AROUND THE GYSIN TRIANGLE I

FRÉDÉRIC DÉGLISE

Abstract. We define and study Gysin morphisms on mixed motives over a perfect field. Our construction extends the case of closed immersions, already known from results of Voevodsky, to arbitrary projective morphisms. We prove several classical formulas in this context, such as the projection and excess intersection formulas, and some more original ones involving residues. We give an application of this construction to duality and motive with compact support.

Introduction

Since Poincaré discovers the first instance of duality in singular homology, mathematicians slowly became aware that most of cohomology theories could be equipped with an exceptional functoriality, covariant, usually referred to as either transfer, trace or more recently Gysin morphism[1]. In homology, this kind of exceptional functoriality exists accordingly. The most famous case is the pullback on Chow groups. Motives of Voevodsky are homological: they are naturally covariant. As they modeled homology theory, they should be equipped with an exceptional functoriality, contravariant. This is what we primarily prove here for smooth schemes over a field. Further, we focus on the two fundamental properties of Gysin morphisms: their functorial nature and their compatibility with the natural functoriality, corresponding to various projection formulas. The reader can already guess the intimate relationship of this theory with the classical intersection theory.

The predecessor of our construction was to be found in the Gysin triangle defined by Voevodsky[2] for motives over a perfect field k: associated with a closed immersion $i : Z \to X$ between smooth k-schemes, Voevodsky constructs a distinguished triangle of mixed motives:

$$M(X - Z) \to M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X - Z)[1].$$

The arrow labeled i^* is the Gysin morphism associated with the closed immersion i. Because this triangle corresponds to the so-called localization long exact sequence in cohomology, fundamental in Chow and higher Chow theory, it has a central position in the theory of mixed motives. In [Dég04] and [Dég08b], we studied its naturality, which corresponds to the projection formulas mentioned in the first paragraph, for the Gysin morphism i^*. Interestingly, we discovered that these formulas had counterpart for the *residue morphism* $\partial_{X,Z}$ appearing in the Gysin triangle[3]. The main technical result of this article (see Theorem 3.9) is the functoriality property

Date: 2005 - last revision: 11/2011.

Partially supported by the ANR (grant No. ANR-07-BLAN-042).

[1] The term *transfer* is more frequently used for finite morphisms, *trace* for structural morphisms of projective smooth schemes over a field, and *Gysin morphisms* for the zero section of a vector bundle, usually understand as part of the Gysin long exact sequence.

[2] See [FSV00, chap. 5, Prop. 3.5.4].

[3] The reader is referred to section 2.4 for a summary of these results.

of the Gysin morphism i^*. But, as in the case of projection formulas, this comes with new formulas for the residue morphism. Let us quote it now:

Theorem. *Let X be a smooth k-scheme, Y (resp. Y') be a smooth closed subscheme of X of pure codimension n (resp. m). Assume the reduced scheme Z associated with $Y \cap Y'$ is smooth of pure codimension d. Put $Y_0 = Y - Z$, $Y'_0 = Y' - Z$, $X_0 = X - Y \cup Y'$.*

Then the following diagram, with i,j,k,l,i' the evident closed immersions, is commutative :

$$\begin{array}{ccccc}
M(X) & \xrightarrow{j^*} & M(Y')(m)[2m] & \xrightarrow{\partial_{X,Y'}} & M(X-Y')[1] \\
{\scriptstyle i^*}\downarrow & (1) & \downarrow{\scriptstyle k^*} & (2) & \downarrow{\scriptstyle (i')^*} \\
M(Y)(n)[2n] & \xrightarrow[l^*]{} & M(Z)(d)[2d] & \xrightarrow{\partial_{Y,Z}} & M(Y_0)(n)[2n+1] \\
 & & \downarrow{\scriptstyle \partial_{Y',Z}} & (3) & \downarrow{\scriptstyle \partial_{X_0,Y_0}} \\
 & & M(Y'_0)(m)[2m+1] & \xrightarrow[-\partial_{X_0,Y'_0}]{} & M(X_0)[2].
\end{array}$$

This theorem can be understood as follows: the commutativity of square (1) in fact gives the functoriality of the Gysin morphism (take $Y' = Z$) ; the commutativity of square (2) shows the Gysin triangle is functorial with respect to the Gysin morphism of a closed immersion. Finally the commutativity of square (3) reveals the differential nature of the residue morphism: it can be seen as an analogue of the change of variable theorem for computing the residue of differential forms.[4]

More generally, our Gysin morphism is associated with any morphism between smooth k-schemes. We go from the case of closed immersions to that of projective morphisms by a nowadays classical method[5]. Using the projective bundle formulas for motives, one easily defines the Gysin morphism for the projection of a projective bundle. As any projective morphism f can be factored as a closed immersion i followed by the projection of a projective bundle p, we can put: $f^* = p^* i^*$. The key point is to show this definition is independent of the factorization. Taking into account the theorem cited above, this reduces to prove that for any section s of the projection p, the following relation holds: $p^* s^* = 1$ (see Prop. 4.2). When the definition is correctly settled, the main properties of the general Gysin morphism follows from the particular case of closed immersions. Let us summarize them for the reader:

- functorial nature (Prop. 4.10),
- projection formula in the transversal case (Prop. 4.15),
- excess intersection formula (Prop. 4.17),
- naturality of the Gysin triangle with respect to Gysin morphisms (Prop. 4.18).

Voevodsky's motives are built in with transfers for finite morphisms, according to the action of finite correspondences. An important property of our Gysin morphisms is that, in the case of a finite morphism, they agree with these transfers (see Th. 4.20).

To end this description of the motivic Gysin morphism, we come back to the point of view at the beginning of the introduction. It was told that the existence of this exceptional functoriality was a consequence of Poincaré duality. In the end

[4]In fact, one can show that the residue morphisms of motives induces the usual residue on differential forms via De Rham realization.

[5]A model for us was the pullback on Chow groups as defined by Fulton in [Ful98].

of this work, we go on the reverse side: Poincaré duality is a consequence of the existence of the Gysin morphism[6]. In fact, we use the tensor structure on the category of mixed motives and construct duality pairings for a smooth projective k-scheme X of dimension n. Let $p : X \rightarrow \mathrm{Spec}(k)$ (resp. $\delta : X \rightarrow X \times_k X$) be the canonical projection (resp. diagonal embedding) of X/k. We obtain duality pairings (cf Theorem 4.24)

$$\eta : \mathbb{Z} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta_*} M(X)(-n)[-2n] \otimes M(X)$$

$$\epsilon : M(X) \otimes M(X)(-n)[-2n] \xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{Z}.$$

which makes $M(X)(-n)[-2n]$ a *strong dual* of $M(X)$ in the sense of Dold-Puppe (see Par. 4.22 for recall on this notion). This result implies the usual formulation of Poincaré duality: the motivic cohomology of X is isomorphic to its motivic homology via cap-product with a homological class, the *fundamental class of X/k*. But this duality result holds more universally: any motive defines both a cohomology and a homology ; the previous duality statement is valid in this generalized setting.

