) a canonical map

$$c_1 : \mathrm{Pic}(X) \longrightarrow H^2(X, \mathcal{E}(1)) \tag{2.2.9.3}$$

called the *first Chern class.* Note that this map is just defined as a map of pointed sets (it obviously preserves zero), but we have more structures, as stated below.

Proposition 2.2.10. *The map* (2.2.9.3) *introduced above is a morphism of abelian groups which is functorial in X with respect to pullbacks.*

Proof. Functoriality is obvious.

The family of Segre embeddings $\sigma_{n,m} : \mathbf{P}^n_S \times \mathbf{P}^m_S \longrightarrow \mathbf{P}^{nm+n+m}_S$ defines a morphism of ind-schemes

$$\mathbf{P}^\infty_S \times_S \mathbf{P}^\infty_S \xrightarrow{\sigma} \mathbf{P}^\infty_S .$$

which in turn defines an H-group structure on $\mathbf{P}^\infty_S$ as an object of $\mathcal{H}(S)$, and put a group structure on $[X, \mathbf{P}^\infty_S]$.

Let λ (resp. λ', λ'') be the canonical dual line bundle on $\mathbf{P}^\infty_S$ (resp. the two canonical dual line bundles on $\mathbf{P}^\infty_S \times_S \mathbf{P}^\infty_S$). An easy computation gives

$$\sigma^* : \mathrm{Pic}(\mathbf{P}^\infty_S) \longrightarrow \mathrm{Pic}(\mathbf{P}^\infty_S \times_S \mathbf{P}^\infty_S)\,,\ \lambda \longmapsto \lambda' \otimes \lambda''$$

which implies the preceding group structure coincides with the usual group structure on the Picard group via (2.2.9.2).

But similarly, from property (iv) in 2.2.6, we obtain the map

$$\sigma^* : H^*(\mathbf{P}^\infty_S; E) = \mathbf{K}[[\mathfrak{c}]] \longrightarrow H^*(\mathbf{P}^\infty_S \times_S \mathbf{P}^\infty_S, E) = \mathbf{K}[[\mathfrak{c}', \mathfrak{c}'']]\,,\ \mathfrak{c} \longmapsto \mathfrak{c}' + \mathfrak{c}''$$

which gives precisely the result we need. □

2.3. **Projective bundle theorem and cycle class maps.**

2.3.1. We consider given a stable theory E and its associated commutative ring spectrum $\mathcal{E}$ (2.1.5).

Recall from 1.5.1 the symmetric monoidal category of $\mathcal{E}$-modules is endowed with a notion of stable $\mathbf{A}^1$-weak equivalence. The associated localized category is denoted by $D_{\mathbf{A}^1}(S, \mathcal{E})$; see 1.5.3. We have an adjoint pair of functors

$$\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{K}} (-) : D_{\mathbf{A}^1}(S, \mathbf{K}) \rightleftarrows D_{\mathbf{A}^1}(S, \mathcal{E}) : \mathcal{U}$$

where $\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{K}} (-)$ is the total left derived fuinctor of the free $\mathcal{E}$-module functor and $\mathcal{U}$ forgets the $\mathcal{E}$-module structure. The functor $\mathcal{U}$ is conservative.

The study of the cohomology theory associated to $\mathcal{E}$ follows obviously from the study of the triangulated category $D_{\mathbf{A}^1}(S, \mathcal{E})$. It follows from the existence of the first Chern class (2.2.9.3) that the results and constructions of [Dég08] apply to $D_{\mathbf{A}^1}(S, \mathcal{E})$. This leads to the following classical results.

2.3.2. Consider now a vector bundle $\mathcal{V}$ of rank n over a smooth S-scheme X, $p : \mathbf{P}(\mathcal{V}) \longrightarrow X$ be the canonical projection. Consider the first Chern class (2.2.9.3)

$$\mathrm{Pic}(\mathbf{P}(\mathcal{V})) \xrightarrow{c_1} H^2(\mathbf{P}(\mathcal{V}), \mathcal{E}(1)) = \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(\mathcal{E}(\mathbf{P}(\mathcal{V})), \mathcal{E}(1)[2]).$$

Thus the canonical dual invertible sheaf $\lambda = \mathcal{O}(-1)$ on $\mathbf{P}(\mathcal{V})$ induces a morphism of $\mathcal{E}$-modules

$$c_1(\lambda) : \mathcal{E}(\mathbf{P}(\mathcal{V})) \longrightarrow \mathcal{E}(1)[2]. \tag{2.3.2.1}$$

This defines a map

$$a_{\mathbf{P}(\mathcal{V})} : \mathcal{E}(\mathbf{P}(\mathcal{V})) \longrightarrow \bigoplus_{i=0}^{n-1} \mathcal{E}(X)(i)[2i] \tag{2.3.2.2}$$

by the formula

$$a_{\mathbf{P}(\mathcal{V})} = \sum_{i=0}^{n-1} c_1(\lambda)^i . p_* . \tag{2.3.2.3}$$

Proposition 2.3.3 (Projective Bundle Formula)**.** *The map* (2.3.2.2) *is an isomorphism in* $D_{\mathbf{A}^1}(S, \mathcal{E})$*.*

Proof. See [Dég08, Theorem 3.2]. □

2.3.4. We can now come to the classical definition of Chern classes. Let X be a smooth scheme and $\mathcal{V}/X$ a vector bundle of rank n. Let λ be the canonical dual line bundle on $\mathbf{P}(\mathcal{V})$.

By virtue of Proposition 2.3.3, the canonical map

$$\bigoplus_{i=0}^{n-1} H^{2j-2i}(X, \mathcal{E}(j-i)). c_1(\lambda)^i \longrightarrow H^{2j}(\mathbf{P}(\mathcal{V}), \mathcal{E}(j))$$

is an isomorphism for all j.

Define the Chern classes $c_i(\mathcal{V})$ of $\mathcal{V}$ in $H^{2i}(X, \mathcal{E}(i))$ by the relations

(a) $\sum_{i=0}^n p^* c_i(\mathcal{V}) . c_1(\lambda)^{n-i} = 0$;
(b) $c_0(\mathcal{V}) = 1$;
(c) $c_i(\mathcal{V}) = 0$ for $i > n$.

These Chern classes are functorial with respect to pullback and extends the first Chern classes given by (2.2.9.3). Following a classical argument, we obtain the additivity for these Chern classes (see [Dég08, Lemma 3.13]):

Lemma 2.3.5. *Let X be a smooth S-scheme, and consider an exact sequence*

$$0 \longrightarrow \mathcal{V}' \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}'' \to 0$$

of vector bundles over X. Then $c_r(\mathcal{V}) = \sum_{i+j=r} c_i(\mathcal{V}') . c_j(\mathcal{V}'')$.

2.3.6. Let $\mathcal{V}$ be a vector bundle of rank $n+1$ over a smooth S-scheme X.

For any integer $r \in [0, n]$, we define the *Lefschetz embedding*

$$\mathfrak{l}_r(\mathbf{P}(\mathcal{V})) : \mathcal{E}(X)(r)[2r] \longrightarrow \mathcal{E}(\mathbf{P}(\mathcal{V})) \tag{2.3.6.1}$$

as the composition

$$\mathcal{E}(X)(r)[2r] \xrightarrow{(*)} \oplus_{i=0}^n \mathcal{E}(X)(i)[2i] \xrightarrow{a_{\mathbf{P}(\mathcal{V})}^{-1}} \mathcal{E}(\mathbf{P}(\mathcal{V})) \tag{2.3.6.2}$$

where $(*)$ is the obvious embedding.

Recall the morphism

$$\pi : \mathcal{E}(\mathbf{P}(\mathcal{V} \oplus \mathcal{O})) \longrightarrow \mathcal{E}(Th\, \mathcal{V}) \tag{2.3.6.3}$$

appearing from the distinguished triangle of Proposition 1.3.6.

Lemma 2.3.7. *Let $\mathcal{V}$ be a vector bundle of rank n over a smooth S-scheme X, and $P = \mathbf{P}(\mathcal{V} \oplus \mathcal{O})$ be its projective completion. The composite morphism*

$$\mathcal{E}(X)(n)[2n] \xrightarrow{\mathfrak{l}_n(P)} \mathcal{E}(\mathbf{P}(\mathcal{V} \oplus \mathcal{O})) \xrightarrow{\pi} \mathcal{E}(Th\, \mathcal{V})$$

is an isomorphism.

Proof. Simply use the distinguished triangle of Proposition 1.3.6, the definition of a_P and the compatibility of the first Chern class with pullback. □

2.3.8. In the situation of the preceding lemma, we will denote by

$$\mathfrak{p}_{\mathcal{V}} : \mathcal{E}(Th\,\mathcal{V}) \longrightarrow \mathcal{E}(X)(n)[2n]$$

the inverse isomorphism of $\pi\,\mathfrak{l}_n(P)$.

Proposition 2.3.9 (Purity Theorem)**.** *Let $i : Z \longrightarrow X$ be a closed immersion of smooth S-schemes of pure codimension n. We denote by $j : U = X - Z \longrightarrow X$ the complementary open immersion. There is a canonical distinguished triangle*

$$\mathcal{E}(U) \xrightarrow{j_*} \mathcal{E}(X) \xrightarrow{i^*} \mathcal{E}(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} \mathcal{E}(U)[1]$$

in $D_{\mathbf{A}^1}(S, \mathcal{E})$.

Proof. By applying the triangulated functor $\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{K}} \mathbf{L}\Sigma^\infty(-)$ to the distinguished triangle of Proposition 1.3.10, we obtain a distinguished triangle

$$\mathcal{E}(U) \longrightarrow \mathcal{E}(X) \longrightarrow \mathcal{E}(Th\,\mathcal{N}_{X,Z}) \longrightarrow \mathcal{E}(U)[1]$$

with $\mathcal{N}_{X,Z}$ the normal vector bundle of the immersion i. We conclude using the isomorphism $\mathfrak{p}_{\mathcal{N}_{X,Z}}$ introduced above. □

2.3.10. The distinguished triangle of the proposition is called the *Gysin triangle associated to the pair* (X, Z), and the map i^* is called the *Gysin morphism associated to i*. The precise definition of the Gysin triangle and its main functorialities are described and proved in [Dég08] in a more general context. We recall the main properties we will need below.

Proposition 2.3.11. *Given a cartesian square of smooth S-schemes*

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

where i and j are closed immersions of pure codimension n, we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{E}(Y) \arrow[r, "j^*"] \arrow[d, "f_*"] & \mathcal{E}(T)(n)[2n] \arrow[r, "\partial_{Y,T}"] \arrow[d, "g_*(m)[2m]"] & \mathcal{E}(Y-T)[1] \arrow[d, "h_*"] \\ \mathcal{E}(X) \arrow[r, "i^*"] & \mathcal{E}(Z)(n)[2n] \arrow[r, "\partial_{X,Z}"] & \mathcal{E}(X-Z)[1] \end{tikzcd}$$

Proof. See [Dég08, Proposition 4.10]. □

Remark 2.3.12. In cohomology, the Gysin morphism introduced above induces a morphism $i_* : H^*(Z, \mathcal{E}) \to H^{*+2n}(X, \mathcal{E}(n))$.

The commutativity of the left hand square above gives the usual projection formula in the transversal case :

(2.3.12.1) $$f^* i_* = j_* g^*.$$

Note that, as explained in [Dég08, Corollary 4.11], the above projection formula implies easily the following projection formula for $\mathcal{E}$-modules

$$(1_Z\,.\,i_*) \circ i^* = i^*\,.\,1_X(n)[2n]\,,$$

which implies the usual projection formula in cohomology:
(2.3.12.2)
$$\forall\, a \in H^i(Z, \mathcal{E}), \forall\, b \in H^i(X, \mathcal{E}), \quad i_*(a\,.\,i^*b) = i_*(a)\,.\,b \in H^{i+2n}(X, \mathcal{E}(n))\ .$$

Definition 2.3.13. Let $i : Z \longrightarrow X$ be a codimension n closed immersion between smooth S-schemes. We put $\eta_X(Z) = i_*(1)$ as an element in $H^{2n}(X, \mathcal{E}(n))$ and call $\eta_X(Z)$ the *fundamental class of Z in X*.

Note that this fundamental class corresponds uniquely to a morphism of $\mathcal{E}$-modules
$$\eta_X(Z) : \mathcal{E}(X) \longrightarrow \mathcal{E}(n)[2n]$$
which we refer also as the fundamental class.

Remark 2.3.14. Suppose that i admits a retraction $p : X \longrightarrow Z$. Let $p_Z : Z \longrightarrow S$ be the structural morphism. Then the projection formula gives
$$i^* = (p_{Z*}\,.\,p_*i_*) \circ i^* = (p_{Z*}i^*)\,.\,p_* = \eta_X(Z)\,.\,p_*.$$
The Gysin morphism in this case is completely determined by the fundamental class $\eta_Z(X)$.

Example 2.3.15. Let $\mathcal{V}$ be a vector bundle of rank n over X, and $P = \mathbf{P}(\mathcal{V} \oplus \mathcal{O})$ be its projective completion. Let $p : P \longrightarrow X$ be the canonical projection and $i : X \longrightarrow P$ the zero section. If λ denotes the canonical dual line bundle on P, the *Thom class of* $\mathcal{V}$ is the cohomological class in $H^{2n}(P, \mathcal{E}(n))$
$$t(\mathcal{V}) = \sum_{i=0}^{n} p^*(c_i(\mathcal{V}))\,.\,c_1(\lambda)^{n-i}.$$
By a purity argument (see [Dég08, 4.14]), one gets $\eta_P(X) = t(\mathcal{V})$.

Suppose now given a section s of $\mathcal{V}/X$ transversal to the zero section s_0 of $\mathcal{V}/X$. Put $Y = s^{-1}(s_0(X))$ and consider the pullback square
$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d, "j"'] & X \arrow[d, "s"] \\ X \arrow[r, "s_0"] & P. \end{tikzcd}$$
From the projection formula and the identities $s^*p^*c_i(\mathcal{V}) = c_i(\mathcal{V})$ and $s^*c_1(\lambda) = 0$, we obtain
$$\eta_X(Y) = i_*j^*(1) = s^*s_{0*}(1) = s^*(t(\mathcal{V})) = c_n(\mathcal{V}).$$

Following [Dég08, Proposition 4.16], we also obtain the excess intersection formula :

Proposition 2.3.16. *Consider a cartesian square of smooth schemes*
$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$
where i and j are closed immersions of respective codimension n and m. Let $e = n - m$ and put $\xi = g^*\mathcal{N}_{X,Z}/\mathcal{N}_{Y,T}$ as a T-vector bundle. Let $c_e(\xi)$ be the e^{th} Chern class of ξ.

