Beilinson motives and the six functors formalism

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Notations

We denote by $\mathscr{S}$ the category of excellent noetherian scheme of finite dimension. Without precision, schemes are considered to be objects of this category.

Monoidal categories (resp. functors) are always assumed to be symmetric.

1. Introduction

Let $\mathscr{T}ri^{\otimes}$ be the 2-category of triangulated monoidal categories, with weakly monoidal triangulated natural transformations as 2-morphisms.

Definition 1.1. A triangulated category satisfying the six functor formalism consists of the following data:

(1) For any scheme S, we consider a triangulated closed monoidal category $\mathcal{T}(S)$, with unit object $\mathbb{1}_S$.

(2) For any morphism $f : T \to S$, a pair of adjoint functors

$$f^* : \mathcal{T}(T) \to \mathcal{T}(S) : f_*$$

such that f^* is monoidal and $S \mapsto \mathcal{T}(S), f \mapsto f^*$ is a contravariant 2-functor from $\mathscr{S}$ to $\mathscr{T}ri^{\otimes}$.

(3) For any separated morphism of finite type $f : T \to S$, a pair of adjoint functors

$$f_! : \mathcal{T}(T) \to \mathcal{T}(S) : f^!$$

such that $S \mapsto \mathcal{T}(S), f \mapsto f_!$ is a 2-functor from the category of schemes with morphisms separated of finite type to $\mathscr{T}ri^{\otimes}$.

These data are assumed to satisty the following properties:

(4) For any separated morphism of finite type, there exists a natural transformation $f_! \to f_*$ compatible with composition which is an isomorphism when f is proper.

Let S be a scheme and $p : \mathbb{P}^1_S \to S$ (resp. $s : S \to \mathbb{P}^1_S$) be the canonical projection (resp. infinite section) of the projective line over S. Define the Tate twist as:

$$\mathbb{1}_S(1) = s^* p^!(\mathbb{1}_S)[-2].$$

For any integer $n \geq 0$, we let $\mathbb{1}_S(n)$ be the n-th tensor power of $\mathbb{1}_S(1)$ and for any object M of $\mathcal{T}(S)$, we put $M(n) = M \otimes \mathbb{1}_S(n)$.

(5) For any smooth quasi-projective morphism f of constant relative dimension n, there exists a natural isomorphism $f^! \to f^*(n)[2n]$ compatible with composition.

(6) For any cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & \Delta & \downarrow{\scriptstyle g} \\ Y & \xrightarrow[f]{} & X, \end{array}$$

in which f is separated of finite type, there exists natural isomorphisms:

$$g^* f_! \longrightarrow f'_! g'^*,$$
$$g'_* f'^! \longrightarrow f^! g_*.$$

(7) For any separated morphism of finite type $f : Y \to X$ in $\mathscr{S}$, there exist natural isomorphisms

$$(f_! K) \otimes_X L \longrightarrow f_!(K \otimes_X f^* L)\,,$$
$$\underline{\mathrm{Hom}}_X(f_!(L), K) \longrightarrow f_* \underline{\mathrm{Hom}}_Y(L, f^!(K))\,,$$
$$f^! \underline{\mathrm{Hom}}_X(L, M) \longrightarrow \underline{\mathrm{Hom}}_Y(f^*(L), f^!(M))\,.$$

The first example of such a formalism was given in [SGA4]. More recently, the six funtors formalism has been constructed by J. Ayoub in [Ayo07] for the stable homotopy category of schemes $SH(S)$ defined by F. Morel and V. Voevodsky.[1]

In the next section, we propose a definition of a rational triangulated category which satisfies the six functors formalism and which we propose as a category of triangulated mixed motives. The justification for this claim is that our category extends the definition of Voevodsky known over (perfect) fields. We refer the interested reader to [CD09] for more details on our construction.

2. Beilinson motives

2.1. Recall that for any scheme S, there exists a ring spectrum $\mathbf{K}_S$ in $SH(S)$ such that:

- For any morphism of schemes $f : T \to S$,

(2.1.1) $$f^*(\mathbf{K}_S) = \mathbf{K}_T.$$

- When S is regular, for any integer n,

(2.1.2) $$\mathrm{Hom}(\Sigma^\infty X_+[n], \mathbf{K}_S) = K_n(S)$$

where the right hand side denotes Quillen algebraic K-theory.

Let us denote by $SH(S, \mathbb{Q})$ the rationalisation of the stable homotopy category.[2] We denote by $\mathbf{K}_S^{\mathbb{Q}}$ the object defined by the above spectrum in $SH(S, \mathbb{Q})$. The idea of the following definition comes from topology:

Definition 2.2. Consider the notations above.