The meaning of this result is that the existence of the Gysin morphism is essentially equivalent to Poincaré duality when one restricts to projective smooth schemes over k (we left the precise statement to the reader).

Organization of the paper. The first section contains our general conventions as well as the description of several realization functors which will constitute our main source of examples.

Section 2 is concerned with the Gysin triangle associated with a closed immersion by Voevodsky. This part contains essentially recall for the reader of the construction of Voevodsky ([Voe02]) together with improvement we have introduced in a previous work ([Dég08b]). The first improvement is the introduction of the purity isomorphism in section 2.2 ; the uniqueness statement was not in *op. cit.* The second one is a detailed analysis of the naturality of the Gysin triangle in section 2.4. Except for this last section, we have recalled all the proofs and included many examples using realization. We hope this exposition improves earlier ones for the benefit of the reader.

Section 3 mainly contains the main theorem of this paper, which was stated at the beginning of the introduction. Its proof uses as an essential ingredient the theory of fundamental classes (section 3.1) which are induced by Gysin morphisms. In the end of section 3, we also have added examples and discussions of the realization of Gysin triangles which make use of fundamental classes.

In section 4, we develop the general Gysin morphism: section 3.1 contains essentially the proof that the definition explained above is independent of the choice of the factorization while section 4.2 states and proves the properties listed above. Section 4.3 explores duality as explained above, and shows how one can deduce a natural construction of a motive with compact support.

Further background and references. Gysin morphisms for motives were already constructed by M. Levine within his framework of mixed motives in [Lev98].[7] The treatment of Levine has many common features with ours. In comparison, our principal contribution consists in the formula involving residues, together with the excess intersection formula.

[6]Though stated in a different language, this was already observed and used in [SGA4, XVIII].

[7]Recall Levine has constructed an equivalence of triangulated monoidal categories between his category of mixed motives and the one of Voevodsky under the assumption of resolution of singularities.

The construction of Gysin morphism on cohomology – which follows from its existence on motives through realization – was also treated directly by Panin in his setting of oriented cohomologies. His theory also has many common features with ours. On the one hand, the theory of Panin is more general as it concerns general oriented cohomology theories while motives corresponds only to the case where the associated formal group law is additive. On the other hand, our setting is more general as it concerns motives rather than cohomology theories: for example, we obtain Gysin morphisms on cohomology without requiring a ring structure (see Example 1.2) and Panin does not consider residues in his setting.

The main ingredient in the proof ofTheorem 3.9 cited in the beginning of the introduction is the so called *double deformation space*. Our source of inspiration is [Ros96, sec. 10 and 13]. One of the referees of this paper made us remark that this kind of spaces were also used by Nenashev in the context of Panin's theory (see [Nen06]).

This work has been available as a preprint for a long time.[8] It has been used in [BVK08] by Barbieri-Viale and Kahn about questions of duality. Ivorra refers to it in [Ivo10] mainly concerning *motivic fundamental classes* (Def. 3.1 here). Our initial interest for the Gysin morphism was motivated by some computation in the coniveau filtration at the level of motives ; we refer the reader to [Dég11a] in this book for this subject. It was noted by one of the referees of this paper that Panin and Yagunov established a duality theorem for motives in [PY09]. One should add the precision that their proof relies on the axiom that Gysin morphisms exists for motives (it is called the *transfer axiom*). The reference given in Example 1 of *op. cit.* is not sufficient for proving this axiom and we note that our construction gives precisely what they need.

We have extended the considerations of the present paper in a more general setting in [Dég08a]: the base can be arbitrary and we work in an abstract setting which allows to consider both motives and MGL-modules – the latter corresponds to generalized oriented cohomologies, see *loc. cit.* for details. The present version is still useful as the proofs are much simpler. Let us mention also the fundamental work [Ayo07] of Ayoub on cross functors. It yields Gysin morphisms through a classical procedure (dating back to [SGA4]). However, one has to take care about questions of orientation which are not treated by Ayoub (a.k.a. Thom isomorphisms). This is done in [CD09b]. On the other hand, the excess intersection formula, as well as formulas involving residues do not follow directly from the 6 functors formalism but from the analysis done here.

A final word concerning Poincaré duality: it was well known that strong duality for motives of smooth projective k-schemes was a consequence of the construction by Voevodsky of a $\otimes$-functor from Chow motives to geometric motives (see [FSV00, chap. 5, 2.1.4]). On the other hand, our direct proof of duality shows the existence of this functor – see Example 4.25(1) – without using the theory of Friedlander and Lawson on moving cycles ([FL98]).[9] Let us mention also that the new idea in our definition of the motive with compact support of a smooth k-scheme is that the Gysin morphism of the diagonal allows to construct a comparison functor from the motive with compact support to the usual motive (see property (iv) after Def. 4.27) – this idea was already used in [CD09a]. Compared to other versions of motive with compact support, one by Voevodsky in [FSV00, chap. 5, §4] and the other

[8]It first appears on the preprint server of the LAGA in 2005.

[9]Explicitly: the proof of Prop. 2.1.4 of [FSV00, chap. 5] refers to [FSV00, chap. 4, 7.1] which uses in particular [FSV00, chap. 4, 6.3] whose proof is a reference to [FL98].

by Huber-Kahn in [HK06, app. B], ours allows one to bypass the assumptions of resolution of singularities for some of the fundamental properties.

Acknowledgments

This paper grew out of a non published part of my thesis and I want to thank my thesis director F. Morel for many discussions on orientation theory. I also want to thank F. Brunault, D.C. Cisinski, B. Kahn, T. Saito, V. Srinivas and J. Wildeshaus for useful discussions during the different stages of writing of this article. Special thanks go to C. Weibel and U. Jannsen for pointing to me the sign issue in the formula numbered (3) in the introduction. Finally, I want to thank the referee for comments which have helped me to clarify and improve the present redaction.

Contents

1. Preliminary

1.1. **Notations and conventions.** We fix a base field k which is assumed to be perfect. The word scheme will stand for any separated k-scheme of finite type, and we will say that a scheme is smooth when it is smooth over the base field. The category of smooth schemes is denoted by $\mathscr{S}m(k)$. Throughout the paper, when we talk about the codimension of a closed immersion, the rank of a projective bundle or the relative dimension of a morphism, we assume it is constant.