Then, the following square is commutative.

$$\begin{tikzcd} \mathcal{E}(Y) \arrow[r, "j^*"] \arrow[d, "f_*"'] & \mathcal{E}(T)(n)[2n] \arrow[r, "\partial_{Y,T}"] \arrow[d, "c_e(\xi)\,.\,g_*(n)[2n]"] & \mathcal{E}(Y-T)[1] \arrow[d, "h_*"] \\ \mathcal{E}(X) \arrow[r, "i^*"] & \mathcal{E}(Z)(n)[2n] \arrow[r, "\partial_{X,Z}"] & \mathcal{E}(X-Z)[1] \end{tikzcd}$$

Remark 2.3.17. In particular, we obtain the excess intersection formula in cohomology.

$$\forall\, a \in H^*(Z, \mathcal{E}), \quad f^* i_*(a) = j_*(c_e(\xi)\,.\,g^*(a))$$

This also implies the self intersection formula for a closed immersion $i : Z \to X$ of codimension n between smooth schemes.

$$\forall\, a \in H^*(Z, \mathcal{E}), \quad i^* i_*(a) = c_n(\mathcal{N}_{X,Z})\,.\,a$$

2.3.18. Let $\mathcal{V}$ be a vector bundle of rank n over X, $p : \mathcal{V} \longrightarrow X$ the canonical projection, and $i : X \longrightarrow \mathcal{V}$ the zero section. Note that $p_* : \mathcal{E}(\mathcal{V}) \longrightarrow \mathcal{E}(X)$ is an isomorphism (by a Mayer-Vietoris argument, one can suppose that $\mathcal{V}$ is trivial, so that this follows from $\mathbf{A}^1$-homotopy invariance). Hence $i_* : \mathcal{E}(X) \longrightarrow \mathcal{E}(\mathcal{V})$ is the reciprocal isomorphism. The self intersection formula implies $\eta_{\mathcal{V}}(X) = p^* c_n(\mathcal{V})$ in $H^{2n}(\mathcal{V}, \mathcal{E}(n))$. Thus, we obtain the computation of the Gysin morphism associated with the zero section : $p_* i^* = c_n(\mathcal{V})\,.\,1_X : \mathcal{E}(X) \to \mathcal{E}(X)(n)[2n]$.

We deduce from that the *Euler long exact sequence in cohomology*

$$\to H^{r-2n}(X, \mathcal{E}(-n)) \xrightarrow{c_n(\mathcal{V}).} H^r(X, \mathcal{E}) \xrightarrow{q^*} H^r(\mathcal{V} - X, \mathcal{E}) \to H^{r-2n+1}(X, \mathcal{E}(-n)) \to$$

Theorem 2.3.19. *Consider a cartesian square of smooth S-schemes*

$$\begin{tikzcd} Z \arrow[r, "k"] \arrow[d, "l"'] & Y' \arrow[d, "j"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

such that i,j,k,l are closed immersions of respective pure codimension n, m, s, t. We consider the open immersions $i' : Y - Z \to X - Y'$, $j' : Y' - Z \to X - Y$ and we put put $d = n + s = m + t$. Then the following diagram is commutative.

$$\begin{tikzcd} \mathcal{E}(X) \arrow[r, "j^*"] \arrow[d, "i^*"'] & \mathcal{E}(Y')(m)[2m] \arrow[r, "\partial_j"] \arrow[d, "k^*(m)[2m]"] & \mathcal{E}(X - Y')[1] \arrow[d, "i'^*[1]"] \\ \mathcal{E}(Y)(n)[2n] \arrow[r, "l^*[2n]"'] & \mathcal{E}(Z)(d)[2d] \arrow[r, "\partial_l(n)[2n]"] \arrow[d, "\partial_k(m)[2m]"] & \mathcal{E}(Y - Z)(n)[2n][1] \arrow[d, "\partial_{i'}[1]"] \\ & \mathcal{E}(Y - Z)(m)[2m][1] \arrow[r, "-\partial_{j'}[1]"'] & \mathcal{E}(X - Y \cup Y')[2] \end{tikzcd}$$

Proof. This is an application of [Dég08, Theorem 4.32]. □

Remark 2.3.20. Indeed, the commutativity of the first two squares asserts the functoriality of the Gysin triangle with respect to the Gysin morphism of a closed immersion. The next square is an associativity result for residues. This theorem also ensures the compatibility of Gysin morphisms with tensor product of $\mathcal{E}$-modules (this will ensure the compatibility of Gysin morphisms with cup product in cohomology).

2.3.21. As a conclusion, we have proved in particular the axioms of Grothendieck for the existence of a cycle class map (cf. paragraph 2 of [Gro58a]): A1 is proved in 2.3.3, A2 in example 2.3.15, A3 in 2.3.19, and A4 in 2.3.12.2. Moreover, the projection fomula 2.3.12.1 implies the axiom A5 of 5 in *loc. cit.*. Hence, following the method of [Gro58a] and the theory of λ-operations, we obtain for any smooth S-scheme X, a unique homomorphism of rings

$$ch : K_0(X)_{\mathbf{Q}} \longrightarrow H^{2*}(X, \mathcal{E}(*)) \tag{2.3.21.1}$$

which is natural in X and such that for any line bundle λ on X, the identity below holds.

$$ch([\lambda]) = \sum_i \frac{1}{i!} c_1(\lambda)^i \tag{2.3.21.2}$$

2.3.22. Let $SH(S)$ be the $\mathbf{P}^1$-stable homotopy category of schemes over S; we refer to [Jar00, DRØ03, Mor04] for different (but equivalent) definitions of $SH(S)$. According to [Mor04, 5.2][9], there is a canonical symmetric monoidal triangulated functor

$$\tilde{R} : SH(S) \longrightarrow D_{\mathbf{A}^1}(S, R) \tag{2.3.22.1}$$

that preserves direct sums. It is essentially defined by sending $\Sigma^\infty_{\mathbf{P}^1}(X_+)$ to the Tate spectrum $\Sigma^\infty(R(X)) = \underline{R}(X)$ for any smooth S-scheme X.

For $R = \mathbf{Q}$, the functor (2.3.22.1) induces an equivalence of categories[10]

$$SH(S)_{\mathbf{Q}} \simeq D_{\mathbf{A}^1}(S, \mathbf{Q}), \tag{2.3.22.2}$$

where $SH(S)_{\mathbf{Q}}$ denotes the localization of $SH(S)$ by the rational equivalences (that are the maps inducing an isomorphism of stable motivic homotopy groups up to torsion); see e.g. [Mor04, Remark 4.3.3 and 5.2]. Hence there is no essential difference to work with $SH(S)_{\mathbf{Q}}$ or with $D_{\mathbf{A}^1}(S, \mathbf{Q})$, which allows to apply here results proved in $SH(S)_{\mathbf{Q}}$. In particular, by virtue of [Rio06, Definition IV.54], there exists an object $KGL_{\mathbf{Q}}$ in $D_{\mathbf{A}^1}(S, \mathbf{Q})$ which represents algebraic K-theory[11]. Hence, for any smooth S-scheme X and any integer n, we have

$$\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{Q})}(\underline{\mathbf{Q}}(X)[n], KGL_{\mathbf{Q}}) = K_n(X)_{\mathbf{Q}}\,. \tag{2.3.22.3}$$

Moreover, Riou defines for any integer k a morphism

$$\mathbf{\Psi}^k : KGL_{\mathbf{Q}} \longrightarrow KGL_{\mathbf{Q}} \tag{2.3.22.4}$$

which induces the usual Adams operation on K-theory (see Definition IV.59 of *loc. cit.*). For an S-scheme X, define

$$K_q(X)^{(p)}_{\mathbf{Q}} = \{x \in K_q(X)_{\mathbf{Q}} \mid \mathbf{\Psi}^k(x) = k^p x \text{ for all } k \in \mathbf{Z}\} \tag{2.3.22.5}$$

[9] In [Mor04], $D_{\mathbf{A}^1}(S, R)$ is defined using non symmetric spectra instead of symmetric spectra. But it follows from Voevodsky's Lemma [Jar00, Lemma 3.13] and from [Hov01, Theorem 10.1] that the two definitions lead to equivalent categories.

[10] The functor (2.3.22.1) and the equivalence of triangulated categories (2.3.22.2) hold without the regularity assumption on S.

[11] This is reasonable because we assumed S to be regular: K-theory is homotopy invariant only for regular schemes.

Recall that *Beilinson motivic cohomology* is defined by the following formula (see [Bei87]).

$$H^q_{\text{Б}}(X, \mathbf{Q}(p)) = K_{2p-q}(X)^{(p)}_{\mathbf{Q}} \tag{2.3.22.6}$$

Remember from [SGA 6, Exposé X, Theorem 5.3.2] that we have

$$H^2_{\text{Б}}(X, \mathbf{Q}(1)) = \mathrm{Pic}(X)_{\mathbf{Q}}\,. \tag{2.3.22.7}$$

By virtue of [Rio06, Theorem IV.72], there exists for each integer p, a projector $\pi_p : KGL_{\mathbf{Q}} \longrightarrow KGL_{\mathbf{Q}}$ such that if we denote by $\mathbf{H}^{(p)}_{\text{Б}}$ the image of π_p, the canonical map

$$\bigoplus_{p\in\mathbf{Z}} \mathbf{H}^{(p)}_{\text{Б}} \longrightarrow KGL_{\mathbf{Q}} \tag{2.3.22.8}$$

is an isomorphism[12]. Furthermore, we have

$$\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{Q})}(\underline{\mathbf{Q}}(X), \mathbf{H}^{(p)}_{\text{Б}}[q]) = H^{2p+q}_{\text{Б}}(X, \mathbf{Q}(p)) \tag{2.3.22.9}$$

for any smooth S-scheme X. The periodicity theorem for algebraic K-theory [Rio06, Proposition IV.2] now translates into the existence of canonical isomorphisms

$$\mathbf{H}^{(0)}_{\text{Б}}(p)[2p] \simeq \mathbf{H}^{(p)}_{\text{Б}}\,. \tag{2.3.22.10}$$

In the sequel of this paper, we will write simply $\mathbf{H}_{\text{Б}}$ for $\mathbf{H}^{(0)}_{\text{Б}}$. The object $\mathbf{H}_{\text{Б}}$ is called the *Beilinson motivic cohomology spectrum.*

Theorem 2.3.23. *There exists a canonical isomorphism in $D(\mathbf{K})$:*

$$\mathbf{R}\mathrm{Hom}(\mathbf{H}_{\text{Б}}, \mathcal{E}) \simeq \mathbf{R}\Gamma(S, \mathcal{E})\,.$$

In particular, we have

$$\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{Q})}(\mathbf{H}_{\text{Б}}, \mathcal{E}[i]) \simeq H^i(S, \mathcal{E}) = \begin{cases} \mathbf{K} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, there is a unique morphism $cl_{\text{Б}} : \mathbf{H}_{\text{Б}} \longrightarrow \mathcal{E}$ inducing the Chern character (2.3.21.1). *This is the unique morphism from $\mathbf{H}_{\text{Б}}$ to $\mathcal{E}$ which preserves the unit.*

Proof. This is a rather straightforward application of the nice results and methods of Riou in [Rio06]. The main remark is that the theory of Chern classes allows to compute the cohomology of Grassmanians (e.g. following the method of [Gro58b]), which in turn shows that we can apply [Rio06, Theorem IV.48]. Hence using corollary 2.2.8, we see that the arguments to prove [Rio06, Theorem V.31] can be followed *mutatis mutandis* to give the expected computation[13].

Note that, given a map $\mathbf{H}_{\text{Б}} \longrightarrow \mathcal{E}$, we get in particular morphisms

$$K_0(X)^{(n)}_{\mathbf{Q}} = \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{Q})}(\underline{\mathbf{Q}}(X), \mathbf{H}_{\text{Б}}(n)[2n]) \longrightarrow H^{2n}(X, \mathcal{E}(n))\,.$$

Hence there is at most one map $\mathbf{H}_{\text{Б}} \longrightarrow \mathcal{E}$ inducing the Chern character (2.3.21.1). The fact that (2.3.21.1) determines a map $\mathbf{H}_{\text{Б}} \longrightarrow \mathcal{E}$ comes from [Rio06, Lemma III.26 and Theorem IV.11] applied to $\mathcal{E}$. □

[12] It follows from [Rio06, Theorem V.31] that this is the unique decomposition of $KGL_{\mathbf{Q}}$ which lifts the Adams decomposition of K-groups (2.3.22.5).

[13] The proof of [Rio06, Theorem V.31] works over any regular base scheme, and holds if we replace motivic cohomology by any oriented $\mathbf{Q}$-linear cohomology theory

2.3.24. The preceding theorem allows to produce cycle class maps in the case where the base S is the spectrum of a field k.

Let $\mathbf{HQ}$ be Voevodsky's motivic cohomology spectrum (see e.g. [RØ08, Lev08]). According to [Rio06, Proposition V.36], the Chern character (which, according to [Rio06, Section 2.6], is the unique map which preserves the unit) $ch : KGL_{\mathbf{Q}} \longrightarrow \mathbf{HQ}$ factorizes through $\mathbf{H}_{\text{Б}}$. Furthermore, it can be shown that the map $\mathbf{H}_{\text{Б}} \longrightarrow \mathbf{HQ}$ is an isomorphism in $D_{\mathbf{A}^1}(S, \mathbf{Q})$: this follows from the coniveau spectral sequence of the K-theory spectrum, which degenerates rationaly; see [FS02, Lev08]). In particular, we get isomorphisms

$$H^q_{\text{Б}}(X, \mathbf{Q}(p)) \simeq H^q(X, \mathbf{Q}(p))\,. \tag{2.3.24.1}$$

We obtain a solid commutative diagram

$$\begin{tikzcd} \mathbf{HQ} \arrow[rr, dashed, "cl"] & & \mathcal{E} \\ & \mathbf{H}_{\text{Б}} \arrow[ul, "\simeq"'] \arrow[ur, "cl_{\text{Б}}"] & \\ & KGL_{\mathbf{Q}} \arrow[uul, "ch"] \arrow[u] \arrow[uur, "ch"'] & \end{tikzcd} \tag{2.3.24.2}$$

which defines the cycle class map

$$cl : \mathbf{HQ} \longrightarrow \mathcal{E}\,. \tag{2.3.24.3}$$

It follows from Theorem 2.3.23 and from [Rio06, Theorem V.31] that cl is the unique map which preserves the unit. It induces functorial maps (the genuine cycle class maps)

$$H^q(X, \mathbf{Q}(p)) \longrightarrow H^q(X, \mathcal{E}(p))\,. \tag{2.3.24.4}$$

These cycle class maps are completely determined by the fact they are functorial and compatible with cup products and with first Chern classes (this is proved by applying [Rio06, Lemma III.26 and Theorem IV.11]).

2.4. **Gysin morphisms.**

2.4.1. We still consider given a stable theory E and its associated commutative ring spectrum $\mathcal{E}$ (2.1.5).

We will now introduce the Gysin morphism of a projective morphism between smooth S-schemes in the setting of $\mathcal{E}$-modules (which corresponds to push forward in cohomology), and recall some of its main properties.

Let $f : Y \longrightarrow X$ be a projective morphism of codimension $d \in \mathbf{Z}$ between smooth S-schemes.

Let us choose a factorisation of f into $Y \xrightarrow{i} \mathbf{P}^n_X \xrightarrow{p} X$, where i is a closed immersion of pure codimension $n + d$, the map p being the canonical projection.

We define the *Gysin morphism associated to* f in $D_{\mathbf{A}^1}(S, \mathcal{E})$

$$f^* : \mathcal{E}(X) \longrightarrow \mathcal{E}(Y)(d)[2d] \tag{2.4.1.1}$$

as the following compositum

(2.4.1.2)

$$f^* = \left[\mathcal{E}(X)(n)[2n] \xrightarrow{\mathfrak{l}_n(\mathbf{P}^n_X)} \mathcal{E}(\mathbf{P}^n_X) \xrightarrow{i^*} \mathcal{E}(Y)(n+d)[2(n+d)]\right](-n)[-2n].$$

One can show f^* is independent of the chosen factorization; see [Dég08, Lemma 5.11].