[1] In the stable homotopy category though, one should be aware that in property (5), one has to replace the twist by a tensor product with a *Thom space*.

[2] The category with same objects but the Hom groups are tensored with $\mathbb{Q}$.

(1) We say an object $\mathbf{E}$ of $SH(S,\mathbb{Q})$ is $\mathbf{K}$-acyclic if $\mathbf{E}\otimes\mathbf{K}_S^{\mathbb{Q}}=0$.
(2) We say a morphism $f:\mathbf{E}\to\mathbf{F}$ in $SH(S,\mathbb{Q})$ is a $\mathbf{K}$-equivalence if a cone of f is $\mathbf{K}$-acyclic.
(3) We say an object M of $SH(S,\mathbb{Q})$ is a *Beilinson motive* if for all $\mathbf{K}$-acyclic spectrum $\mathbf{E}$, $\mathrm{Hom}(\mathbf{E},M)=0$.

We let $DM_{\text{Б}}(S)$ be the full subcategory of $SH(S,\mathbb{Q})$ made by the Beilinson motives.

According to the theory of Bousfield localization, the category $DM_{\text{Б}}(S)$ can be described as the localization of the category $SH(S,\mathbb{Q})$ with respect to $\mathbf{K}$-equivalences. Moreover, we get an adjunction of triangulated categories:

$$L_{\text{Б}}:SH(S,\mathbb{Q})\leftrightarrows DM_{\text{Б}}(S):\mathcal{O}_{\text{Б}}$$

where $\mathcal{O}_{\text{Б}}$ is the natural forgetful functors. As the $\mathbf{K}$-equivalences are stable by base change (using (2.1.1)) and tensor product, we get using the main result of [Ayo07] the following theorem:

Theorem 2.3 ([CD09, §13.2])**.** *The triangulated category $DM_{\text{Б}}$ satisfies the six functors formalism.*

Note moreover that $L_{\text{Б}}$ is monoidal and commutes with operations such as f^* and $f_!$.

2.4. Let S be any regular scheme. We will consider on $K_n(S)\otimes\mathbb{Q}$ the γ-filtration together with its graded pieces which give a canonical decomposition:

(2.4.1) $$K_n(S)\otimes\mathbb{Q}=\bigoplus_{i\in\mathbb{N}}Gr_\gamma^i\big(K_n(S)\otimes\mathbb{Q}\big).$$

We will use the following theorem of J. Riou:

Theorem 2.5 ([Rio06])**.** *Let S be a scheme. There exists a canonical decomposition in $SH(S,\mathbb{Q})$ of the form:*

(2.5.1) $$\mathbf{K}_S=\bigoplus_{i\in\mathbb{Z}}K_S^{(i)}$$

stable by base change and such that, whenever S is regular, for any integer $n\in\mathbb{Z}$, the induced decomposition on the cohomology represented by $\mathbf{K}_S$ coincide with (2.4.1) *through the identification* (2.1.2).

According to Riou, we define the Beilinson spectrum over any scheme S as $\mathbf{H}_{\text{Б},S}=\mathbf{K}_S^{(0)}$. Note that Bott periodicity for K-theory implies that (2.5.1) can be rewritten as:

(2.5.2) $$\mathbf{K}_S=\bigoplus_{i\in\mathbb{Z}}\mathbf{H}_{\text{Б},S}(i)[2i]$$

where $\mathbf{H}_{\text{Б},S}(i)$ is the i-th Tate twist in $SH(S,\mathbb{Q})$.

The following result is a key point of our construction:

Proposition 2.6 ([CD09, 13.1.5, 13.1.6])**.** *The spectrum* $\mathbf{H}_{\text{Б},S}$ *admits a ring structure in* $SH(S,\mathbb{Q})$ *such that its multiplication map*

$$\mu : \mathbf{H}_{\text{Б},S} \wedge \mathbf{H}_{\text{Б},S} \to \mathbf{H}_{\text{Б},S}$$

is an isomorphism.