Given a vector bundle E over X, and P the associated projective bundle with projection $p : P \to X$, we will call *canonical line bundle* on P the canonical invertible sheaf λ over P characterized by the property that $\lambda \subset p^{-1}(E)$. Similarly, we will call *canonical dual line bundle* on P the dual of λ.

We say that a morphism is *projective* if it admits a factorization into a closed immersion followed by the projection of a projective bundle.[10]

We let $DM_{gm}(k)$ be the category of geometric motives (resp. effective geometric motives) introduced in [FSV00, chap. 5]. If X is a smooth scheme, we denote by $M(X)$ the effective motive associated with X in $DM_{gm}(k)$.

For a morphism $f : Y \to X$ of smooth schemes, we will simply put $f_* = M(f)$. Moreover for any integer r, we sometimes put $\mathbb{Z}((r)) = \mathbb{Z}(r)[2r]$ in large diagrams. When they are clear from the context (for example in diagrams), we do not indicate twists or shifts on morphisms.

1.2. **Realization.** To make our constructions more explicit to the reader, we will fix a realization functor of geometric motives into an abelian category $\mathcal{A}$. This will be a contravariant functor

$$H : DM_{gm}(k)^{op} \to \mathscr{A}$$

sending exact triangles to exact sequences. To this realization functor is associated a canonical *twisted cohomology*, for any smooth scheme X an any pair of integers $(i, n) \in \mathbb{Z}^2$:

$$H^{i,n}(X) = H\big(M(X)(-i)[-n]\big).$$

Example 1.1. We will consider the following explicit realization functors:

(1) *Motivic cohomology.*– Put $H_{\mathcal{M}}(M) = \mathrm{Hom}_{DM_{gm}(k)}(M, \mathbb{Z})$ – see Paragraph 2.5 for more details.

(2) *Mixed Weil cohomology.*– A realization functor H as above can be associated with any classical Weil cohomology: Betti and De Rham cohomology in characteristic 0, rigid cohomology in characteristic p, rational l-adic étale cohomology (after extension to the algebraic closure) in any characteristic different from l. See [CD07] for more details.

(3) *Galois realization.*– Let $\bar{k}$ be a separable closure of k, G be the Galois group of $\bar{k}/k$ and l a prime invertible in k. We consider the abelian category $\mathbb{Z}_l[G] - \mathrm{mod}$ integral l-adic representations of G and the *Galois realization* functor:

$$H_{G,l} : DM_{gm}(k) \to \mathbb{Z}_l[G] - \mathrm{mod}$$

characterized by the relation:

$$H^{i,n}_{G,l}(X) = H^i_{\text{ét}}\big(X_{\bar{k}}, \mathbb{Z}_l(n)\big)$$

where the right hand side is the étale cohomology of the scheme $X_{\bar{k}} = X \otimes_k \bar{k}$ with coefficients in the Tate twist $\mathbb{Z}_l(n)$, equipped with its continuous action of G.

It can be obtained using the construction of A. Huber [Hub00, Hub04] or that of F. Ivorra [Ivo10]. However, in the case considered here, we can give a direct construction as follows. For any integer $s > 0$, define the following composite functor:

$$\begin{aligned} R_s : & DM^{eff}_{gm}(k) \xrightarrow{(1)} DM^{eff}_{-}(k) \xrightarrow{.\otimes^{\mathrm{L}} \mathbb{Z}/l^r\mathbb{Z}} DM^{eff}_{-}(k, \mathbb{Z}/l^s\mathbb{Z}) \\ & \xrightarrow{a_{\text{ét}}} DM^{eff}_{-,\text{ét}}(k, \mathbb{Z}/l^s\mathbb{Z}) \xrightarrow{(2)} D^-(\mathbb{Z}/l^s[G] - \mathrm{mod}) \end{aligned}$$

where the map (1) is obtained from [FSV00, chap. 5, Th. 3.2.6], $a_{\text{ét}}$ is induced by the functor taking a Nisnevich sheaf with transfers to the

[10] Beware this is not the convention of [EGA2] unless the aim of the morphism admits an ample line bundle.

associated étale sheaf with transfers (see [FSV00, chap. 5, Prop. 3.3.1]) and the map (3) is the quasi-isomorphism given by Suslin-Voevodsky's rigidity theorem (*i.e.* [FSV00, chap. 5, Prop. 3.3.3]).

Then for any effective geometric motive M, we put:

$$H_{G,/l^s}(M) := \mathrm{Hom}(R_s(M), \mathbb{Z}/l^s\mathbb{Z})$$
$$H_{G,l}(M) := \varprojlim_s H_{G,/l^s}(M).$$

Because $R_s(\mathbb{Z}(1))$ is isomorphic to the invertible étale sheaf of l^s-roots of unity μ_{l^s}, one gets $H_{G,l}(\mathbb{Z}(1)) = \mathbb{Z}_l(-1)$ and the above definition uniquely extends to the category of non effective geometrical motives.

Note that in all these examples, $\mathscr{A}$ is endowed with a canonical monoidal structure such that $H(\mathbb{Z}) = \mathbb{1}$ is the unit object. We define a canonical functor with values in the category of abelian groups:

$$\gamma : \mathscr{A} \to \mathscr{A}b, A \mapsto \mathrm{Hom}_{\mathscr{A}}(\mathbb{1}, A)$$

Then for any geometric motive M, the morphism of abelian groups

$$\mathrm{Hom}_{DM_{gm}(k)}(M, \mathbb{Z}) \to \mathrm{Hom}_{\mathscr{A}}(H(\mathbb{Z}), H(M))$$

associated with the functor H defines a canonical functorial *regulator map*:

$$\rho_M : H_{\mathcal{M}}(M) \to \gamma H(M). \tag{1.1.a}$$

In the case of a Mixed Weil cohomology, α is just the obvious forgetful functor. On the contrary, in the case of the Galois realization, $\gamma(E)$ is the G-invariant part of the representation E. In other words, the regulator map lands in the G-invariant part of cohomology as expected.

Further, H is weakly monoidal: there is a canonical morphism $H(M) \otimes H(N) \to H(M \otimes N)$ in $\mathcal{A}$. It is not an isomorphism but it induces a bigraded ring structure on the cohomology H^{**} which coincides with the usual one in all the particular cases introduced above. The regulator is obviously multiplicative with respect to this product.