Proposition 2.4.2. *Consider $Z \xrightarrow{g} Y \xrightarrow{f} X$ be projective morphisms, of codimension n and m respectively, between smooth S-schemes. Then the following triangle commutes.*

$$\begin{tikzcd} \mathcal{E}(X) \arrow[rr, "(fg)^*"] \arrow[dr, "f^*"'] & & \mathcal{E}(Z)(n+m)[2(n+m)] \\ & \mathcal{E}(Y)(m)[2m] \arrow[ur, "g^*(n)[2n]"'] & \end{tikzcd}$$

Proof. See [Dég08, Proposition 5.14]. $\square$

Proposition 2.4.3. *Consider a cartesian square of smooth S-schemes*

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[d, "p"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

such that f (resp. g) is a projective morphism of codimension n (resp. m). Let ξ be the excess bundle over T associated to that square, and let $e = n - m$ be its rank (cf. [Ful98, proof of Proposition 6.6]*). Then, $f^*p_* = \big(c_e(\xi) \,.\, q_*(m)[2m]\big)g^*$ as maps from $\mathcal{E}(Z)$ to $\mathcal{E}(Y)(n)[2n]$.*

Proof. See [Dég08, Proposition 5.17]. $\square$

Proposition 2.4.4. *Consider a cartesian square of smooth S-schemes*

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

such that f and g are projective morphism of respective relative codimension p and q, and such that i and j are closed immersion of respective codimension n and m. Denote by $h : Y - T \longrightarrow X - Z$ the morphism induced by f. Then the following square is commutative (in which the two arrows $\partial_{X,Z}$ and $\partial_{Y,T}$ are the one appearing in the obvious Gysin triangles).

$$\begin{tikzcd} \mathcal{E}(T)(m+q)[2m+2q] \arrow[r, "\partial_{Y,T}(p)[2p]"] & \mathcal{E}(Y-T)(p)[2p+1] \\ \mathcal{E}(Z)(n)[2n] \arrow[u, "g^*(n)[2n]"] \arrow[r, "\partial_{X,Z}"] & \mathcal{E}(X-Z)[1] \arrow[u, "h^*[1]"'] \end{tikzcd}$$

Proof. See [Dég08, Proposition 5.15]. $\square$

2.5. **Poincaré duality.**

2.5.1. We first recall the abstract definition of duality in monoidal categories. Let $\mathcal{C}$ be a symmetric monoidal category. We let $\mathbf{1}$ and $\otimes$ denote respectively the unit object and the tensor product of $\mathcal{C}$. An object X of $\mathcal{C}$ is said to have a *strong dual* if there exists an object $X^\vee$ of $\mathcal{C}$ and two maps

$$\eta : \mathbf{1} \longrightarrow X^\vee \otimes X \quad \text{and} \quad \varepsilon : X \otimes X^\vee \longrightarrow \mathbf{1}$$

such that the following diagrams commute.

$$\begin{tikzcd} X \arrow[r, "X\otimes\eta"] \arrow[dr, "1_X"'] & X\otimes X^\vee\otimes X \arrow[d, "\varepsilon\otimes X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X^\vee \arrow[r, "\eta\otimes X^\vee"] \arrow[dr, "1_{X^\vee}"'] & X^\vee\otimes X\otimes X^\vee \arrow[d, "X^\vee\otimes\varepsilon"] \\ & X^\vee \end{tikzcd} \tag{2.5.1.1}$$

For any objects Y and Z of $\mathcal{C}$, we then have a canonical bijection

$$\mathrm{Hom}_{\mathcal{C}}(Z\otimes X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Z, X^\vee\otimes Y). \tag{2.5.1.2}$$

In other words, $X^\vee\otimes Y$ is in this case the internal Hom of the pair (X,Y) for any Y. In particular, such a strong dual, together with the maps ε and η, is unique up to a unique isomorphism. It is clear that for any symmetric monoidal category $\mathcal{D}$ and any monoidal functor $F:\mathcal{C}\longrightarrow\mathcal{D}$, if X has a strong dual $X^\vee$, then $F(X)$ has a strong dual canonically isomorphic to $F(X^\vee)$. If $X^\vee$ is a strong dual of X, then X is a strong dual of $X^\vee$.

Let $\mathcal{T}$ be a closed symmetric monoidal triangulated category[14]. Denote by $\mathbf{Hom}$ its internal Hom. For any objects X and Y in $\mathcal{T}$ the evaluation map

$$X\otimes\mathbf{Hom}(X,\mathbf{1})\longrightarrow\mathbf{1}$$

tensored with the identity of Y defines by adjunction a map

$$\mathbf{Hom}(X,\mathbf{1})\otimes Y\longrightarrow\mathbf{Hom}(X,Y). \tag{2.5.1.3}$$

The object X has a strong dual if and only if this map is an isomorphism for all objects Y in $\mathcal{T}$, and in this case, we have $X^\vee=\mathbf{Hom}(X,\mathbf{1})$: this follows from the fact that, essentially by definition, X has a strong dual if and only if there exists an object $X^\vee$ in $\mathcal{T}$, such that the functor $X^\vee\otimes(-)$ is right adjoint to the functor $(-)\otimes X$, so that $X^\vee\otimes(-)$ has to be canonically isomorphic to the functor $\mathbf{Hom}(X,-)$ (the canonical isomorphism being precisely (2.5.1.3)). For Y fixed, the map (2.5.1.3) is a morphism of triangulated functors. Hence the objects X such that (2.5.1.3) is an isomorphism form a full triangulated subcategory of $\mathcal{T}$. In other words, the full subcategory $\mathcal{T}_{dual}$ of $\mathcal{T}$ that consists of the objects which have a strong dual is a thick triangulated subcategory of $\mathcal{T}$.

If moreover $\mathcal{T}$ has small sums, then to say that any object of $\mathcal{T}_{dual}$ is compact is equivalent to say that the unit $\mathbf{1}$ is compact. This is proved as follows. Suppose that $\mathbf{1}$ is compact, and let X be an object of $\mathcal{T}$ which has a strong dual $X^\vee$. Then for any small family $(Y_\lambda)_{\lambda\in\Lambda}$ of objects of $\mathcal{T}$, we get the following identifications.

$$\begin{aligned}\bigoplus_{\lambda\in\Lambda}\mathrm{Hom}_{\mathcal{T}}\big(X,Y_\lambda\big) &\simeq \bigoplus_{\lambda\in\Lambda}\mathrm{Hom}_{\mathcal{T}}\big(\mathbf{1},X^\vee\otimes Y_\lambda\big)\\ &\simeq \mathrm{Hom}_{\mathcal{T}}\big(\mathbf{1},\bigoplus_{\lambda\in\Lambda}(X^\vee\otimes Y_\lambda)\big)\\ &\simeq \mathrm{Hom}_{\mathcal{T}}\big(\mathbf{1},X^\vee\otimes\bigoplus_{\lambda\in\Lambda}Y_\lambda\big)\\ &\simeq \mathrm{Hom}_{\mathcal{T}}\big(X,\bigoplus_{\lambda\in\Lambda}Y_\lambda\big).\end{aligned}$$

[14]We just mean that the category $\mathcal{T}$ is endowed with a symmetric monoidal structure and with a triangulated category structure, such that for any object X of $\mathcal{T}$, the functor $Y\longmapsto X\otimes Y$ is triangulated.

The converse is obvious. Note that it can happen that a compact object of $\mathcal{T}$ doesn't have any strong dual; a counter-example can be found in [Rio05]. We will produce another counter-example below, as a consequence of a comparison theorem: for any complete discrete valuation ring V (of characteristic zero with perfect residue field), there are compact objects of $D_{\mathbf{A}^1}(\mathrm{Spec}\,(V)\,,\mathbf{Q})$ which don't have any strong dual; see 3.2.7.

Example 2.5.2. Recall that $D(\mathbf{K})$ denotes the derived category of $\mathbf{K}$-vector spaces. This is a closed symmetric monoidal triangulated category with tensor product $\otimes_{\mathbf{K}}$ and derived (internal) Hom $\mathbf{R}\mathrm{Hom}_{\mathbf{K}}$. Note that for a complex of $\mathbf{K}$-vector spaces C, the following conditions are equivalent:

(a) C is compact in $D(\mathbf{K})$;
(b) C has a strong dual in $D(\mathbf{K})$;
(c) the $\mathbf{K}$-vector space $\bigoplus_n H^n(C)$ is finite dimensional;
(d) C is isomorphic in $D(\mathbf{K})$ to a bounded complex of $\mathbf{K}$-vector spaces which is degreewise finite dimensional.

2.5.3. We consider again a stable theory E and its associated commutative ring spectrum $\mathcal{E}$ (2.1.5).

Let X be a smooth and projective S-scheme of pure dimension d, and denote by $p : X \longrightarrow S$ the canonical projection, $\delta : X \longrightarrow X \times_S X$ the diagonal embedding.

Then we can define pairings

$$\eta : \mathcal{E} \xrightarrow{p^*} \mathcal{E}(X)(-d)[-2d] \xrightarrow{\delta_*} \mathcal{E}(X \times_S X)(-d)[-2d] = \mathcal{E}(X)(-d)[-2d] \otimes^{\mathbf{L}}_{\mathcal{E}} \mathcal{E}(X)$$

$$\varepsilon : \mathcal{E}(X) \otimes^{\mathbf{L}}_{\mathcal{E}} \mathcal{E}(X)(-d)[-2d] = \mathcal{E}(X \times_S X)(-d)[-2d] \xrightarrow{\delta^*(-d)[-2d]} \mathcal{E}(X) \xrightarrow{p_*} \mathcal{E}.$$

Theorem 2.5.4 (Poincaré duality)**.** *The pair (ε, η) defined above turns the object $\mathcal{E}(X)(-d)[-2d]$ into the strong dual of $\mathcal{E}(X)$.*

Proof. This follows from the functoriality properties of the Gysin morphism; see [Dég08, Theorem 5.23]. □

2.5.5. It can happen that $\mathcal{E}(X)$ has a strong dual for a non projective smooth S-scheme. A classical example is the case where $X = \overline{X} - D$, for a smooth and projective S-scheme $\overline{X}$ and a *relative strict normal crossings divisor* D in $\overline{X}$ (which means here that D is a divisor in S with irreducible components D_i, $i \in I$, such that D is a reduced closed subscheme of $\overline{X}$, and such that for any subset $J \subset I$, $D_J = \cap_{j \in J} D_j$ is smooth over S, and of codimension $\#J$ in X). The case where D is irreducible comes from Proposition 2.3.9 and Theorem 2.5.4, applied to $\overline{X}$ and D, and the general case follows by an easy induction on the number of irreducible components of D. As we already noticed, we cannot expect the object $\mathcal{E}(X)$ to have a strong dual for an arbitrary S-scheme X (3.2.7). However, when S is the spectrum of a perfect field, $\mathcal{E}(X)$ has a strong dual for any smooth S-scheme X; see 2.7.11.

2.6. **Homological realization.**

2.6.1. Let E be a stable theory, and $\mathcal{E}$ its associated commutative ring spectrum. Recall $D(\mathbf{K})$ denotes the (unbounded) derived category of the category of $\mathbf{K}$-vector spaces.

We define the *homological realization functor associated to* $\mathcal{E}$ to be

$$D_{\mathbf{A}^1}(S, \mathcal{E}) \longrightarrow D(\mathbf{K}) \quad , \qquad M \longmapsto \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M) \tag{2.6.1.1}$$

(where $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}$ denotes the total right derived functor of the Hom functor; see (1.5.3.6)). This functor is right adjoint to the functor

$$D(\mathbf{K}) \longrightarrow D_{\mathbf{A}^1}(S, \mathcal{E}) \quad , \qquad C \longmapsto \mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{K}} \mathbf{L}\Sigma^{\infty}(C_S) \tag{2.6.1.2}$$

(where C_S denotes the constant sheaf associated to C). As the functor (2.6.1.2) is obviously a symmetric monoidal functor, the homological realization functor (2.6.1.1) is a lax symmetric monoidal functor. This means that for any $\mathcal{E}$-modules M and N, there are coherent and natural maps

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M) \otimes_{\mathbf{K}} \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, N) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M \otimes^{\mathbf{L}}_{\mathcal{E}} N) \tag{2.6.1.3}$$

and

$$\mathbf{K} \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{E})\,. \tag{2.6.1.4}$$

We define the category $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ to be the localizing subcategory (cf. 1.1.14) of the triangulated category $D_{\mathbf{A}^1}(S, \mathcal{E})$ generated by the objects which have a strong dual.

Note that any isomorphism $\mathcal{E}(1) \simeq \mathcal{E}$ (cf. 2.1.6) induces an isomorphism $M(1) \simeq M$ in $D_{\mathbf{A}^1}(S, \mathcal{E})$ for any $\mathcal{E}$-module M. We deduce that the category $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ is stable by Tate twists. Moreover, if M and N have strong duals, their tensor product $M \otimes^{\mathbf{L}}_{\mathcal{E}} N$ share the same property. In other words, $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ is generated by a family of objects which is stable by tensor product. This implies that the category $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ itself is stable by tensor product in $D_{\mathbf{A}^1}(S, \mathcal{E})$. As a consequence, $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ is a symmetric monoidal category, and the inclusion functor from $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ into $D_{\mathbf{A}^1}(S, \mathcal{E})$ is symmetric monoidal. It is also obvious that any object M of $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ which has a strong dual $M^{\vee}$ in $D_{\mathbf{A}^1}(S, \mathcal{E})$ has a strong dual in $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ which happens to be $M^{\vee}$ itself. There is a rather nice feature of the category $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$: an object of $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ is compact if and only if it has a strong dual. The reason why this category $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ remains interesting is that, by virtue of Poincaré duality (2.5.4), for any smooth and projective S-scheme X, the $\mathcal{E}$-module $\mathcal{E}(X)$ is in $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$.

We finaly get a homological realization functor

$$D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E}) \longrightarrow D(\mathbf{K}) \quad , \qquad M \longmapsto \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M) \tag{2.6.1.5}$$

by restriction of 2.6.1.1.

Theorem 2.6.2. *If E is a mixed Weil theory, then the homological realization functor*

$$D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E}) \longrightarrow D(\mathbf{K}) \quad , \qquad M \longmapsto \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M)$$

is an equivalence of symmetric monoidal triangulated categories. As a consequence, an object M of $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ is compact if and only if $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M)$ is compact. Moreover, for any $\mathcal{E}$-module M in $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$, there is a canonical isomorphism

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) \simeq \mathbf{R}\mathrm{Hom}_{\mathbf{K}}(\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M), \mathbf{K}).$$

In particular, if M is compact, then we have canonical isomorphisms

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M^{\vee}) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, M)^{\vee}.$$

Proof. The first step in the proof consists to see that the Künneth formula implies that for any compact objects M and N of $D_{\mathbf{A}^1}(S, \mathcal{E})$, the pairing

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) \otimes_{\mathbf{K}} \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(N, \mathcal{E}) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M \otimes^{\mathbf{L}}_{\mathcal{E}} N, \mathcal{E})$$

is an isomorphism (it is sufficient to check this on a family of compact generators, which is true by assumption for the family that consists of the objects of type $\mathcal{E}(X)$ for any smooth S-scheme X).