2.7. Recall that the category $SH(S,\mathbb{Q})$ is the homotopy category of a monoidal model category $Sp(S,\mathbb{Q})$. One deduces from the previous theorem that $\mathbf{H}_{\text{Б},S}$ there exists a (commutative) monoid $\bar{\mathbf{H}}_{\text{Б},S}$ in $Sp(S,\mathbb{Q})$ which coincides in $SH(S,\mathbb{Q})$ with $\mathbf{H}_{\text{Б},S}$.[3] This allows to define the triangulated category $\mathbf{H}_{\text{Б},S}-\mathrm{mod}$ of $\mathbf{H}_{\text{Б},S}$-modules.[4] By construction, we get a canonical adjunction:

$$L_{\mathbf{H}_{\text{Б}}} : SH(S,\mathbb{Q}) \leftrightarrows \mathbf{H}_{\text{Б},S}-\mathrm{mod} : \mathcal{O}_{\mathbf{H}_{\text{Б}}}.$$

such that $L_{\mathbf{H}_{\text{Б}}}(\mathbf{E}) = \mathbf{E} \wedge \mathbf{H}_{\text{Б},S}$. As a corollary of the previous result, we get the following theorem:

Theorem 2.8 ([CD09, 13.2.9])**.** *Consider the notations above. There exists a canonical functor* $\varphi : DM_{\text{Б}}(S) \to \mathbf{H}_{\text{Б},S}-\mathrm{mod}$ *which fits into the commutative diagram:*

$$\begin{array}{ccc} SH(S,\mathbb{Q}) & \xrightarrow{\quad L_{\mathbf{H}_{\text{Б}}} \quad} & \mathbf{H}_{\text{Б},S}-\mathrm{mod}\,. \\ & {}_{L_{\text{Б}}}\searrow \qquad \nearrow_{\varphi} & \\ & DM_{\text{Б}}(S) & \end{array}$$

Moreover, φ *is an equivalence of triangulated monoidal categories.*

Corollary 2.9. *For any regualr scheme* S *and any couple of integers* $(n,p) \in \mathbb{Z}^2$*, one has:*

$$\mathrm{Hom}_{DM_{\text{Б}}(S)}(\mathbb{1}_S, \mathbb{1}_S(p)[n]) = K^{(p)}_{2p-n}(S).$$

For a non necessarily regular scheme S, we will define *Beilinson motivic cohomology* of S as the left hand side in the above identification.

Example 2.10. Let X be a smooth S-scheme. Define the (homological) motive of X/S as $M(X) = L_{\text{Б}}(\Sigma^\infty X_+)$.

If in addition, X/S is projective of constant dimension d, then one shows $M(X)$ is strongly dualisable with strong dual $M(X)(-d)[-2d]$.

Assuming that S is regular, one can define the category $\mathcal{M}^{rat}(S)$ of Chow motives as usual. Applying the previous corollary, one gets a fully faithful functor:

$$\mathcal{M}^{rat}(S)^{op} \to DM_{\text{Б}}(S), h(X) \mapsto M(X).$$

Corollary 2.11. *Let* S *be any scheme,* $\mathbf{E}$ *be an object of* $SH(S,\mathbb{Q})$ *and* $u : \mathbf{S}^0 \to \mathbf{H}_{\text{Б},S}$ *be the unit of ring spectrum* $\mathbf{H}_{\text{Б},S}$*. Then the following conditions are equivalent:*

(i) $\mathbf{E}$ *is a Beilinson motive.*

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[3]One says also that $\mathbf{H}_{\text{Б},S}$ is a *strict* ring spectrum.

[4]One constructs according to Schwede and Shipley a model category on the category of modules over $\bar{\mathbf{H}}_{\text{Б},S}$; $\mathbf{H}_{\text{Б},S}-\mathrm{mod}$ is its homotopy category.

(ii) $\mathbf{E}$ *admits a structure of an* $\mathbf{H}_{\mathbb{B},S}$*-module in* $SH(S,\mathbb{Q})$.
(iii) *The morphism* $u \wedge Id_{\mathbf{E}} : \mathbf{E} \to \mathbf{H}_{\mathbb{B},S} \wedge \mathbf{E}$ *is an isomorphism.*

Moreover, when these conditions are satisfied, the structure of an $\mathbf{H}_{\mathbb{B},S}$*-module on* $\mathbf{E}$ *is unique.*[5]

3. Proper descent and Voevodsky motives

3.1. Consider again a scheme S.

Let us recall that Voevodsky has introduced the h-topology on the category $\mathscr{S}_S^{ft}$ of finite type S-schemes: its coverings are made of the universal topological epimorphism $f : W \to X$.[6] We let $\mathrm{Sh}_h(S,\mathbb{Q})$ be the category of sheaves of $\mathbb{Q}$-vector spaces on $\mathscr{S}_S^{ft}$ for the h-topology.