Let us finish with a less usual example:

Example 1.2. *Motivic cohomology with coefficients.*– Recall one can define a triangulated category $DM(k)$ which contains both $DM_{gm}(k)$ and $DM_-^{eff}(k)$ as full triangulated categories – this is the analog of the stable homotopy category ; see [RØ08] or [CD09a]. Take any object E of the category $DM(k)$ and any geometric motive M in $DM_{gm}(k)$, we put $H_E(M) = \mathrm{Hom}_{DM_{gm}(k)}(M, E)$.

In this last case, there is *a priori* no regulator map. However, we will see in Example 2.8 that for any scheme X the bigraded abelian group $H^{**}(X)$ carries a natural bigraded module structure over the ring $H^{**}_{\mathcal{M}}(X)$ such that the regulator map is $H^{**}_{\mathcal{M}}(X)$-linear.

2. The Gysin triangle

2.1. **Relative motives.**

Definition 2.1. We call closed (resp. open) pair any couple (X, Z) (resp. (X, U)) such that X is a smooth scheme and Z (resp. U) is a closed (resp. open) subscheme of X.

Let (X, Z) be an arbitrary closed pair. We will say (X, Z) is smooth if Z is smooth. For an integer n, we will say that (X, Z) has codimension n if Z has (pure) codimension n in X.

A morphism of open or closed pairs $(Y,B) \to (X,A)$ is a couple of morphisms (f,g) which fits into the commutative diagram of schemes

$$\begin{array}{ccc} B & \hookrightarrow & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \hookrightarrow & X. \end{array}$$

If the pairs are closed, we also require that this square is topologically cartesian[11].

We add the following definitions :

- The morphism (f,g) is said to be cartesian if the above square is cartesian as a square of schemes.
- A morphism (f,g) of closed pairs is said to be excisive if f is étale and g_{red} is an isomorphism.
- A morphism (f,g) of smooth closed pairs is said to be transversal if it is cartesian and the source and target have the same codimension.

We will denote conventionally open pairs as fractions (X/U).

Definition 2.2. Let (X,Z) be a closed pair. We define the relative motive $M_Z(X)$ — sometimes denoted by $M(X/X-Z)$ — associated with (X,Z) to be the object in $DM_{gm}(k)$ induced by the complex

$$... \to 0 \to [X-Z] \to [X] \to 0 \to ...$$

where $[X]$ is placed in degree 0.

Relative motives are functorial with respect to morphisms of closed pairs. In fact, $M_Z(X)$ is functorial with respect to morphisms of the associated open pair $(X/X-Z)$. For example, if $Z \subset T$ are closed subschemes of X, we get a morphism $M_T(X) \to M_Z(X)$.

If $j : (X-Z) \to X$ denotes the complementary open immersion, we obtain a canonical distinguished triangle in $DM_{gm}(k)$:

$$M(X-Z) \xrightarrow{j_*} M(X) \to M_Z(X) \to M(X-Z)\,[1]. \tag{2.2.a}$$

Remark 2.3. The relative motive in $DM_{gm}(k)$ defined here corresponds under the canonical embedding to the relative motive in $DM_-^{eff}(k)$ defined in [Dég04, def. 2.2].

The following proposition sums up the basic properties of relative motives. It follows directly from [Dég04, 1.3] using the previous remark. Note moreover that in the category $DM_{gm}(k)$, each property is rather clear, except **(Exc)** which follows from the embedding theorem [FSV00, chap. 5, 3.2.6] of Voevodsky.

Proposition 2.4. *Let (X,Z) be a closed pair. The following properties of relative motives hold:*

(Red) Reduction*: If we denote by Z_0 the reduced scheme associated with Z then:*

$$M_Z(X) = M_{Z_0}(X)\,.$$

(Exc) Excision*: If $(f,g) : (Y,T) \to (X,Z)$ is an excisive morphism then $(f,g)_*$ is an isomorphism.*

[11] *i.e.* cartesian as a square of topological spaces ; in other words, $B_{red} = (A \times_X Y)_{red}$.

(MV) Mayer-Vietoris *: If $X = U \cup V$ is an open covering of X then we obtain a canonical distinguished triangle of shape:*

$$M_{Z\cap U\cap V}(U\cap V) \xrightarrow{M(j_U)-M(j_V)} M_{Z\cap U}(U)\oplus M_{Z\cap V}(V) \xrightarrow{M(i_U)+M(i_V)} M_Z(X) \longrightarrow M_{Z\cap U\cap V}(U\cap V)\,[1].$$

The morphism i_U, i_V, j_U, j_V stands for the obvious cartesian morphisms of closed pairs induced by the corresponding canonical open immersions.

(Add) Additivity*: Let Z' be a closed subscheme of X disjoint from Z. Then the morphism induced by the inclusions*

$$M_{Z\sqcup Z'}(X) \to M_Z(X)\oplus M_{Z'}(X)$$

is an isomorphism.

(Htp) Homotopy*: Let $\pi : (\mathbb{A}^1_X, \mathbb{A}^1_Z) \to (X,Z)$ denote the cartesian morphism induced by the projection. Then π_* is an isomorphism.*

2.2. **Purity isomorphism.**

2.5. Consider an integer $i \geq 0$. According to Voevodsky the i-th twisted motivic complex over k is defined as *Suslin's singular simplicial complex* of the cokernel of the natural map of sheaves with transfers $\mathbb{Z}^{tr}(\mathbb{A}^i_k - 0) \to \mathbb{Z}^{tr}(\mathbb{A}^i_k)$, shifted by $2i$ degrees on the left (cf [SV00] or [FSV00]). Motivic cohomology of a smooth scheme X in degree $n \in \mathbb{Z}$ and twists i is defined following Beilinson's idea as the Nisnevich hypercohomology groups of this complex $H^n_{\mathrm{Nis}}(X, \mathbb{Z}(i))$.

One of the fundamental properties of the construction of Voevodsky is the following isomorphism

(2.5.a) $$\epsilon_X : CH^i(X) \xrightarrow{\sim} H^n_{\mathrm{Nis}}(X, \mathbb{Z}(i))$$

which is compatible with pullbacks. The construction of this isomorphism initially appeared in the Corollary 2.4 of the preprint [Voe96] under the assumption $char(k) = 0$. Using the cancellation theorem of Voevodsky published in [Voe10], one removes this assumption using essentially the same argument.[12]

According to [FSV00, chap. 5, 3.2.6], we also get an isomorphism

(2.5.b) $$H^{n,i}_{\mathcal{M}}(X) := \mathrm{Hom}_{DM_{gm}(k)}(M(X), \mathbb{Z}(i)[n]) \simeq H^n_{\mathrm{Nis}}(X, \mathbb{Z}(i))$$

where $\mathbb{Z}(i)$ in the middle term stands (by the usual abuse of notation) for the i-th Tate geometric motive. In what follows, we will identify cohomology classes in motivic cohomology with morphisms in $DM_{gm}(k)$ according to this isomorphism.