We will now prove that the homological realization functor (2.6.1.5) is a symmetric monoidal functor. The only thing to prove is in fact that the map (2.6.1.3) is an isomorphism whenever M and N are in $D^{\vee}_{\mathbf{A}^1}(S,\mathcal{E})$. As $D^{\vee}_{\mathbf{A}^1}(S,\mathcal{E})$ is generated by its compact objects, it is sufficient to check this property when M and N are compact. But in this case, M and N have strong duals, so that we get the following isomorphisms.

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},M)\otimes_{\mathbf{K}}\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},N) &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M^{\vee},\mathcal{E})\otimes_{\mathbf{K}}\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(N^{\vee},\mathcal{E})\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M^{\vee}\otimes^{\mathbf{L}}_{\mathcal{E}}N^{\vee},\mathcal{E})\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}((M\otimes^{\mathbf{L}}_{\mathcal{E}}N)^{\vee},\mathcal{E})\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},M\otimes^{\mathbf{L}}_{\mathcal{E}}N)\end{aligned}$$

We are now able to prove that the homological realization functor (2.6.1.5) is fully faithful. Using the fact $\mathcal{E}$ is compact in $D_{\mathbf{A}^1}(S,\mathcal{E})$ (see the end of 1.5.3), we reduce to problem to showing fully faithfulness on compact objects. Let M and N be compact objects of $D^{\vee}_{\mathbf{A}^1}(S,\mathcal{E})$. We already noticed that they both have strong duals $M^{\vee}$ and $N^{\vee}$ respectively. Note also that symmetric monoidal functors preserve strong duals, so that we get the following computations.

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M,N) &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},M^{\vee}\otimes^{\mathbf{L}}_{\mathcal{E}}N)\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},M)^{\vee}\otimes_{\mathbf{K}}\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},N)\\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{K}}(\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},M),\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},N))\end{aligned}$$

To prove the essential surjectivity, it is sufficient to check that a generating family of $D(\mathbf{K})$ is in the essential image of the homological realization functor. But this is obvious, as the object $\mathbf{K}$ (seen as a complex concentrated in degree 0) generates $D(\mathbf{K})$.

The other assertions of the theorem are obvious consequences of this equivalence of symmetric monoidal triangulated categories. □

2.6.3. Assume that E is a mixed Weil theory. Denote as usual by $\mathcal{E}$ the commutative ring spectrum associated to E.

By virtue of Example 2.5.2, we know that $\mathbf{R}\Gamma(X,\mathcal{E})$ is compact in $D(\mathbf{K})$ if and only if $H^*(X,\mathcal{E})$ is a finite dimensional vector space. This is how Theorem 2.6.2 implies a finiteness result for $H^*(X,\mathcal{E})$ whenever $\mathcal{E}(X)$ has a strong dual in $D_{\mathbf{A}^1}(S,\mathcal{E})$.

For any object M of $D_{\mathbf{A}^1}(S,\mathcal{E})$ and any integers p and q, we get a pairing

$$\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(\mathcal{E},M(-p)[-q])\otimes_{\mathbf{K}}\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(M,\mathcal{E}(p)[q])\longrightarrow\mathbf{K}\tag{2.6.3.1}$$

inducing an isomorphism

$$\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(M,\mathcal{E}(p)[q])\simeq\mathrm{Hom}_{\mathbf{K}}(\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(\mathcal{E},M(-p)[-q]),\mathbf{K}).\tag{2.6.3.2}$$

whenever M is in $D^{\vee}_{\mathbf{A}^1}(S,\mathcal{E})$.

If M has strong dual, the pairing (2.6.3.1) thus happens to be a perfect pairing between finite dimensional $\mathbf{K}$-vector spaces.

For a smooth S-scheme X, define the *homology of X with coefficients in $\mathcal{E}$* by the formula

$$H_q(X, \mathcal{E}(p)) = \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(\mathcal{E}(p)[q], \mathcal{E}(X)) \tag{2.6.3.3}$$

We get a canonical pairing

$$H_q(X, \mathcal{E}(p)) \otimes_{\mathbf{K}} H^q(X, \mathcal{E}(p)) \longrightarrow \mathbf{K} \tag{2.6.3.4}$$

which happens to be perfect whenever $\mathcal{E}(X)$ has a strong dual (e.g. when X is projective). For a smooth and projective S-scheme X of pure dimension d, Poincaré duality gives an isomorphism

$$H^{2d-q}(X, \mathcal{E}(d-p)) \simeq H_q(X, \mathcal{E}(p)) \tag{2.6.3.5}$$

so that we get a perfect pairing

$$H^{2d-q}(X, \mathcal{E}(d-p)) \otimes_{\mathbf{K}} H^q(X, \mathcal{E}(p)) \longrightarrow \mathbf{K}\ , \quad \alpha \otimes \beta \longmapsto \langle \alpha, \beta \rangle\,. \tag{2.6.3.6}$$

Note that according to the definition of the duality pairing 2.5.3 and the Künneth formula, this pairing has the familiar form :

$$\langle \alpha, \beta \rangle = p_*(\alpha.\beta)$$

where $p_* : H^{2d}(X, \mathcal{E}(d)) \to H^0(S, \mathcal{E}(0)) = \mathbf{K}$ is the Gysin morphism associated to the canonical projection of X/S — the so called *trace morphism*.

For a smooth S-scheme X of pure dimension d, we can also define the *cohomology with compact support with coefficients in $\mathcal{E}$* by the formula

$$\mathbf{R}\Gamma_c(X, \mathcal{E}(p)) = \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{E}(X)(p-d)[-2d])\,. \tag{2.6.3.7}$$

Setting $H^q_c(X, \mathcal{E}(p)) = H^q(\mathbf{R}\Gamma_c(X, \mathcal{E}(p)))$, we obtain

$$H^q_c(X, \mathcal{E}(p)) = \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(\mathcal{E}, \mathcal{E}(X)(p-d)[q-2d]) = H_{2d-q}(X, \mathcal{E}(d-p))\,. \tag{2.6.3.8}$$

It follows from Proposition 2.4.2 that the cohomology with compact support is functorial (in a contravariant way) with respect to projective morphisms, and functorial (in a covariant way) with respect to equidimensional morphisms. We also have a map

$$\varepsilon : \mathcal{E}(X) \otimes^{\mathbf{L}}_{\mathcal{E}} \mathcal{E}(X)(-d)[-2d] \xrightarrow{\delta^*(-d)[-2d]} \mathcal{E}(X) \xrightarrow{p_*} \mathcal{E} \tag{2.6.3.9}$$

which defines by transposition a map

$$\mathcal{E}(X)(p-d)[-2d] \longrightarrow \mathbf{RHom}_{\mathcal{E}}(\mathcal{E}(X), \mathcal{E}(p))\,. \tag{2.6.3.10}$$

Note that $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathbf{RHom}_{\mathcal{E}}(\mathcal{E}(X), \mathcal{E}(p))) = \mathbf{R}\Gamma(X, \mathcal{E}(p))$. Hence we obtain a morphism

$$\mathbf{R}\Gamma_c(X, \mathcal{E}(p)) \longrightarrow \mathbf{R}\Gamma(X, \mathcal{E}(p)) \tag{2.6.3.11}$$

which is functorial with respect to projective morphisms of S-schemes (thanks to the good functorial properties of the Gysin morphisms), and an isomorphism whenever X is projective. We also get a canonical pairing of complexes

$$\mathbf{R}\Gamma_c(X, \mathcal{E}(p)) \otimes_{\mathbf{K}} \mathbf{R}\Gamma(X, \mathcal{E}(d-p)[2d]) \longrightarrow \mathbf{K} \tag{2.6.3.12}$$

defined by the canonical map

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{E}(X)(p-d)[-2d]) \otimes_{\mathbf{K}} \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}(X), \mathcal{E}(d-p)[2d]) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{E})\,.$$

This gives rise to a pairing

$$H^q_c(X, \mathcal{E}(p)) \otimes_{\mathbf{K}} H^{2d-q}(X, \mathcal{E}(d-p)) \longrightarrow \mathbf{K} \tag{2.6.3.13}$$

which happens to be perfect if $\mathcal{E}(X)$ has a strong dual in $D_{\mathbf{A}^1}(S,\mathcal{E})$.

Note that Poincaré duality gives rise to the following classical computation; see e.g. [And04, 3.3.3].

Corollary 2.6.4 (Lefschetz trace formula)**.** *Let X and Y be smooth and projective S-schemes of pure dimension d_X and d_Y respectively, Then, given two integers p and q, for*

$$\alpha \in H^{2d_Y+q}(X\times_S Y,\mathcal{E}(d_Y+p)) \quad \textit{and} \quad \beta \in H^{2d_X-q}(Y\times_S X,\mathcal{E}(d_X-p))\,,$$

we have the equality

$$\langle \alpha, {}^t\beta\rangle = \sum_i (-1)^i \mathrm{tr}(\beta\circ\alpha | H^i(X,\mathcal{E}))\,,$$

where ${}^t\beta \in H^{2d_X-q}(X\times_S Y,\mathcal{E}(d_X-p))$ is the class corresponding to β through the pullback by the isomorphism $X\times_S Y\simeq Y\times_S X$, $\langle\,.\,,\,.\,\rangle$ is the Poincaré duality pairing, and $\beta\circ\alpha$ denotes the composition of α and β as cohomological correspondences.

Theorem 2.6.5. *Let E and E' be a mixed Weil theory and a stable theory respectively. Denote by $\mathcal{E}$ and $\mathcal{E}'$ the commutative ring spectra associated to E and E' respectively. Let $u : E \longrightarrow E'$ be a morphism of sheaves of differential graded $\mathbf{K}$-algebras. We assume that the induced map*

$$H^1(\mathbf{G}_m, E) \longrightarrow H^1(\mathbf{G}_m, E')$$

is not trivial. Then there exists a commutative ring spectrum $\mathcal{E}''$ and two morphisms of ring spectra (which means morphisms of monoids in the category of symmetric Tate spectra)

$$\mathcal{E} \xrightarrow{a} \mathcal{E}'' \xleftarrow{b} \mathcal{E}'$$

with the following properties:

(a) *The map $\mathcal{E}' \xrightarrow{b} \mathcal{E}''$ is an isomorphism in $D_{\mathbf{A}^1}(S,\mathbf{K})$.*
(b) *For any smooth S-scheme X, and any integer n, the following diagram commutes (in which the vertical arrows are the canonical isomorphisms).*

$$\begin{tikzcd} H^n(X,E) \arrow[rr, "u"] \arrow[d] & & H^n(X,E') \arrow[d] \\ H^n(X,\mathcal{E}) \arrow[r, "a"] & H^n(X,\mathcal{E}'') & H^n(X,\mathcal{E}') \arrow[l, "b"', "\simeq"] \end{tikzcd}$$

(c) *The maps a and b^{-1} define for any smooth S-scheme X maps*

$$H^q(X,\mathcal{E}(p)) \longrightarrow H^q(X,\mathcal{E}'(p)) \quad \textit{and} \quad H^q_c(X,\mathcal{E}(p)) \longrightarrow H^q_c(X,\mathcal{E}'(p))$$

which are compatible with cup products and cycle class maps. If moreover $\mathcal{E}(X)$ has a strong dual (e.g., from 2.5.5, if X is the complement of a relative strict normal crossings divisor in a smooth and projective S-scheme), then these maps are bijective.

In particular, if moreover for any smooth S-scheme X, the $\mathcal{E}$-module $\mathcal{E}(X)$ has a strong dual in $D_{\mathbf{A}^1}(S,\mathcal{E})$, E' is a mixed Weil theory and the map u is a quasi-isomorphism of complexes of Nisnevich sheaves.

Proof. We have to come back to the very construction of the ring spectrum associated to a stable theory given in 2.1.5. Define

$$L = \mathrm{Hom}^*(\mathbf{K}(1), E)_S \quad \text{and} \quad L' = \mathrm{Hom}^*(\mathbf{K}(1), E')_S.$$

We know that the symmetric Tate spectra $\mathcal{E}$ and $\mathcal{E}'$ are defined respectively by the sheaves of complexes

$$\mathcal{E}_n = \mathbf{Hom}(L^{\otimes n}, E) \quad \text{and} \quad \mathcal{E}'_n = \mathbf{Hom}(L'^{\otimes n}, E').$$

Define a third ring spectrum $\mathcal{E}'' = (\mathcal{E}''_n, \tau_n)$ as follows. Put $\mathcal{E}''_n = \mathbf{Hom}(L^{\otimes n}, E')$. We have maps

$$L \longrightarrow L' \longrightarrow \mathbf{Hom}(\mathbf{K}(1), E')$$

from which we construct maps of type

$$\tau'_n : \mathbf{K}(1) \otimes_{\mathbf{K}} L' \otimes_{\mathbf{K}} \mathbf{Hom}(L'^{\otimes n}, E) \longrightarrow \mathbf{Hom}(L'^{\otimes n}, E)$$

following the same steps as for the construction of the map (2.1.5.6). The structural maps

$$\tau_n : \mathcal{E}''_n(1) \longrightarrow \mathcal{E}''_{n+1}$$

are defined by transposition of the maps τ'_n. One can then check that $\mathcal{E}''$ is a commutative ring spectrum. The map a is induced by the maps

$$a_n : \mathbf{Hom}(L^{\otimes n}, E) \longrightarrow \mathbf{Hom}(L^{\otimes n}, E')$$

which correspond to the composition with u, and the map b is induced by the maps

$$b_n : \mathbf{Hom}(L'^{\otimes n}, E') \longrightarrow \mathbf{Hom}(L^{\otimes n}, E')$$

which corresponds to the composition with the map $L \longrightarrow L'$ obtained from u by functoriality. These define the expected morphisms of ring spectra.

Property (a) comes obviously from the fact the map $L \longrightarrow L'$ has to be a quasi-isomorphism according to the assumption on u. Indeed, this implies the maps b_n are all quasi-isomorphisms as well. In particular, the total left derived functor of the base change functor induced by b is an equivalence of triangulated categories

$$D_{\mathbf{A}^1}(S, \mathcal{E}') \simeq D_{\mathbf{A}^1}(S, \mathcal{E}'')\,.$$

As a consequence, the total left derived functor of the base change functor induced by a

$$D_{\mathbf{A}^1}(S, \mathcal{E}) \longrightarrow D_{\mathbf{A}^1}(S, \mathcal{E}') \quad , \qquad M \longmapsto \mathcal{E}'' \otimes^{\mathbf{L}}_{\mathcal{E}} M$$

is a triangulated functor which preserves small direct sums, and it is also symmetric monoidal. We claim that this induces by restriction a fully faithful symmetric monoidal triangulated functor

$$D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E}) \longrightarrow D_{\mathbf{A}^1}(S, \mathcal{E}')\,.$$

To see this, we first remark that $\mathcal{E}$ is a compact generator of $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$: applying Theorem 2.6.2 to E implies that $D^{\vee}_{\mathbf{A}^1}(S, \mathcal{E})$ is equivalent to $D(\mathbf{K})$. As the base change functor sends $\mathcal{E}$ to $\mathcal{E}'' \simeq \mathcal{E}'$, it is sufficient to prove that the induced maps

$$\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E})}(\mathcal{E}, \mathcal{E}[n]) \longrightarrow \mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathcal{E}')}(\mathcal{E}', \mathcal{E}'[n])$$

are bijective. For $n \neq 0$, the two terms are null, and for $n = 0$, this map is a morphism of $\mathbf{K}$-algebras from $\mathbf{K}$ to itself, so that it has to be an identity.