Voevodsky then defines the category of (rational) h-motives $\underline{DM}_h^{eff}(S,\mathbb{Q})$ as the $\mathbb{A}^1$-localization of the derived category of the abelian category $\mathrm{Sh}_h(S,\mathbb{Q})$. Any S-scheme X of finite type defines an object of $\mathrm{Sh}_h(S,\mathbb{Q})$ denoted by $\mathbb{Q}^h(X)$. We then define the Tate twist $\mathbb{Q}_S^h(1)$ in $\underline{DM}_h^{eff}(S,\mathbb{Q})$ as the cokernel of the split monomorphism $\mathbb{Q}^h(S) \to \mathbb{Q}^h(\mathbb{P}_S^1)$ defined by the inclusion of the infinite S-point.

In fact, one can show that $\underline{DM}_h(S,\mathbb{Q})$ is the homotopy category of a suitable Quillen model category on the category of complexes on $\mathrm{Sh}_h(S,\mathbb{Q})$. Moreover, this model category is monoidal: it defines a (derived) closed monoidal structure on $\underline{DM}_h(S,\mathbb{Q})$. Moreover, we can define the so called $\mathbb{P}^1$*-stabilisation* of this category: this is the universal homotopy category $\underline{DM}_h(S,\mathbb{Q})$ of a monoidal model category given with a left derived monoidal functor

$$\Sigma^\infty : \underline{DM}_h^{eff}(S,\mathbb{Q}) \longrightarrow \underline{DM}_h(S,\mathbb{Q})$$

such that $\Sigma^\infty \mathbb{Q}_S^h(1)$ is $\otimes$-invertible.

One can recognize in this construction the steps needed to define the stable homotopy category $SH(S)$: in the former, one simply starts from complexes of $\mathbb{Q}$-sheaves for the h-topology on $\mathscr{S}_S^{ft}$ instead of simplicial sheaves of sets for the Nisnevich topology on smooth S-schemes. The analogy between the tow constructions allow to define a canonical triangulated monoidal functor:

$$a_h : SH(S) \to \underline{DM}_h(S,\mathbb{Q})$$

which factors through the rational stable homotopy category. One of the main theorem of [CD09] is the following:

Theorem 3.2. *There exists a unique functor* $\psi : DM_{\mathbb{B}}(S) \to \underline{DM}_h(S,\mathbb{Q})$ *which makes the following diagram (essentially) commutative:*

$$\begin{array}{ccc} SH(S,\mathbb{Q}) & \xrightarrow{\quad a_h \quad} & \underline{DM}_h(S,\mathbb{Q}). \\ & {\scriptstyle L_{\mathbb{B}}} \searrow \qquad \nearrow {\scriptstyle \psi} & \\ & DM_{\mathbb{B}}(S) & \end{array}$$

[5]And can be lifted in the monoidal category of symmetric spectra.

[6]That is the topology of X is the final topology relative to f, and this property remains true after any base change. The basic examples of such coverings: faithfully flat morphisms, proper surjective morphisms.

Moreover, ψ is fully faithful and monoidal.

In fact, ψ sends the Beilinson motive $M_S(X)$ of a smooth S-scheme X to the object $\mathbb{Q}_S^h(X)$ and the essential image of ψ is made by the localizing subcategory of the triangulated category $\underline{DM}_h(S)$ generated by the objects $\mathbb{Q}_S^h(X)(i)$ for a smooth S-scheme X and an integer $i \in \mathbb{Z}$.

3.3. Consider a spectrum $\mathbf{E}$ over a scheme S. Given a scheme X/S of finite type, with structural morphism f, we define the cohomology of X with coefficients in $\mathbf{E}$ as:

$$\mathbf{E}^{n,p}(X) = \mathrm{Hom}_{SH(X,\mathbb{Q})}\big(\Sigma^\infty X_+, f^*(\mathbf{E})(p)[n]\big), (n,p) \in \mathbb{Z}^2.$$

This definition can be extended to simplicial objects of $\mathscr{S}_S^{ft}$ and defines in fact a contravariant functor.

One says that $\mathbf{E}$ satisfies h-descent if for any smooth S-scheme X and any h-cover $\pi : \mathcal{V}_\bullet \to X$ the induced morphism:

$$\pi^* : \mathbf{E}^{n,p}(X) \to \mathbf{E}^{n,p}(\mathcal{V}_\bullet)$$

is an isomorphism. One can reformulate the previous theorem by the equivalence of the following conditions for a rational spectrum $\mathbf{E}$:

(i) $\mathbf{E}$ is a Beilinson motive.

(iv) $\mathbf{E}$ satisfies h-descent.

Note in particular that Beilinson motivic cohomology satisfies h-descent – thus proper and faithfully flat descent.

References

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