Note in particular that any cycle class modulo rational equivalence corresponds to a unique morphism in $DM_{gm}(k)$ according to (2.5.a). Thus, given a vector bundle E over a smooth scheme X and an integer $i \geq 0$, we can define a morphism in $DM_{gm}(k)$

(2.5.c) $$\mathfrak{c}_i(E) : M(X) \to \mathbb{Z}(i)[2i]$$

which corresponds under the preceding isomorphisms to the i-th Chern class of E in the Chow group. For short, we call this morphism the *i-th motivic Chern class* of E.

[12]Recall it follows from the computation of the E_1-term of the coniveau spectral sequence for motivic cohomology, using the identification of the motivic cohomology of a field with its Milnor K-theory in the relevant degrees. The compatibility with pullback (and product) then follows from a careful study (cf for example [Dég02, 8.3.4] or [Dég11b]).

Remark 2.6. Taking into account the preceding isomorphisms, the regulator map (1.1.a) defines a canonical cycle class map:

$$CH^i(X) \simeq H^{2i,i}_{\mathcal{M}}(X) \xrightarrow{\rho} \gamma H^{2i,i}(X).$$

In particular, one gets Chern classes for the cohomology H^{**} and its easy to check they coincide with the usual Chern classes in the case of a realization attached to one of the classical Mixed Weil cohomology.

2.7. There is a canonical isomorphism

$$\mathbb{Z}(i) \otimes \mathbb{Z}(j) \to \mathbb{Z}(i+j)$$

(cf [SV00]) which induces a product on motivic cohomology, sometimes called the cup-product. It can be described as follows. Let X be a smooth scheme, $\delta : X \to X \times_k X$ be the diagonal embedding and $f : M(X) \to \mathcal{M}$, $g : M(X) \to \mathcal{N}$ be two morphisms with target a geometric motive. We define the *exterior product* of f and g, denoted by $f \boxtimes_X g$ or simply $f \boxtimes g$, as the composite

$$M(X) \xrightarrow{\delta_*} M(X) \otimes M(X) \xrightarrow{f \otimes g} \mathcal{M} \otimes \mathcal{N}. \tag{2.7.a}$$

In the case where $\mathcal{M} = \mathbb{Z}(i)[n]$, $\mathcal{N} = \mathbb{Z}(j)[m]$, $f \boxtimes g$ is just the cup-product of f and g seen as cohomology classes, once we have identified $\mathbb{Z}(i)[n] \otimes \mathbb{Z}(j)[m]$ with $\mathbb{Z}(i+j)[n+m]$ by the isomorphism above.

Example 2.8. Let X be a smooth scheme and α a cohomology class in $H^{n,i}_{\mathcal{M}}(X)$. Identifying α to a morphism in $DM_{gm}(k)$ as above and using the exterior product, one defines a morphism:

$$\alpha \boxtimes_X 1_{X*} : M(X) \to M(X)(i)[n].$$

Applying the realization functor H, one obtains a morphism of abelian groups:

$$H^{*-n,*-i}(X) \to H^{*,*}(X), x \mapsto \alpha.x.$$

One obtains in this way a bigraded $H^{**}_{\mathcal{M}}(X)$-module structure on H^{**}. Of course, when $H = H_{\mathcal{M}}$, this is nothing else than the module structure derived form the ring structure.

Remark 2.9. According to our construction, any formula in the Chow group involving pullbacks and intersections of Chern classes induces a corresponding formula for the morphisms of type (2.5.c).

2.10. We finally recall the projective bundle theorem (cf [FSV00, chap. 5, 3.5.1]). Let P be a projective bundle of rank n over a smooth scheme X, λ its canonical dual line bundle and $p : P \to X$ the canonical projection. The projective bundle theorem of Voevodsky says that the morphism

$$M(P) \xrightarrow{\sum_{i \leq n} \mathfrak{c}_1(\lambda)^i \boxtimes p_*} \bigoplus_{i=0}^{n} M(X)((i)) \tag{2.10.a}$$

is an isomorphism.

Thus, we can associate with P a family of split monomorphisms indexed by an integer $r \in [0, n]$ corresponding to the decomposition of its motive :

$$\mathfrak{l}_r(P) : M(X)(r)[2r] \to \oplus_{i \leq n} M(X)(i)[2i] \to M(P)\,. \tag{2.10.b}$$

2.11. Let E be a vector bundle of rank n over a smooth scheme X, and put $Q = \mathbb{P}(E)$, $P := \mathbb{P}(E \oplus \mathbb{A}^1_1)$. This is the projective completion of E/X and it admits a canonical section $s : X \to P$. We then get a commutative diagram of immersions:

$$\begin{array}{ccc} P - X & \xrightarrow{j} & P \\ {\scriptstyle \nu'}\uparrow & & \| \\ Q & \xrightarrow{\nu} & P \end{array} \tag{2.11.a}$$

where j (resp. ν) is the canonical open (resp. closed) immersion. According to [EGA2, 8.6.4], $P - X$ admits a canonical structure of a line bundle over Q such that ν' corresponds to the zero section. Thus the induced morphism on motives ν'_* in an isomorphisms by the $\mathbb{A}^1_k$-homotopy property. Using now the projective bundle isomorphism for P and Q as recalled in the previous paragraph, we get:

Lemma 2.12. *Given the notations above, the following maps gives an isomorphism of distinguished triangles in $DM_{gm}(k)$:*

$$\begin{array}{ccccccc} M(P-X) & \longrightarrow & M(P) & \xrightarrow{\pi_P} & M_X(P) & \xrightarrow{+1} & \\ {\scriptstyle \nu'_* \circ \mathfrak{l}_*(Q)}\uparrow & & {\scriptstyle \mathfrak{l}_*(P)}\uparrow & & \uparrow{\scriptstyle \pi_P \circ \mathfrak{l}_n(P)} & & \\ \oplus_{i<n} M(X)((i)) & \hookrightarrow & \oplus_{i \geq n} M(X)((i)) & \twoheadrightarrow & M(X)((n)) & \xrightarrow{+1} & \end{array}$$

where the first (resp. second) vertical map is the obvious (split) inclusion (resp. projection).