Properties (b) and (c) follow immediately from this fully faithfulness (the compatibility with cycle class maps follows from Theorem 2.3.23). □

2.7. **Cohomology of motives.**

2.7.1. In this section, the base scheme S is the spectrum of a perfect field k.

We consider given a stable cohomology theory E, as well as its associated ring spectrum $\mathcal{E}$. Let $TD_{\mathbf{A}^1}(k,\mathcal{E})$ the localizing subcategory of $D_{\mathbf{A}^1}(k,\mathcal{E})$ generated by objects of type $\mathcal{E}(p)[q]$, $p,q \in \mathbf{Z}$.

Proposition 2.7.2. *The functor*

$$TD_{\mathbf{A}^1}(k,\mathcal{E}) \longrightarrow D(\mathbf{K}) \ , \quad M \longmapsto \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},M)$$

is an equivalence of symmetric monoidal triangulated categories.

Proof. This functor is a right adjoint to the symmetric monoidal functor

$$D(\mathbf{K}) \longrightarrow TD_{\mathbf{A}^1}(k,\mathcal{E})$$

which sends a complex C to $\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{K}} \Sigma^\infty C$. It is sufficient to prove that the latter is an equivalence of categories. This follows essentially from the Homotopy axiom W1: this implies that this functor is fully faithful on the set of compact generators given by the unit object of $D(\mathbf{K})$, which is sent to $\mathcal{E}$. As $\mathcal{E}(p) \simeq \mathcal{E}$ for any integer p, and as $\mathcal{E}$ is compact in $TD_{\mathbf{A}^1}(k,\mathcal{E})$, we get the essential surjectivity by definition of $TD_{\mathbf{A}^1}(k,\mathcal{E})$. □

Corollary 2.7.3. *For any object M of $TD_{\mathbf{A}^1}(k,\mathcal{E})$, we have a canonical isomorphism*

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M,\mathcal{E}) \simeq \mathbf{R}\mathrm{Hom}_{\mathbf{K}}(\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},M),\mathbf{K}).$$

Proof. This follows from a straightforward translation from the equivalence of categories given by Proposition 2.7.2. □

Proposition 2.7.4. *The $\mathcal{E}$-module $\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ}$ is in $TD_{\mathbf{A}^1}(k,\mathcal{E})$.*

Proof. We know that $\mathbf{HQ} \simeq \mathbf{H}_{\mathrm{B}}$ is a direct factor of the K-theory spectrum $KGL_{\mathbf{Q}}$. Hence it is sufficient to prove that $\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} KGL_{\mathbf{Q}}$ is in $TD_{\mathbf{A}^1}(k,\mathcal{E})$, which follows immediately from [DI05, Theorem 6.2]. □

2.7.5. Remember from 2.3.24 we have an isomorphism $\mathbf{H}_{\mathrm{B}} \simeq \mathbf{HQ}$ in the category $D_{\mathbf{A}^1}(k,\mathbf{Q})$. Let $cl : \mathbf{HQ} \longrightarrow \mathcal{E}$ be the cycle class map (2.3.24.3). It induces by adjunction a $\mathcal{E}$-linear map

$$\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ} \longrightarrow \mathcal{E}\,.$$

Proposition 2.7.6. *The map $\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ} \longrightarrow \mathcal{E}$ is an isomorphism in the category $D_{\mathbf{A}^1}(k,\mathcal{E})$.*

Proof. We know from Theorem 2.3.23 that there is a canonical isomorphism in $D(\mathbf{K})$:

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ},\mathcal{E}) = \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\mathbf{HQ},\mathcal{E}) \simeq \mathbf{K}$$

(where $\mathbf{K}$ is seen as a complex concentrated in degree 0). By virtue of Proposition 2.7.4, we can apply Corollary 2.7.3 to $\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ}$ to obtain an isomorphism

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ},\mathcal{E}) \simeq \mathbf{R}\mathrm{Hom}_{\mathbf{K}}(\mathbf{R}\mathrm{Hom}(\mathcal{E},\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ}),\mathbf{K})\,,$$

This implies that we have an isomorphism in $D(\mathbf{K})$:

$$\mathbf{K} \simeq \mathbf{R}\mathrm{Hom}(\mathcal{E},\mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ})\,.$$

As $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},\mathcal{E}) \simeq \mathbf{K}$, and as, by Proposition 2.7.2, the homological realization functor $\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E},-)$ is an equivalence of categories from $TD_{\mathbf{A}^1}(k,\mathcal{E})$ to $D(\mathbf{K})$, to prove that the map $\mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathbf{HQ} \longrightarrow \mathcal{E}$ is an isomorphism, we are reduced to check that it is not trivial, which is obvious, by definition of the cycle class map cl. □

2.7.7. The canonical map from $\mathcal{E}$ to $\mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathbf{HQ}$ is an inverse of the isomorphism of Proposition 2.7.6, whence it is an isomorphism as well. We deduce from this the following result.

Proposition 2.7.8. *The functor*

$$D_{\mathbf{A}^1}(k,\mathbf{HQ}) \longrightarrow D_{\mathbf{A}^1}(k,\mathcal{E})\ ,\ M \longmapsto \mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}} M$$

is a symmetric monoidal triangulated functor.

Proof. As we are working with rational coefficients, using [Lur07, proposition 4.3.21] (see also [BM03, Hin97]), we can see that there is a commutative monoid structure[15] on the derived tensor product $\mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathbf{HQ}$. Proposition 2.7.6 tells us that the canonical map $\mathcal{E} \longrightarrow \mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathbf{HQ}$ is an isomorphism in the homotopy category of of commutative ring spectra (defined by stable $\mathbf{A}^1$-equivalences). Notice that, by virtue of [CD09a, Proposition 6.35], we can apply [SS00, Theorem 4.3] to see that $\mathcal{E} \longrightarrow \mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathbf{HQ}$ induces an equivalence of symmetric monoidal triangulated categories

$$D_{\mathbf{A}^1}(k,\mathcal{E}) \simeq D_{\mathbf{A}^1}(k,\mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathbf{HQ})\,.$$

The base change functor along $\mathbf{HQ} \longrightarrow \mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathbf{HQ}$ thus gives a symmetric monoidal triangulated functor

$$D_{\mathbf{A}^1}(k,\mathbf{HQ}) \longrightarrow D_{\mathbf{A}^1}(k,\mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathbf{HQ}) \simeq D_{\mathbf{A}^1}(k,\mathcal{E})\,.$$

The formula

$$\mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}} M \simeq \mathcal{E}\otimes^{\mathbf{L}}_{\mathbf{Q}}\mathbf{HQ}\otimes^{\mathbf{L}}_{\mathbf{HQ}} M$$

shows that the functor we constructed above is (isomorphic to) the functor considered in the proposition. □

2.7.9. Let $DM(k)$ be the triangulated category of mixed motives over k; see [CD09a, Example 7.15] for its construction. This is a symmetric monoidal triangulated category (as the homotopy category of a stable symmetric monoidal model category), and it is generated, as a triangulated category, by its compact objects. Moreover, the full subcategory of compact objects in $DM(k)$ is canonically equivalent to Voevodsky's triangulated category of mixed motives $DM_{gm}(k)$, constructed in [Voe00]. Different (but equivalent) constructions of $DM(k)$ are given by [RØ08, Theorem 35], and the relation with $DM_{gm}(k)$ is described in [RØ08, Section 2.3]; a systematic study of the triangulated categories $DM(S)$ will appear in [CD09b]. We will denote by $DM(k,\mathbf{Q})$ the rational version of $DM(k)$, and by $DM_{gm}(k,\mathbf{Q})$

[15] Our purpose is to deal with symmetric monoidal structures on homotopy categories of modules over a commutative monoid. A natural setting for this is the notion of E_∞-algebra. But, as we are working with rational coefficients, it is possible to strictify any E_∞-algebra into a commutative monoid, so that we have chosen to remain coherent with the rest of these notes, by considering genuine commutative monoids. One could also avoid any complication by working directly with symmetric monoidal ∞-categories [Lur07].

the rational version of $DM_{gm}(k)$. By virtue of [RØ08, Theorem 68], there is a canonical equivalence of symmetric monoidal triangulated categories[16]

$$\mathcal{U} : DM(k, \mathbf{Q}) \xrightarrow{\sim} D_{\mathbf{A}^1}(k, \mathbf{HQ}) \tag{2.7.9.1}$$

which sends the motive of X twisted by p to the object $\mathbf{HQ} \otimes^{\mathbf{L}}_{\mathbf{Q}} \Sigma^\infty(\mathbf{Q}(X))(p)$ (for X/k smooth, and $p \in \mathbf{Z}$); it is induced by the forgetful functor from the category of Nisnevich sheaves with transfers to the category of Nisnevich sheaves on Sm/k.

Theorem 2.7.10. *The motives of shape $M_{gm}(X)(p)$, for X smooth and projective, and $p \in \mathbf{Z}$, form a set of compact generators in $DM(k, \mathbf{Q})$. In particular, an object of $DM(k, \mathbf{Q})$ is compact if and only if it has a strong dual.*

Proof. This is proven using de Jong's resolution of singularities by alterations [dJ96]; see the proof of [RØ08, Theorem 68]. □

Corollary 2.7.11. *The following equality holds.*

$$D^\vee_{\mathbf{A}^1}(k, \mathcal{E}) = D_{\mathbf{A}^1}(k, \mathcal{E})\,.$$

If moreover E is a mixed Weil theory, then the homological realization functor (2.6.1.1) *defines an equivalence of symmetric monoidal triangulated categories*

$$D_{\mathbf{A}^1}(k, \mathcal{E}) \simeq D(\mathbf{K})\,.$$

In particular, for any smooth k-scheme X, $\mathcal{E}(X)$ has a strong dual, so that (2.6.3.13) *is a perfect pairing between finite dimensional vector spaces.*

Proof. The first assertion follows immediately from Theorem 2.7.10. Theorem 2.6.2 then ends the proof. □

Corollary 2.7.12. *Assume that E is a mixed Weil theory. For any $\mathbf{K}$-linear stable theory E' defined on smooth k-schemes, a morphism of sheaves of differential graded $\mathbf{K}$-algebras $E \longrightarrow E'$ is a quasi-isomorphism (in the category of complexes of Nisnevich sheaves) if and only if the induced map $H^1(\mathbf{G}_m, E) \longrightarrow H^1(\mathbf{G}_m, E')$ is not trivial.*

Proof. Apply Theorem 2.6.5 and Corollary 2.7.11. □

Corollary 2.7.13. *Assume that, for any smooth and projective k-schemes X and Y, the Künneth map*

$$\bigoplus_{p+q=n} H^p(X, E) \otimes_{\mathbf{K}} H^q(Y, E) \xrightarrow{\sim} H^n(X \times_k Y, E)\ .$$

is an isomorphism.

Then E is a mixed Weil theory.

Proof. We claim that for any compact objects M and N of $D_{\mathbf{A}^1}(k, \mathcal{E})$, the map

$$\mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M, \mathcal{E}) \otimes_{\mathbf{K}} \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(N, \mathcal{E}) \longrightarrow \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(M \otimes^{\mathbf{L}}_{\mathcal{E}} N, \mathcal{E})$$

is an isomorphism: it is sufficient to check this on a set of compact generators, which is true by assumption, by virtue of Theorem 2.7.10. □

[16] The equivalence of categories (2.7.9.1) is proved in [RØ08] using resolution of singularities by de Jong alterations [dJ96]; however, it will be shown in [CD09b] that such an equivalence of triangulated categories holds over a geometrically unibranch base scheme, by very different methods (without any kind of resolution of singularities).

Theorem 2.7.14. *Let E be a mixed Weil theory on smooth k-schemes, and $\mathcal{E}$ its associated commutative ring spectrum. Then the motivic homological realization functor*

$$DM(k, \mathbf{Q}) \longrightarrow D(\mathbf{K}) \quad , \qquad M \longmapsto \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}, \mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathcal{U}(M))$$

is a symmetric monoidal triangulated functor which preserves compact objects. In particular, if $D^b(\mathbf{K})$ denotes the bounded derived category of the category of finite dimensional $\mathbf{K}$-vector spaces, it induces by restriction a symmetric monoidal triangulated functor

$$R_{\mathcal{E}} : DM_{gm}(k, \mathbf{Q}) \longrightarrow D^b(\mathbf{K})$$

such that, for any smooth k-scheme X, one has canonical isomorphisms

$$R_{\mathcal{E}}(M_{gm}(X)^{\vee}) \simeq R_{\mathcal{E}}(M_{gm}(X))^{\vee} \simeq \mathbf{R}\Gamma(X, \mathcal{E})\,.$$

Proof. Under the equivalence of categories (2.7.9.1) this functor corresponds to the composition of the functor of Proposition 2.7.8 with the homological realization functor (2.6.1.1). Hence the first assertion follows from Corollary 2.7.11. In particular, this functor preserves strong duals. Theorem 2.7.10 now implies it sends $DM_{gm}(k, \mathbf{Q})$ to $D^b(\mathbf{K})$. If X is a smooth k-scheme, we have a natural isomorphism

$$\mathbf{HQ} \otimes^{\mathbf{L}}_{\mathbf{Q}} \Sigma^{\infty}\mathbf{Q}(X) \simeq \mathcal{U}(M_{gm}(X))\,.$$

We deduce from Proposition 2.7.6 that

$$\mathcal{E}(X) \simeq \mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathbf{HQ} \otimes^{\mathbf{L}}_{\mathbf{Q}} \Sigma^{\infty}\mathbf{Q}(X) \simeq \mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathcal{U}(M_{gm}(X))\,,$$

which implies that

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}, \mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} \mathcal{U}(M_{gm}(X))) &\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}, \mathcal{E}(X)) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\mathcal{E}}(\mathcal{E}, \mathcal{E}(X))\,.\end{aligned}$$

By Theorem 2.6.2, we get isomorphisms

$$R_{\mathcal{E}}(M_{gm}(X)^{\vee}) \simeq R_{\mathcal{E}}(M_{gm}(X))^{\vee} \simeq \mathbf{R}\Gamma(X, \mathcal{E})\,,$$

which ends the proof. □

Remark 2.7.15. The functor $R_{\mathcal{E}}$ induces cycle class maps

$$H^q(X, \mathbf{Q}(p)) \longrightarrow H^q(X, \mathcal{E}(p)) = H^q(X, E)(p)$$

which coincide with the cycle class maps introduced in 2.3.24. These cycle class maps are compatible with first Chern classes, hence with Gysin maps (by the categorical construction of these; see [Dég08]).

The reader might have noticed that, in the definition of a mixed Weil cohomology, we didn't ask the differential graded algebra E to be concentrated in non negative degrees. It would be natural to ask the cohomology groups $H^n(X, E)$ to vanish for any (affine) smooth scheme X and any negative integer n (which is true in practice). We conjecture this vanishing property to hold in general.

The existence of cycle class maps compatible with cup products and with Gysin morphisms finally proves that the cohomology groups $H^n(X, E)$, for X smooth and projective over k, define a Weil cohomology in the sense of [And04, Definition 3.3.1.1], modulo the vanishing property discussed above.

2.7.16. The proof of Theorem 2.7.14 relies on Proposition 2.7.8 and on the description of $DM(k, \mathbf{Q})$ as the homotopy category of modules on the rational motivic cohomology spectrum. Another strategy to prove Theorem 2.7.14 is to identify $DM(k, \mathbf{Q})$ with the “orientable part” of $D_{\mathbf{A}^1}(S, \mathbf{Q})$. This is achieved using an unpublished result of F. Morel [Mor06], which computes the rational motivic sphere spectrum in terms of motivic cohomology spectrum; see Theorem 2.7.18. More precisely, another proof of Theorem 2.7.14 is given by Corollary 2.7.24, equality (2.7.25.1), and Theorem 2.7.26 below. Moreover, Morel's result gives a very straightforward proof of the existence and unicity of the cycle class map for a stable theory; see Remark 2.7.22. We will now outline this alternative point of view.

2.7.17. Let S be a scheme. The permutation isomorphism

$$\tau : \underline{\mathbf{Q}}(1)[1] \otimes^{\mathbf{L}}_{\mathbf{Q}} \underline{\mathbf{Q}}(1)[1] \longrightarrow \underline{\mathbf{Q}}(1)[1] \otimes^{\mathbf{L}}_{\mathbf{Q}} \underline{\mathbf{Q}}(1)[1] \tag{2.7.17.1}$$

satisfies the equation $\tau^2 = 1$ in $D_{\mathbf{A}^1}(S, \mathbf{Q})$. Hence it defines an element ϵ in $\mathrm{End}_{D_{\mathbf{A}^1}(S,\mathbf{Q})}(\underline{\mathbf{Q}})$ which also satisfies the relation $\epsilon^2 = 1$. We define two projectors

$$e_+ = \frac{1-\epsilon}{2} \quad \text{and} \quad e_- = \frac{1+\epsilon}{2}\,. \tag{2.7.17.2}$$

As the triangulated category $D_{\mathbf{A}^1}(S, \mathbf{Q})$ is pseudo abelian, we can define two objects by the formulæ:

$$\mathbf{Q}_+ = \mathrm{Im}\, e_+ \quad \text{and} \quad \mathbf{Q}_- = \mathrm{Im}\, e_-\,. \tag{2.7.17.3}$$

Then for an object M of $D_{\mathbf{A}^1}(S, \mathbf{Q})$, we set

$$M_+ = \mathbf{Q}_+ \otimes^{\mathbf{L}}_{\mathbf{Q}} M \quad \text{and} \quad M_- = \mathbf{Q}_- \otimes^{\mathbf{L}}_{\mathbf{Q}} M\,. \tag{2.7.17.4}$$

It is obvious that for any objects M and N of $D_{\mathbf{A}^1}(S, \mathbf{Q})$, one has

$$\mathrm{Hom}_{D_{\mathbf{A}^1}(S,\mathbf{Q})}(M_i, N_j) = 0 \quad \text{for } i, j \in \{+, -\} \text{ with } i \neq j. \tag{2.7.17.5}$$

Denote by $D_{\mathbf{A}^1}(S, \mathbf{Q}_+)$ (resp. $D_{\mathbf{A}^1}(S, \mathbf{Q}_-)$) the full subcategory of $D_{\mathbf{A}^1}(S, \mathbf{Q})$ made of objects which are isomorphic to some M_+ (resp. some M_-) for an object M in $D_{\mathbf{A}^1}(S, \mathbf{Q})$. Then (2.7.17.5) implies that the direct sum functor induces an equivalence of triangulated categories

$$D_{\mathbf{A}^1}(S, \mathbf{Q}_+) \times D_{\mathbf{A}^1}(S, \mathbf{Q}_-) \simeq D_{\mathbf{A}^1}(S, \mathbf{Q})\,. \tag{2.7.17.6}$$

Assume now that S is a regular scheme. Recall from 2.3.22 the Beilinson motivic cohomology spectrum $\mathbf{H}_{\mathcal{B}}$. A deep result announced by F. Morel in [Mor06] takes the following form (taking into account the equivalence of categories (2.3.22.3)).

Theorem 2.7.18. *We have a canonical identification $\mathbf{Q}_+ = \mathbf{H}_{\mathcal{B}}$. Moreover, if -1 is a sum of squares in $\mathcal{O}(S)$, then $\underline{\mathbf{Q}} = \mathbf{H}_{\mathcal{B}}$.*

A proof will be given in [CD09b].

2.7.19. For a general scheme S, we define the *triangulated category of Morel-Beilinson motives* to be

$$DM_{\mathcal{B}}(S) = D_{\mathbf{A}^1}(S, \mathbf{Q}_+)\,. \tag{2.7.19.1}$$

Note that according to [Ayo07], the Grothendieck six operations are defined on the categories $D_{\mathbf{A}^1}(S, \mathbf{Q})$. As all these operations commute with Tate twists, it is obvious that they preserve Morel-Beilinson motives. Hence the categories $DM_{\mathcal{B}}(S)$ for various schemes S are stable by the six operations as subcategories of $D_{\mathbf{A}^1}(S, \mathbf{Q})$.

In particular, $DM_{\mathrm{B}}(S)$ is a symmetric monoidal triangulated category, and the canonical functor from $D_{\mathbf{A}^1}(S,\mathbf{Q})$ to $DM_{\mathrm{B}}(S)$ is a symmetric monoidal triangulated functor.

2.7.20. Suppose now that S is a regular scheme. Consider a stable theory E defined on smooth S-schemes, and let $\mathcal{E}$ be its associated commutative ring spectrum.

Proposition 2.7.21. *We have* $\mathcal{E} = \mathcal{E}_+$.

Proof. This is a translation from Lemma 2.2.4. □

Remark 2.7.22. Theorem 2.7.18 and Proposition 2.7.21 give another proof of Theorem 2.3.23: the unit map $\underline{\mathbf{Q}} = \mathbf{Q}_+ \oplus \mathbf{Q}_- \longrightarrow \mathcal{E}$ factors uniquely through $\underline{\mathbf{Q}}_+ = \mathbf{H}_{\mathrm{B}}$, which gives the cycle class map $\mathbf{H}_{\mathrm{B}} \longrightarrow \mathcal{E}$ (it clearly preserves the unit, so that it has to be the map obtained from the Chern character by Theorem 2.7.18). This construction has the advantage of giving directly the compatibilities of the cycle class map with the algebra structures.

2.7.23. Define $DM^{\vee}_{\mathrm{B}}(S)$ as the localizing subcategory (1.1.14) of $DM_{\mathrm{B}}(S)$ generated by the objects which have a strong dual (e.g. $\underline{\mathbf{Q}}(X)_+(p)$ for a smooth and projective S-scheme X and an integer p; see [Ayo07, Rön05]).

Corollary 2.7.24. *If E is a mixed Weil theory, then the* motivic homological realization functor

$$DM^{\vee}_{\mathrm{B}}(S) \longrightarrow D(\mathbf{K}) \quad , \qquad M \longmapsto \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\mathbf{H}_{\mathrm{B}}, \mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} M)$$

is a symmetric monoidal triangulated functor.

Proof. By virtue of Theorem 2.7.18 and of the preceding proposition, this functor is isomorphic to the composition of the symmetric monoidal triangulated functor

$$DM^{\vee}_{\mathrm{B}}(S) \longrightarrow D^{\vee}_{\mathbf{A}^1}(S,\mathcal{E}) \quad , \qquad M \longmapsto \mathcal{E} \otimes^{\mathbf{L}}_{\mathbf{Q}} M$$

with the homological realization functor (2.6.1.5). Theorem 2.6.2 concludes. □

2.7.25. Assume now S is the spectrum of a perfect field k.

It follows then from [Rio05] that we have

$$DM^{\vee}_{\mathrm{B}}(\mathrm{Spec}\,(k)) = DM_{\mathrm{B}}(\mathrm{Spec}\,(k))\,. \tag{2.7.25.1}$$

Theorem 2.7.26 (F. Morel)**.** *There exists a canonical equivalence of symmetric monoidal triangulated categories*

$$DM_{\mathrm{B}}(\mathrm{Spec}\,(k)) \simeq DM(k,\mathbf{Q})\,.$$

Proof. We know (e.g. from [RØ08]) that we have a canonical symmetric monoidal triangulated functor

$$D_{\mathbf{A}^1}(\mathrm{Spec}\,(k)\,,\mathbf{Q}) \longrightarrow DM(k,\mathbf{Q}) \quad , \qquad M \longmapsto M_{tr} \tag{2.7.26.1}$$

which preserves Tate twists, direct sums, and compact objects. By virtue of [Voe00, Corollary 2.1.5], the functor (2.7.26.1) vanishes on $D_{\mathbf{A}^1}(\mathrm{Spec}\,(k)\,,\mathbf{Q}_-)$, so that it induces a symmetric monoidal triangulated functor

$$DM_{\mathrm{B}}(\mathrm{Spec}\,(k)) \longrightarrow DM(k,\mathbf{Q}) \quad , \qquad M \longmapsto M_{tr}\,. \tag{2.7.26.2}$$

It then follows from Theorem 2.7.18 and [Rio06, Theorem V.31] that for a given smooth k-scheme X and two integers p and q, the induced map

$$\mathrm{Hom}_{DM_{\mathrm{B}}(\mathrm{Spec}(k))}(\underline{\mathbf{Q}}(X)_+, \mathbf{H}_{\mathrm{B}}(p)[q]) \longrightarrow \mathrm{Hom}_{DM(k,\mathbf{Q})}(\underline{\mathbf{Q}}(X)_{tr}, \underline{\mathbf{Q}}_{tr}(p)[q])$$

is in fact the isomorphism (2.3.24.1). By 2.7.25.1, this implies that the functor (2.7.26.2) is fully faithful on compact objects which proves the full faithfulness. The essential surjectivity follows from the fact that, by the very construction of $DM(k, \mathbf{Q})$, the objects of shape $\underline{\mathbf{Q}}(X)_{tr}(p)[q]$ generate $DM(k, \mathbf{Q})$. □

Remark 2.7.27. It will be proved in [CD09b] that Morel's Theorem 2.7.18 implies that Theorem 2.7.26 is true over any geometrically unibranch base scheme.

3. Some classical mixed Weil cohomologies

3.1. **Algebraic and analytic de Rham cohomologies.**

3.1.1. Suppose k is a field of characteristic 0. Let X be a smooth k-scheme. We denote by $\Omega^1_{X/k}$ the locally free sheaf of algebraic differential forms on X over k. Then the de Rham complex is the complex of $\mathcal{O}_X$-modules obtained from the exterior $\mathcal{O}_X$-algebra generated by $\Omega^1_{X/k}$:

$$\Omega^*_{X/k} = \bigwedge \Omega^1_{X/k}.$$

Remember from [Gro66, Gro68] that the algebraic de Rham cohomology of X is defined to be

$$H^*_{dR}(X) = H^*_{\mathrm{Zar}}(X, \Omega^*_{X/k}).$$

We will show here that de Rham cohomology is canonically represented by a mixed Weil theory.

3.1.2. Let X/k be an affine smooth scheme. We simply put $\Omega_{dR}(X) = \Gamma(X, \Omega^*_{X/k})$. Then $\Omega_{dR}(X)$ is a commutative graded differential algebra and it defines a presheaf of commutative differential graded k-algebras

$$\Omega_{dR} : X \longmapsto \Omega_{dR}(X).$$

In this context, the Künneth formula is obvious: the canonical map

$$\Omega_{dR}(X) \otimes_k \Omega_{dR}(Y) \longrightarrow \Omega_{dR}(X \times_k Y)$$

is an isomorphism.

As $\Omega^*_{X/k}$ is a complex of coherent sheaves on X and X is affine, the vanishing theorem of Serre [EGA III, 1.3.1] and the spectral sequence

$$E^1_{p,q} = H^p_{\mathrm{Zar}}(X, \Omega^q_{X/k}) \Rightarrow H^{p+q}_{dR}(X)$$

implies

$$H^*_{dR}(X) = H^*(\Omega_{dR}(X)).$$

3.1.3. The complex Ω_{dR} satisfies étale descent on smooth k-schemes, thus Nisnevich descent. This means the following.

Let $X = \mathrm{Spec}\,(A)$ and $Y = \mathrm{Spec}\,(B)$ be smooth affine schemes and $f : Y \longrightarrow X$ an étale morphism. Then, $\Omega_{dR}(Y) = \Omega_{dR}(X) \otimes_A B$. Suppose f is an étale cover. The augmented Čech complex $\check{C}^+_*(Y/X)$ is associated to the differential graded A-algebra

$$T^+_A(B) = (A \longrightarrow B \longrightarrow B \otimes_A B \longrightarrow B \otimes_A B \otimes_A B \longrightarrow \dots)$$

Thus, $\Omega_{dR}(\check{C}^+_*(Y/X)) = \Omega_{dR}(Y) \otimes_A T^+_A(B)$.

As f is faithfully flat, it is a morphism of effective descent with respect to the fibred category of quasi-coherent modules (see [SGA 1, Exposé VIII, Theorem 1.1]), so that the complex $T_A^+(B)$ is acyclic. For any integer $r \geqslant 0$, $\Omega^r_{dR}(Y)$ is flat over A, thus $\Omega^r_{dR}(Y) \otimes_A T_A^+(B)$ is acyclic. Hence the spectral sequence of a bounded bicomplex shows the complex $\mathrm{Tot}[\Omega_{dR}(\check{C}^+_*(Y/X))]$ is acyclic. This implies the étale descent for algebraic de Rham cohomology; see [Art71]. We deduce easily from the computations above that for any distinguished square as (1.1.2.1) which consists of smooth affine k-schemes, we get a short exact sequence

$$0 \longrightarrow \Omega_{dR}(X) \longrightarrow \Omega_{dR}(U) \oplus \Omega_{dR}(V) \longrightarrow \Omega_{dR}(U \times_X V) \longrightarrow 0.$$

Hence Ω_{dR} has the B.-G.-property with respect to the Nisnevich topology on the category of affine smooth k-schemes.

3.1.4. Finally, the following computations are easy:

$$H^n_{dR}(\mathbf{A}^1_k) = \begin{cases} k & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H^n_{dR}(\mathbf{G}_m) = \begin{cases} k & \text{if } n = 0 \\ k.d\log & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $d\log$ is the differential form defined by $d\log(t) = dt/t$. In conclusion, we have proved:

Proposition 3.1.5. *The presheaf Ω_{dR} is a mixed Weil theory.*

3.1.6. We denote by $\mathcal{E}_{dR}$ the corresponding commutative ring spectrum. Recall the canonical map

$$H^*_{dR}(X) \longrightarrow H^*_{\mathrm{Nis}}(X, \Omega_{dR}) \simeq H^*(X, \mathcal{E}_{dR})$$

is an isomorphism for any smooth k-scheme X.