2.13. Consider a smooth closed pair (X, Z). Let $N_Z X$ (resp. $B_Z X$) be the normal bundle (resp. blow-up) of (X, Z) and $P_Z X$ be the projective completion of $N_Z X$. We denote by $B_Z(\mathbb{A}^1_X)$ the blow-up of $\mathbb{A}^1_X$ with center $\{0\} \times Z$. It contains as a closed subscheme the trivial blow-up $\mathbb{A}^1_Z = B_Z(\mathbb{A}^1_Z)$. We consider the closed pair $(B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z)$ over $\mathbb{A}^1_k$. Its fiber over 1 is the closed pair (X, Z) and its fiber over 0 is $(B_Z X \cup P_Z X, Z)$. Thus we can consider the following deformation diagram :

$$(X, Z) \xrightarrow{\bar{\sigma}_1} (B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z) \xleftarrow{\bar{\sigma}_0} (P_Z X, Z). \tag{2.13.a}$$

This diagram is functorial in (X, Z) with respect to cartesian morphisms of closed pairs. Note finally that, on the closed subschemes of each closed pair, $\bar{\sigma}_0$ (resp. $\bar{\sigma}_1$) is the 0-section (resp. 1-section) of $\mathbb{A}^1_Z/Z$.

The existence statement in the following result already appears in [Dég08b, 2.2.5] but the uniqueness statement is new :

Theorem 2.14. *Let n be a natural integer.*

There exists a unique family of isomorphisms of the form

$$\mathfrak{p}_{(X,Z)} : M_Z(X) \to M(Z)(n)[2n]$$

indexed by smooth closed pairs of codimension n such that :

(1) *for every cartesian morphism $(f, g) : (Y, T) \to (X, Z)$ of smooth closed pairs of codimension n, the following diagram is commutative :*

$$\begin{array}{ccc} M_T(Y) & \xrightarrow{(f,g)_*} & M_Z(X) \\ {\scriptstyle \mathfrak{p}_{(Y,T)}}\downarrow & & \downarrow{\scriptstyle \mathfrak{p}_{(X,Z)}} \\ M(T)(n)[2n] & \xrightarrow{g_*(n)[2n]} & M(Z)(n)[2n]. \end{array}$$

(2) *Let X be a smooth scheme and P be the projective completion of a vector bundle E/X of rank n. Consider the closed pair (P,X) corresponding to the 0-section of E/X. Then $\mathfrak{p}_{(P,X)}$ is the inverse of the morphism*

$$M(X)(n)[2n] \xrightarrow{\mathfrak{t}_n(P)} M(P) \xrightarrow{\pi_P} M_X(P)\,.$$

where we used the notations of Lemma 2.12.

Proof. Uniqueness : Consider a smooth closed pair (X,Z) of codimension n.

Applying property (1) to the deformation diagram (2.13.a), we obtain the commutative diagram :

$$\begin{array}{ccccc}
M(X,Z) & \xrightarrow{\bar{\sigma}_{1*}} & M\big(B_Z(\mathbb{A}^1_X),\mathbb{A}^1_Z\big) & \xleftarrow{\bar{\sigma}_{0*}} & M(P_ZX,Z) \\
\downarrow{\scriptstyle \mathfrak{p}_{(X,Z)}} & & \downarrow{\scriptstyle \mathfrak{p}_{(B_Z(\mathbb{A}^1_X),\mathbb{A}^1_Z)}} & & \downarrow{\scriptstyle \mathfrak{p}_{(P_ZX,Z)}} \\
M(Z)(n)[2n] & \xrightarrow{s_{1*}} & M(\mathbb{A}^1_Z)(n)[2n] & \xleftarrow{s_{0*}} & M(Z)(n)[2n]
\end{array}$$

Using homotopy invariance, s_{0*} and s_{1*} are isomorphisms. Thus in this diagram, all the morphisms are isomorphisms. Now, the second property of the purity isomorphisms determines uniquely $\mathfrak{p}_{(P_ZX,Z)}$, thus $\mathfrak{p}_{(X,Z)}$ is also uniquely determined.

Consider the existence part. Case (2) follows from the preceding lemma. On the other hand, one can prove (see [MV99, Prop. 2.24, p. 115] or [Még08b, section 2.2]) that the morphisms of closed pairs $\bar{\sigma}_0$ and $\bar{\sigma}_1$ of the deformation diagram (2.13.a) induces isomorphisms of relatives motives. Thus one defines the purity isomorphism as the following composite map:
(2.14.a)

$$M(X,Z) \xrightarrow{\bar{\sigma}_{1*}} M\big(B_Z(\mathbb{A}^1_X),\mathbb{A}^1_Z\big) \xrightarrow{\bar{\sigma}_{0*}^{-1}} M(P_ZX,Z) \xrightarrow{\mathfrak{p}_{(P_ZX,Z)}} M(Z)(n)[2n].$$

□

Remark 2.15. The second point of the above proposition appears as a normalization condition. It will be reinforced later (cf Remark 4.3).

Definition 2.16. Given any smooth closed pair (X,Z) of codimension n, the isomorphism $\mathfrak{p}_{(X,Z)}$ of the previous theorem is called the *purity isomorphism* associated with (X,Z).

Example 2.17. Given a closed pair (X,Z), the abelian group

$$H^{r,i}_Z(X) := H\big(M_ZX(-i)[-r]\big)$$

is the cohomology of X with support in Z. The purity isomorphism induces, when (X,Z) is smooth of codimension n, an isomorphism:

$$\mathfrak{p}^*_{(X,Z)} : H^{r,i}(Z) \xrightarrow{\sim} H^{r+2n,i+n}_Z(X).$$

Note that the assertion of uniqueness in the previous theorem can be extended to the realization H as follows. The family of isomorphism $(\mathfrak{p}^*_{(X,Z)})$ for smooth closed pairs of codimension n is uniquely characterized by the properties:

(1) The morphism $\mathfrak{p}^*_{(?)}$ is natural with respect to cartesian morphisms of smooth closed pairs of codimension n.
(2) When P/X is the projective completion of a vector bundle of rank n, (P,X) the closed pair corresponding to the canonical section, $\mathfrak{p}^*_{(P,X)}$ is the inverse

of the following isomorphism:

$$\begin{array}{ccccc} H^{**}_X(P) & \longrightarrow & H^{**}(P) & \longrightarrow & H^{**}(X) \\ & & \sum_{i\leq n} p^*(x_i).c_1(\lambda)^i & \longmapsto & x_n \end{array}$$

where the first map is the obvious one forgetting the support, and the second one uses the projective bundle formula.

In fact, the above properties follow directly from the previous theorem ; for the uniqueness statement, we use the same argument as in the proof but uses cohomology instead of motives. This remark allows one to identify $\mathfrak{p}^*_{(X,Z)}$ with the purity isomorphisms when they are known – in particular in the case where H is associated with a mixed Weil theory.

The purity isomorphism satisfies the following additivity properties:

Proposition 2.18. *Let (X, Z) be a smooth closed pair of codimension n.*

(1) *Assume $X = X_1 \sqcup X_2$. We put $Z_i = X_i \cap Z$ for $i = 1, 2$. Then the following diagram is commutative:*

$$\begin{array}{ccc} M_Z(X) & \xrightarrow{\mathfrak{p}_{(X,Z)}} & M(Z)(n)[2n] \\ \sim\uparrow & & \uparrow\sim \\ M_{Z_1}(X_1) \oplus M_{Z_1}(X_1) & \xrightarrow{\mathfrak{p}_{(X_1,Z_1)}+\mathfrak{p}_{(X_2,Z_2)}} & M(Z_1)(n)[2n] \oplus M(Z_2)(n)[2n] \end{array}$$

where the left vertical maps are the natural isomorphisms.

(2) *Assume $Z = Z_1 \sqcup Z_2$. Then the following diagram is commutative:*

$$\begin{array}{ccc} M_Z(X) & \xrightarrow{\mathfrak{p}_{(X,Z)}} & M(Z)(n)[2n] \\ \sim\downarrow & & \uparrow\sim \\ M_{Z_1}(X) \oplus M_{Z_2}(X) & \xrightarrow{\mathfrak{p}_{(X_1,Z_1)}+\mathfrak{p}_{(X_2,Z_2)}} & M(Z_1)(n)[2n] \oplus M(Z_2)(n)[2n] \end{array}$$

where the left vertical map is the isomorphism from property (Add) of Proposition 2.4.

Proof. The first point follows easily from Formula (2.14.a) and the fact $B_Z(\mathbb{A}^1_X) = B_{Z_1}(\mathbb{A}^1_{X_1}) \sqcup B_{Z_2}(\mathbb{A}^1_{X_2})$.

Let us prove the second point. We use again the notations introduced in the deformation diagram (2.13.a) First we note that the following diagram is commutative:

$$\begin{array}{ccccc} M(X,Z) & \xrightarrow{\bar\sigma_{1*}} & M\big(B_Z(\mathbb{A}^1_X), \mathbb{A}^1_Z\big) & \xleftarrow{\bar\sigma_{0*}} & M(P_Z X, Z) \\ \sim\downarrow & & \sim\downarrow & & \sim\downarrow \\ \begin{array}{c} M(X,Z_1) \\ \oplus \\ M(X,Z_2) \end{array} & \xrightarrow{\bar\sigma^{(1)}_{1*}+\bar\sigma^{(2)}_{1*}} & \begin{array}{c} M\big(B_Z(\mathbb{A}^1_X), \mathbb{A}^1_{Z_1}\big) \\ \oplus \\ M\big(B_Z(\mathbb{A}^1_X), \mathbb{A}^1_{Z_2}\big) \end{array} & \xleftarrow{\bar\sigma^{(1)}_{0*}+\bar\sigma^{(2)}_{0*}} & \begin{array}{c} M(P_Z X, Z_1) \\ \oplus \\ M(P_Z X, Z_2) \end{array} \end{array}$$

where $\bar\sigma^{(i)}_0$ (resp. $\bar\sigma^{(i)}_1$) is the obvious restriction of the morphism of closed pairs $\bar\sigma_0$ (resp. $\bar\sigma_1$) and the vertical maps are given by property (Add). Thus we are reduced to the case where $X = P_Z X$. But this case follows from the first point as $P_Z X = P_{Z_1} X \sqcup P_{Z_2} X$. □

Remark 2.19. One can extend the definition of the purity isomorphism to the case of a smooth closed pair (X, Z) such that the codimension of Z in X is not pure. Let $(Z_i)_{i \in I}$ be the family of connected components of Z and $n : I \to \mathbb{N}$ be the function associating to any such component its codimension in X. We put:

$$M(Z)(n)[2n] := \bigoplus_{i \in I} M(Z_i)(n(i))[2n(i)]$$

Then one define the purity isomorphism for the closed pair (X, Z) as follows:

$$\mathfrak{p}_{(X,Z)} : M_Z(X) \xrightarrow{\sim} \oplus_{i \in I} M_{Z_i}(X) \xrightarrow{\sum_i \mathfrak{p}_{(X,Z_i)}} M(Z)(n)[2n].$$

The preceding proposition ensures that this definition is consistent with the original one, in case Z is not connected but has pure codimension.

2.3. **The Gysin triangle.**

Definition 2.20. Let (X, Z) be a smooth closed pair of codimension n. Denote by j (resp. i) the open immersion $(X - Z) \to X$ (resp. closed immersion $Z \to X$).

Using the purity isomorphism of Definition 2.16, we deduce from the distinguished triangle (2.2.a) the following distinguished triangle in $DM_{gm}(k)$, called the Gysin triangle of (X, Z)

$$M(X - Z) \xrightarrow{j_*} M(X) \xrightarrow{i^*} M(Z)(n)[2n] \xrightarrow{\partial_{X,Z}} M(X - Z)[1].$$

This triangle (resp. the morphism i^*, $\partial_{X,Z}$) is called the *Gysin triangle* (resp. *Gysin morphism*, *residue*) associated with (X, Z) (or i). Sometimes we use the notation $\partial_i = \partial_{X,Z}$.

Example 2.21. We consider again the situation of Paragraph 2.11: (P, X) is a closed pair of codimension n corresponding to the canonical section $s : X \to P$ of the projective completion P/X of the vector bundle E/X. According to Lemma 2.12 and the definition of the purity isomorphism $\mathfrak{p}_{(P,X)}$, the Gysin triangle associated with (P, X) is split: $\partial_{P,X} = 0$.

Let us denote by λ (resp. $p : P \to X$) be the canonical dual invertible sheaf (resp. projection) of P/X. We define the Thom class of E in $CH^n(P)$ as the class

$$t(E) = \sum_{i=0}^{n} p^*(c_{n-i}(E)).c_1(\lambda)^i. \tag{2.21.a}$$

It has the following characterizing properties:

(a) the restriction of $t(E)$ to $Q = \mathbb{P}(E)$ is 0,
(b) the coefficients of $c_1(\lambda)^n$ in $t(E)$ is 1.

According to paragraph 2.5, this class corresponds to a morphism:

$$\mathfrak{t}(E) : M(P) \to \mathbb{Z}(n)[2n].$$

Then Properties (a) and (b) above together with Lemma 2.12 gives the following equality:

$$s^* = \mathfrak{t}(E) \boxtimes_P p_* \tag{2.21.b}$$

where $\boxtimes$ is the exterior product defines in Paragraph 2.7.