3.1.7. Suppose that k is an algebraically closed field of characteristic zero, complete with respect to an archimedean (resp. non archimedean) absolute value $|-|$. Then we can associate to any smooth k-scheme X an analytic space (resp. a rigid analytic space) X^{an}. Let $\Omega^*_{X^{an}}$ be the analytic de Rham complex of X^{an} (seen as a sheaf of complexes). This defines a presheaf Ω^{an}_{dR} of differential graded k-algebras on Sm/k by the formula

$$\Omega^{an}_{dR}(X) = \Omega^*_{X^{an}}(X^{an})\,.$$

The *analytic de Rham cohomology* of a smooth scheme X is defined as the hypercohomology of X^{an} with coefficients in the sheaf $\Omega^*_{X^{an}}$.

$$H^*_{dR}(X^{an}) = H^*(X^{an}, \Omega^*_{X^{an}})$$

As X^{an} is Stein (resp. quasi-Stein) whenever X is affine, Cartan's Theorem B (resp. Kiehl's analog of this theorem) implies that for an affine smooth k-scheme X, one has

$$H^*_{dR}(X^{an}) = H^*(\Omega^*_{X^{an}}(X^{an}))\,.$$

As analytic de Rham cohomology satisfies étale descent and is $\mathbf{A}^1$-homotopy invariant, this implies that Ω^{an}_{dR} has the B.-G.-property on affine smooth k-schemes, and is $\mathbf{A}^1$-local. In fact, the complex Ω^{an}_{dR} is even a stable theory[17] so that, by virtue

[17] We leave this as an exercise for the reader; the arguments used below to prove that rigid cohomology is a stable theory (essentially the proof of Theorem 3.2.3) might give a hint.

of Corollary 2.7.12, the canonical map

$$\Omega_{dR} \longrightarrow \Omega_{dR}^{an}$$

is a quasi-isomorphism locally for the Nisnevich topology. In other words, we get Grothendieck's theorem [Gro66] (resp. Kiehl's theorem [Kie67a]): for any smooth k-scheme X, the canonical map

$$H^*_{dR}(X) \longrightarrow H^*_{dR}(X^{an})$$

is an isomorphism.

3.2. **Variations on Monsky-Washnitzer cohomology.**

3.2.1. We consider here a complete discrete valuation ring V with fraction field of characteristic zero K and perfect residue field k. We set $S = \mathrm{Spec}\,(V)$, $\eta = \mathrm{Spec}\,(K)$, and $s = \mathrm{Spec}\,(k)$. We have an open immersion $j : \eta \longrightarrow S$ and a closed immersion $i : s \longrightarrow S$. For a (smooth) S-scheme X, we write X_η and X_s for the generic fiber and the special fiber of X respectively.

3.2.2. Consider a smooth affine S-scheme $X = \mathrm{Spec}\,(A)$.

We denote by $A^\dagger$ the weak completion of A with respect to the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ stands for the maximal ideal of V; see [MW68, Definition 1.1]. Recall $A^\dagger$ is a formally smooth V-algebra [MW68, Theorem 2.6]. Denote by $\Omega^*(A^\dagger/V)$ the complex of differential forms of $A^\dagger$ relative to V. It can be defined as the universal $\mathfrak{m}$-separated differential graded V-algebra associated to A; see [MW68, Theorem 4.2]. More precisely, it is obtained from the algebraic de Rham complex of $A^\dagger$ over V by the formula

$$\Omega^*(A^\dagger/V) = \Omega^*_{A^\dagger/V} / \cap_{i=0}^{\infty} \mathfrak{m}^i \Omega^*_{A^\dagger/V}\,.$$

The *Monsky-Washnitzer complex* of X is defined as

$$E_{MW}(X) = \Omega^*(A^\dagger/V) \otimes_V K \simeq A^\dagger \otimes_A \Omega^*_{A/V} \otimes_V K\,,$$

and the *Monsky-Washnitzer cohomology* of X is

$$H^n_{MW}(X) = H^n(E_{MW}(X))\,.$$

(see [MW68, vdP86]).

Theorem 3.2.3. *The Monsky-Washnitzer complex is a stable theory on smooth affine S-schemes.*

Proof. The complex $E_{MW}(X)$ can be compared with Berthelot's rigid cohomology; see [Ber97b, Proposition 1.10]. More precisely, once a closed embedding $X \longrightarrow \mathbf{A}^n_S$ is chosen, let W denotes the schematic closure of X in $\mathbf{P}^n_S$, and $\hat{W}$ denotes the formal $\mathfrak{m}$-adic completion of W. The proof of [Ber97b, Proposition 1.10] consists then to check that we have a canonical isomorphisms of complexes of K-vector spaces

$$E_{MW}(X) \simeq \varinjlim_{V} \Gamma(V, \Omega^*_V) \simeq \Gamma(]W[_{\hat{W}}, j^\dagger \Omega^*_{\hat{W}})\,,$$

where V ranges over the strict neighbourhoods of the tube of X in $\hat{W}$, and that the canonical map

$$H^n(\Gamma(]W[_{\hat{W}}, j^\dagger\Omega^*_{\hat{W}})) \longrightarrow H^n(\mathbf{R}\Gamma(]W[_{\hat{W}}, j^\dagger\Omega^*_{\hat{W}})) = H^n_{rig}(X_s/K)$$

is an isomorphism. In other words, $E_{MW}(X)$ is (up to a canonical quasi-isomorphism) the rigid complex associated to the embbedings

$$X_s \longrightarrow W \longrightarrow \hat{W}\,.$$

Using [Ber97b, Proposition 2.2], one can extend this comparison results to cohomology with support: for any closed subscheme Z of X, one has a canonical isomorphism

$$H^n_{MW,Z}(X) \simeq H^n_{rig,Z_s}(X_s/K)$$

(where $H^{i+1}_{MW,Z}(X)$ denotes the ith cohomology group of the cone of the map $E_{MW}(X) \longrightarrow E_{MW}(X-Z)$). Hence to prove étale excision for Monsky-Washnitzer cohomology, we are reduce to prove étale excision for rigid cohomology. This follows immediately from the étale descent theorem for rigid cohomology, proved by Chiarellotto and Tsuzuki [CT03].

We also have the following computations:

$$H^n_{MW}(\mathbf{A}^1_S) = \begin{cases} K & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases} \qquad H^n_{MW}(\mathbf{G}_m) = \begin{cases} K & \text{if } n=0 \\ K.d\log & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

(where $d\log$ is the differential form on $V[t,t^{-1}]^\dagger$ defined by $d\log(t) = dt/t$).

It remains to prove that the Künneth map

$$E_{MW}(X) \otimes_K E_{MW}(Y) \longrightarrow E_{MW}(X \times_S Y)$$

is a quasi-isomorphism for any affine smooth S-scheme X and for $Y = \mathbf{A}^1_S$ or $Y = \mathbf{G}_m$. If $Y = \mathbf{A}^1_S$, this follows from Monsky and Washnitzer Homotopy Invariance Theorem [MW68, Theorem 5.4]. The case of $Y = \mathbf{G}_m$ is solved by considering the Gysin long exact sequence associated to the closed immersion

$$i : X = X \times \{0\} \longrightarrow X \times \mathbf{A}^1$$

which is constructed explicitely from [Mon68, Theorem 3.5]:

$$\cdots \longrightarrow H^{n-2}_{MW}(X) \longrightarrow H^n_{MW}(X \times \mathbf{A}^1) \longrightarrow H^n_{MW}(X \times \mathbf{G}_m) \longrightarrow H^{n-1}_{MW}(X) \longrightarrow \cdots$$

The homotopy invariance of Monsky-Washnitzer cohomology allows then to split canonically the long exact sequence above (using the projection of $X \times \mathbf{G}_m$ onto X), and we finally get an isomorphism of graded $H^*_{rig}(X/K)$-modules

$$H^*_{MW}(X \times \mathbf{G}_m) \simeq H^{*-1}_{MW}(X)\,.\,d\log \oplus H^*_{MW}(X)\,.$$

This implies immediately the Künneth formula above for $Y = \mathbf{G}_m$. □

3.2.4. We define a presheaf of commutative differential graded K-algebras $j_* E_{dR}$ on Sm/S by the formula below.

$$j_* E_{dR}(X) = \Omega_{dR}(X_\eta)$$

It follows immediately from Proposition 3.1.5 that $j_* E_{dR}$ is a mixed Weil cohomology on affine smooth S-schemes.

3.2.5. Consider a smooth affine S-scheme $X = \mathrm{Spec}\,(A)$. By definition of the Monsky-Washnitzer complex, we have a natural morphism of differential graded algebras

$$sp_X : j_* E_{dR}(X) = \Omega^*_{A/V} \otimes_V K \longrightarrow A^\dagger \otimes_A \Omega^*_{A/V} \otimes_V K = E_{MW}(X) \tag{3.2.5.1}$$

called the *specialisation map*. This defines a morphism of presheaves of differential graded algebras

$$sp : j_* E_{dR} \longrightarrow E_{MW} \,. \tag{3.2.5.2}$$

Denote by $j_*\mathcal{E}_{dR}$ (resp. by $\mathcal{E}_{MW}$) the commutative ring spectra associated to j_*E_{dR} and E_{MW} respectively. It is clear that we have, for any affine smooth S-scheme X, the following identifications in the derived category of the category of K-vector spaces.

$$\mathbf{R}\Gamma(X, j_*\mathcal{E}_{dR}) \simeq \Omega^*_{dR}(X_\eta) \quad \text{and} \quad \mathbf{R}\Gamma(X, \mathcal{E}_{MW}) \simeq E_{MW}(X) \,. \tag{3.2.5.3}$$

Theorem 3.2.6. *There is a specialisation map*

$$sp : j_*\mathcal{E}_{dR} \longrightarrow \mathcal{E}_{MW}$$

in $D_{\mathbf{A}^1}(S, K)$ which is compatible with cup product, and induces isomorphisms

$$\mathbf{R}\Gamma(X_\eta, \mathcal{E}_{dR}) \xrightarrow{sp_X} \mathbf{R}\Gamma(X, \mathcal{E}_{MW}) \quad \textit{and} \quad \mathbf{R}\Gamma_c(X_\eta, \mathcal{E}_{dR}) \xrightarrow{sp_{X,c}} \mathbf{R}\Gamma_c(X, \mathcal{E}_{MW})$$

in $D(K)$ for any smooth S-scheme X such that $j_*\mathcal{E}_{dR}(X)$ has a strong dual in $D_{\mathbf{A}^1}(S, j_*\mathcal{E}_{dR})$ (e.g. X might be projective or the complement of a relative strict normal crossings divisor in a smooth and projective S-scheme).

Proof. Apply Theorem 2.6.5 to (3.2.5.2) to get directly the map sp from $j_*\mathcal{E}_{dR}$ to $\mathcal{E}_{MW}$ and the isomorphism sp_X.

We also obtain isomorphisms

$$\mathbf{R}\Gamma_c(X, j_*\mathcal{E}_{dR}) \simeq \mathbf{R}\Gamma_c(X, \mathcal{E}_{MW}) \,.$$

Using the fact $j_*\mathcal{E}_{dR}(X)$ has a strong dual in $D_{\mathbf{A}^1}(S, j_*\mathcal{E}_{dR})$, we have the following computations (we assume X/S is of dimension d).

$$\begin{aligned}\mathbf{R}\Gamma_c(X, j_*\mathcal{E}_{dR}) &\simeq \mathbf{R}\Gamma(X, j_*\mathcal{E}_{dR}(-d)[-2d])^\vee \\ &\simeq \mathbf{R}\Gamma(X_\eta, \mathcal{E}_{dR}(-d)[-2d])^\vee \\ &\simeq \mathbf{R}\Gamma_c(X_\eta, \mathcal{E}_{dR})\end{aligned}$$

These identifications give the expected isomorphism $sp_{X,c}$. □

Corollary 3.2.7. *For any non empty smooth S-scheme X with empty special fiber, $\underline{\mathbf{Q}}(X)$ does not have any strong dual in $DM_{\mathrm{B}}(S)$.*

Proof. Given such an S-scheme X, it is clear that the specialisation map

$$\mathbf{R}\Gamma(X_\eta, \mathcal{E}_{dR}) \xrightarrow{sp_X} \mathbf{R}\Gamma(X, \mathcal{E}_{MW}) = 0$$

is not an isomorphism. But if $\underline{\mathbf{Q}}(X)$ had a strong dual in $DM_{\mathrm{B}}(S)$, then $j_*\mathcal{E}_{dR}(X)$ would have a strong dual in $D_{\mathbf{A}^1}(S, j_*\mathcal{E}_{dR})$ as well, so that, by virtue of Theorem 3.2.6, sp_X would be an isomorphism in $D(K)$. □

3.2.8. Recall from [Ayo07, Rön05, CD09b] that we have two pairs of adjoint functors

$$j_! : D^{\mathit{eff}}_{\mathbf{A}^1}(\eta, \mathbf{Q}) \leftrightarrows D^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{Q}) : j^* \tag{3.2.8.1}$$

$$\mathbf{L}i^* : D^{\mathit{eff}}_{\mathbf{A}^1}(S, \mathbf{Q}) \leftrightarrows D^{\mathit{eff}}_{\mathbf{A}^1}(s, \mathbf{Q}) : i_* \tag{3.2.8.2}$$

such that $j_!$ and i_* are fully faithful, and such that for any object M of $D^{\mathit{eff}}_{\mathbf{A}^1}(S,K)$, there is a canonical distinguished triangle:

$$j_!j^*(M) \longrightarrow M \longrightarrow i_*\mathbf{L}i^*(M) \longrightarrow j_!j^*(M)[1]\,. \tag{3.2.8.3}$$

As we obviously have $j^*(E_{MW}) = 0$ (this just means the Monsky-Washnitzer cohomology of an affine smooth V-scheme with empty special fiber is trivial), we deduce that

$$E_{MW} \simeq i_*\mathbf{L}i^*(E_{MW})\,. \tag{3.2.8.4}$$

Let X be a smooth affine k-scheme. Using [Ara01, Theorem 1.3.1], there exists a smooth and affine V-scheme $Y = \operatorname{Spec}(A)$ such that $X = Y_s$. In other words, we get $\mathbf{Q}(X) = \mathbf{L}i^*\mathbf{Q}(Y)$. This leads to the following computations.

$$\begin{aligned}
\mathbf{R}\Gamma(X, \mathbf{L}i^*E_{MW}) &\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(X), \mathbf{L}i^*E_{MW}) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\mathbf{L}i^*\mathbf{Q}(Y), \mathbf{L}i^*E_{MW}) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(Y), i_*\mathbf{L}i^*E_{MW}) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\mathbf{Q}(Y), E_{MW}) \\
&\simeq \mathbf{R}\Gamma(Y, E_{MW}) \\
&\simeq E_{MW}(Y)
\end{aligned} \tag{3.2.8.5}$$

Note this isomorphism is functorial with respect to Y, X being identified with Y_s.

The cohomology theory represented by $\mathbf{L}i^*E_{MW}$ in $D^{\mathit{eff}}_{\mathbf{A}^1}(s,\mathbf{Q})$ can be described as a stable cohomology theory as follows. The main difficulty for this is to represent it by a sheaf of commutative differential graded K-algebras. This is achieved by having a closer look at the definition of the functor $\mathbf{L}i^*$ of (3.2.8.2): this is the total left derived functor of the functor

$$i^* : \mathit{Comp}(\mathit{Sh}(\mathit{Sm}/S,\mathbf{Q})) \longrightarrow \mathit{Comp}(\mathit{Sh}(\mathit{Sm}/s,\mathbf{Q})) \tag{3.2.8.6}$$

which preserves colimits and sends $\mathbf{Q}(X)$ to $\mathbf{Q}(X_s)$. The functor (3.2.8.6) is a left Quillen functor with respect to the model structures defined by Proposition 1.1.15. Hence $\mathbf{L}i^*E_{MW}$ is defined by applying (3.2.8.6) to a $\mathfrak{V}$-cofibrant resolution of E_{MW}, where $\mathfrak{V}$ is the category of smooth and affine V-schemes. We can consider a quasi-isomorphism $p : E'_{MW} \longrightarrow E_{MW}$, with E'_{MW} a commutative monoid which is $\mathfrak{V}$-cofibrant as a complex of sheaves (using the model structure of [Lur07, proposition 4.3.21], whose assumptions are trivially checked in the $\mathbf{Q}$-linear setting). We then put $E_{rig} = i^*E'_{MW}$. By definition, we have a canonical isomorphism

$$E_{rig} \simeq \mathbf{L}i^*E_{MW}\,.$$

We will call E_{rig} the *rigid cohomology complex*. By construction, for any smooth affine k-scheme X, we have an isomorphism

$$\mathbf{R}\mathrm{Hom}(\mathbf{Q}(X), \mathbf{L}i^*E_{MW}) \simeq E_{rig}(X)\,. \tag{3.2.8.7}$$

Proposition 3.2.9. *For any smooth affine k-scheme X, there is a functorial isomorphism*

$$H^n(E_{rig}(X)) \simeq H^n_{rig}(X/K)\,,$$

where $H^n_{rig}(X/K)$ denotes Berthelot's rigid cohomology of X. This comparison map is compatible with cup product.

Proof. We know that, given a smooth affine S-scheme Y, there is a functorial isomorphism

$$H^n_{MW}(Y) \simeq H^n_{rig}(Y_s/K)\,,$$

which is compatible with cup product; see [Ber97b, Proposition 1.10]. As, by definition, $E_{rig} = \mathbf{L}i^* E_{MW}$, given a smooth affine k-scheme X, once a smooth affine S-scheme Y with special fiber isomorphic to X is chosen, we obtain from the isomorphims (3.2.8.5) that

$$H^n(E_{rig}(X)) \simeq H^n_{rig}(X/K)\,.$$

It remains to prove this isomorphism is independent of the lift Y, and is functorial in X. Let $f : X \longrightarrow X'$ be a morphism of smooth affine k-schemes. Choose two smooth affine S-schemes Y and Y' endowed with two isomorphisms $Y_s \simeq X$ and $Y'_s \simeq X'$ (which exist, thanks to [Ara01, Theorem 1.3.1]). By virtue of [Ara01, Theorem 2.1.3], there exists a commutative diagram of S-schemes

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[dl, "i"'] \arrow[d, "i_\varepsilon"] & X' \arrow[d, "i'"] \\ Y & Y_\varepsilon \arrow[l, "\varepsilon"] \arrow[r, "g"'] & Y' \end{tikzcd}$$

with i (resp. i_ε, resp. i') being a closed immersion which identifies X (resp. X, resp. X') with the special fiber of Y (resp. of Y_ϵ, resp. of Y'), and with $\varepsilon : Y_\varepsilon \longrightarrow Y$ étale and inducing the identity on the special fibers. Then the naturality of the isomorphisms (3.2.8.5) gives the following commutative diagram

$$\begin{tikzcd} & E_{rig}(X) & E_{rig}(X') \arrow[l, "f^*"'] \\ E_{MW}(Y) \arrow[ur, "\simeq"] \arrow[r, "\varepsilon^*"'] & E_{MW}(Y_\varepsilon) \arrow[u, "\simeq"'] & E_{MW}(Y') \arrow[l, "g^*"] \arrow[u, "\simeq"'] \end{tikzcd}$$

in which the non-horizontal maps are the canonical isomorphisms. $\square$

Theorem 3.2.10. *The sheaf of commutative differential graded algebras E_{rig} is a mixed Weil cohomology on smooth k-schemes.*

Proof. As E_{rig} is fibrant by definition, it is $\mathbf{A}^1$-homotopy invariant and has the B.-G. property. Using Theorem 3.2.3 and the comparison isomorphisms (3.2.8.5), we see that E_{rig} is a stable cohomology theory. It thus remains to prove the Künneth Formula. This comes immediately from the comparison with Berthelot's rigid cohomology (Proposition 3.2.9), the latter being known to satisfy the Künneth formula; see [Ber97a]. $\square$

Scholium 3.2.11. Let us denote by $\mathcal{E}_{rig}$ the commutative ring spectrum associated to the mixed Weil cohomology E_{rig}.

Theorem 3.2.6 can be made a little more precise in the following way. Recall from [Rön05, Ayo07, CD09b] that we have a pair of adjoint triangulated functors

(3.2.11.1) $$\mathbf{L}i^* : D_{\mathbf{A}^1}(S, \mathbf{Q}) \leftrightarrows D_{\mathbf{A}^1}(s, \mathbf{Q}) : i_*$$

satisfying the following properties.

(i) The functor $\mathbf{L}i^*$ is symmetric monoidal and preserves Tate twists.

(ii) For any smooth S-scheme X, we have $\mathbf{L}i^*\underline{\mathbf{Q}}(X) = \underline{\mathbf{Q}}(X_s)$.
(iii) The functor i_* is fully faithful.
(iv) For any objects M of $D_{\mathbf{A}^1}(S, \mathbf{Q})$ and any object N of $D_{\mathbf{A}^1}(s, \mathbf{Q})$, we have a canonical isomorphism

$$M \otimes^{\mathbf{L}}_{\mathbf{Q}} i_*(N) \simeq i_*(\mathbf{L}i^*(M) \otimes^{\mathbf{L}}_{\mathbf{Q}} N)\,. \tag{3.2.11.2}$$

It follows from property (ii) and the definition of $\mathcal{E}_{rig}$ that we have an isomorphism

$$\mathcal{E}_{MW} \simeq i_*\mathcal{E}_{rig}\,, \tag{3.2.11.3}$$

so that we have a specialization map

$$sp : j_*\mathcal{E}_{dR} \longrightarrow i_*\mathcal{E}_{rig} \tag{3.2.11.4}$$

in $D_{\mathbf{A}^1}(S, K)$. Moreover, we obtain from properties (i) and (iv) the following identifications for a smooth S-scheme X of pure dimension d.

$$\begin{aligned}
\mathbf{R}\Gamma_c(X, \mathcal{E}_{MW}) &\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\underline{\mathbf{Q}}, \mathcal{E}_{MW} \otimes^{\mathbf{L}}_{\mathbf{Q}} \underline{\mathbf{Q}}(X)(-d)[-2d]) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\underline{\mathbf{Q}}, i_*(\mathcal{E}_{rig}) \otimes^{\mathbf{L}}_{\mathbf{Q}} \underline{\mathbf{Q}}(X)(-d)[-2d]) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\underline{\mathbf{Q}}, i_*(\mathcal{E}_{rig} \otimes^{\mathbf{L}}_{\mathbf{Q}} i^*(\underline{\mathbf{Q}}(X)))(-d)[-2d]) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(i^*(\underline{\mathbf{Q}}), \mathcal{E}_{rig} \otimes^{\mathbf{L}}_{\mathbf{Q}} \underline{\mathbf{Q}}(X_s)(-d)[-2d]) \\
&\simeq \mathbf{R}\mathrm{Hom}_{\mathbf{Q}}(\underline{\mathbf{Q}}, \mathcal{E}_{rig} \otimes^{\mathbf{L}}_{\mathbf{Q}} \underline{\mathbf{Q}}(X_s)(-d)[-2d]) \\
&\simeq \mathbf{R}\Gamma_c(X_s, \mathcal{E}_{rig})
\end{aligned}$$

By virtue of Theorem 3.2.6, the specialisation map (3.2.11.4) induces isomorphisms

$$\mathbf{R}\Gamma(X_\eta, \mathcal{E}_{dR}) \xrightarrow{sp_X} \mathbf{R}\Gamma(X_s, \mathcal{E}_{rig}) \quad \text{and} \quad \mathbf{R}\Gamma_c(X_\eta, \mathcal{E}_{dR}) \xrightarrow{sp_{X_s,c}} \mathbf{R}\Gamma_c(X_s, \mathcal{E}_{rig})$$

in $D(K)$ for any smooth S-scheme X such that $j_*\mathcal{E}_{dR}(X)$ has a strong dual in $D_{\mathbf{A}^1}(S, j_*\mathcal{E}_{dR})$.

It can be proven that E_{rig} is quasi-isomorphic to the restriction of Besser's rigid complex (see [Bes00, Definition 4.13]) to the category of smooth k-schemes. In other words, the object $\mathcal{E}_{rig}$ represents Berthelot's rigid cohomology in $D_{\mathbf{A}^1}(s, \mathbf{Q})$. In the case where X is smooth and projective over S, using the comparison isomorphism relating rigid cohomology and crystalline cohomology (see [Ber97b, Proposition 1.9]), Theorem 3.2.6 gives back the comparison isomorphism of Berthelot and Ogus [BO83].

3.3. Étale cohomology.

3.3.1. For sake of completeness, we will finish by explaining how ℓ-adic cohomology fits in the picture of mixed Weil cohomologies as they are defined here.

Consider a countable perfect field k, and choose a separable closure $\overline{k}$ of k. For a smooth k-scheme X, write $\overline{X} = X \otimes_k \overline{k}$. Let ℓ be a prime which is distinct from the characteristic of k.

Deligne [Del80] defines for any smooth k-scheme X a commutative differential graded $\mathbf{Q}_\ell$-algebra which computes the ℓ-adic cohomology of $\overline{X}$. We will modify slightly some steps of his construction to ensure its functoriality.

3.3.2. Consider a pro-simplicial set $\mathbf{X} =$ "$\varprojlim$" X_α. We can then define its singular cohomology with coefficients in $\mathbf{Z}/\ell^n$ by the formula

$$H^i(\mathbf{X}, \mathbf{Z}/\ell^n) = \varinjlim H^i(X_\alpha, \mathbf{Z}/\ell^n)\,. \tag{3.3.2.1}$$

We will say that $\mathbf{X}$ is *essentially ℓ-finite* if the groups $H^i(\mathbf{X}, \mathbf{Z}/\ell^n)$ are finite.

For an essentially ℓ-finite pro-simplicial set $\mathbf{X}$, formula (5.2.1.7) of [Del80] defines a commutative differential graded $\mathbf{Q}_\ell$-algebra $A(\mathbf{X})$ such that

$$H^i(A(\mathbf{X})) = \mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} \varprojlim_n H^i(\mathbf{X}, \mathbf{Z}/\ell^n)\,. \tag{3.3.2.2}$$

This construction is (contravariantly) functorial in $\mathbf{X}$.

3.3.3. For an étale surjective morphism $X' \longrightarrow X$, define $\check{C}(X'/X)$ to be the Čech simplicial scheme defined by the formula

$$\check{C}(X'/X)_n = \underbrace{X' \times_X \cdots \times_X X'}_{n+1 \text{ times}}\,.$$

Note that the map $\check{C}(X'/X) \longrightarrow X$ is an étale hypercovering.

Given a smooth $\overline{k}$-scheme X, define an *étale fundamental system* $\mathcal{X}$ of X to be a tower of morphisms of smooth $\overline{k}$-schemes indexed by integers $\alpha \geqslant 0$

$$\mathcal{X} = \Big[\cdots \longrightarrow X_{\alpha+1} \longrightarrow X_\alpha \longrightarrow \cdots \longrightarrow X_1 \longrightarrow X_0 = X\Big]$$

such that $X_\alpha \longrightarrow X$ is étale surjective for all $\alpha \geqslant 0$, and such that any étale surjective map $U \longrightarrow X$ factors through X_α for α big enough. Such an étale fundamental system of X defines a pro-simplicial scheme "$\varprojlim$" $\check{C}(X_\alpha/X)$, whence a pro-simplicial set

$$\pi(\mathcal{X}) = \text{“}\varprojlim\text{”}\, \pi_0(\check{C}(X_\alpha/X)) \tag{3.3.3.1}$$

which is essentially ℓ-finite, and such that there is a canonical isomorphism

$$H^i(A(\pi(\mathcal{X}))) \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathbf{Q}_\ell) \tag{3.3.3.2}$$

(see [Del80, 5.2.2]). Given a non-empty finite family of étale fundamental systems $\mathcal{X} = \{\mathcal{X}^1, \ldots, \mathcal{X}^n\}$, we define an étale fundamental system $\mathcal{X}_{tot}$ whose α^{th} stage is defined as the fiber product of the α^{th} stages of the $\mathcal{X}^i$'s over X. Given a non-empty subset $\mathcal{X}'$ of $\mathcal{X}$, it can be described as $\mathcal{X}' = \{\mathcal{X}^{i_1}, \ldots, \mathcal{X}^{i_m}\}$, with $i_k \neq i_l$ whenever $k \neq l$, with $m \leqslant n$. We then have a canonical morphism of pro-schemes $\mathcal{X}_{tot} \longrightarrow \mathcal{X}'_{tot}$ induced by the projections $\mathcal{X}_{tot} \longrightarrow \mathcal{X}^{i_k}$. Taking the filtering projective limit of all the pro-simplicial sets $\pi(\mathcal{X}_{tot})$, where $\mathcal{X}$ ranges over the non-empty finite families of étale fundamental systems of X, defines a pro-simplicial set. We define

$$A(X) = \varinjlim_{\mathcal{X}} A(\pi(\mathcal{X}_{tot}))\,. \tag{3.3.3.3}$$

As filtering colimits are exacts, we deduce from (3.3.3.2) that we have a canonical isomorphism

$$H^i(A(X)) = \varinjlim_{\mathcal{X}} H^i(A(\pi(\mathcal{X}_{tot}))) \xrightarrow{\sim} H^i_{\text{ét}}(X, \mathbf{Q}_\ell)\,. \tag{3.3.3.4}$$

We claim formula (3.3.3.3) defines a presheaf of commutative differential graded $\mathbf{Q}_\ell$-algebras A on the category of smooth $\overline{k}$-schemes. Consider a morphism $f : X \longrightarrow Y$ of smooth $\overline{k}$-schemes. Any non-empty finite family of étale fundamental systems $\mathcal{Y} = \{\mathcal{Y}^1, \ldots, \mathcal{Y}^n\}$ of Y defines by pullback a non-empty finite family of étale fundamental systems $f^*(\mathcal{Y}) = \{X \times_Y \mathcal{Y}^1, \ldots, X \times_Y \mathcal{Y}^n\}$ of X, with a canonical morphism of pro-schemes $f^*(\mathcal{Y})_{tot} \longrightarrow \mathcal{Y}_{tot}$. This induces a map

$$A(\pi(\mathcal{Y}_{tot})) \longrightarrow A(\pi(f^*(\mathcal{Y})_{tot})) \longrightarrow A(X)\,.$$

By passing to the colimit of the $A(\pi(\mathcal{Y}_{tot}))$'s, we get the expected map

$$f^* : A(Y) \longrightarrow A(X) .$$

3.3.4. Define a presheaf of commutative differential graded $\mathbf{Q}_\ell$-algebras $E_{\text{ét},\ell}$ on Sm/k by the formula

$$E_{\text{ét},\ell}(X) = A(\overline{X}) . \tag{3.3.4.1}$$

Then one has

$$H^n(E_{\text{ét},\ell}(X)) = H^n_{\text{ét}}(\overline{X}, \mathbf{Q}_\ell) . \tag{3.3.4.2}$$

In particular, $E_{\text{ét},\ell}$ satisfies étale descent, whence it has the B.-G.-property. The well known properties of étale cohomology proved by Artin and Grothendieck thus imply:

Theorem 3.3.5. *$E_{\text{ét},\ell}$ is a mixed Weil theory over smooth k-schemes.*

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LAGA, CNRS (UMR 7539), Institut Galilée, Université Paris 13, Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

E-mail address: cisinski@math.univ-paris13.fr

URL: http://www.math.univ-paris13.fr/~cisinski/

LAGA, CNRS (UMR 7539), Institut Galilée, Université Paris 13, Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

E-mail address: deglise@math.univ-paris13.fr

URL: http://www.math.univ-paris13.fr/~deglise/