Remark 2.22. The Gysin triangle defined above agrees with that of [FSV00], chap. 5, prop. 3.5.4. Indeed, in the proof of 3.5.4, Voevodsky constructs an isomorphism which he denotes by $\alpha_{(X,Z)}$. He then uses it as we use the purity isomorphism to construct his triangle. It is not hard to check that this isomorphism $\alpha_{(X,Z)}$ satisfies the two conditions of Theorem 2.14 and thus coincides with the purity isomorphism from the uniqueness statement.

2.23. In the assumptions of definition 2.20, if we apply the realization functor H to the Gysin triangle, one gets a long exact sequence

$$\cdots \to H^{n-2c,i-c}(Z) \xrightarrow{i_*} H^{n,i}(X) \xrightarrow{j^*} H^{n,i}(X-Z) \xrightarrow{\partial_{X,Z}} H^{n+1-2c,i-c}(Z) \to \cdots$$

classically called the *localization long exact sequence*. Note in particular that given a cohomology class $u \in H^{n,i}(X-Z)$, the residue $\partial_{X,Z}(u)$ is precisely the obstruction to extend u to the scheme X. This fact will be discussed in section 3.3 and Example 4.11.

Within the framework of Example 2.21, Formula (2.21.b) gives after realization the following classical formula in cohomology:

$$\forall x \in H^{n,i}(X), s_*(x) = \mathfrak{t}(E).p^*(x),$$

using the module structure of Example 2.8.

Remark 2.24. A particular case of the localization long exact sequence defined above is the one induced in motivic cohomology according to Example 1.1(1). One nice fact about our construction is that we readily get that the regulator map (1.1.a) is compatible with the localization long exact sequence.

Example 2.25. Let us assume that $H = H_{G,l}$ is the l-adic Galois realization. Consider a smooth geometrically connected projective curve C and D be a finite set of closed points in C. Put $U = C - D$, $j : U \to C$ being the canonical immersion. Then (C, D) is a smooth closed pair of codimension 1 and the localization sequence gives in particular the following short exact sequence of l-adic Galois representations:

$$0 \to H^1(C_{\bar{k}}, \mathbb{Z}_l(1)) \xrightarrow{j^*} H^1(U_{\bar{k}}, \mathbb{Z}_l(1)) \xrightarrow{\partial_{C,D}} H^0(D_{\bar{k}}, \mathbb{Z}_l) \to 0$$

Let J be the Jacobian of $C_{\bar{k}}$. According to the classical theorem in étale cohomology, the group on the left hand side is the l-adic Tate module $T_l(J)$ given with its canonical action of G. The right hand side is the permutation $\mathbb{Z}_l[G]$-module induced by the set $D_{\bar{k}}$ with its natural action of G.

Thus the Galois module $H^1(U_{\bar{k}}, \mathbb{Z}_l(1))$ is presented through the localization exact sequence as an extension of the Tate module $T_l(J)$ by the permutation module $\mathbb{Z}_l[D_{\bar{k}}]$, one map being given by the residue $\partial_{C,D}$.

Recall also that the *generalized Jacobian* J' of $C_{\bar{k}}$ relative to the divisor associated with D (each point having multiplicity 1) is defined as an extension of algebraic groups of the following shape:

$$0 \to (\bar{k}^\times)^D/(\bar{k}^\times) \xrightarrow{\nu} J' \to J \to 0$$

(see [Ser84, chap. V]). According to [Del77, Arcata, VI, 2, §(2.3)], the l-adic Tate module of J' agrees with the étale cohomology group with compact support: $H^1_c(U_{\bar{k}}, \mathbb{Z}_l(1))$.

Thus according to the duality results in étale cohomology (cf [Del77, Dualité]), the l-adic Galois module $H^1(U_{\bar{k}}, \mathbb{Z}_l(1))$ is the $\mathbb{Z}_l$-dual of the generalized Jacobian J' of (C, D) and the residue map is the $\mathbb{Z}_l$-dual of the inclusion ν.

Let us finally remark that the additivity properties of the purity isomorphism obtained in Proposition 2.18 immediately translate as follows:

Proposition 2.26. *Let (X, Z) be a smooth closed pair of codimension n and $i : Z \to X$ the corresponding closed immersion.*

(1) *Assume* $X = X_1 \sqcup X_2$. *For* $s = 1, 2$, *we put* $Z_s = X_s \cap Z$ *and denote by* $i_s : Z_s \to X_s$ *the canonical immersion. Then one obtains:*

$$i^* = i_1^* + i_2^*,$$
$$\partial_{X,Z} = \partial_{X_1,Z_1} + \partial_{X_2,Z_2}$$

(2) *Assume* $Z = Z_1 \sqcup Z_2$. *For* $s = 1, 2$, *we denote by* $i_s : Z_s \to X$ *and* $j_s : (X - Z) \to (X - Z_s)$ *the canonical immersions. Then, one obtains:*

$$i^* = i_1^* + i_2^*,$$
$$\forall s = 1, 2,\ j_{s*} \circ \partial_{X,Z} = \partial_{X,Z_s}.$$

2.4. **Base change formulas.** This subsection is devoted to recall some results we obtained previously in [Dég04] and [Dég08b] about the following type of morphism :

Definition 2.27. Let (X, Z) (resp. (Y, T)) be a smooth closed pair of codimension n (resp. m). Let $(f, g) : (Y, T) \to (X, Z)$ be a morphism of closed pairs.

We define the morphism $(f, g)_!$ as the following composite :

$$M(T)(m)[2m] \xrightarrow{\mathfrak{p}_{(Y,T)}^{-1}} M(Y, T) \xrightarrow{(f,g)_*} M(X, Z) \xrightarrow{\mathfrak{p}_{(X,Z)}} M(Z)(n)[2n].$$

In the situation of this definition, let $i : Z \to X$ and $k : T \to Y$ be the obvious closed embeddings and $h : (Y - T) \to (X - Z)$ be the restriction of f. Then we obtain from our definitions the following commutative diagram :

(2.27.a)
$$\begin{array}{ccccccc}
M(Y-T) & \longrightarrow & M(Y) & \xrightarrow{j^*} & M(T)(m)[2m] & \xrightarrow{\partial_{Y,T}} & M(Y-T)\,[1] \\
\downarrow & & \downarrow{\scriptstyle f_*} & {\scriptstyle (1)} & \downarrow{\scriptstyle (f,g)_!} & {\scriptstyle (2)} & \downarrow{\scriptstyle h_*} \\
M(X-Z) & \longrightarrow & M(X) & \xrightarrow{i^*} & M(Z)(n)[2n] & \xrightarrow{\partial_{X,Z}} & M(X-Z)\,[1]
\end{array}$$

The commutativity of square (1) corresponds to a *refined projection formula*. The word refined is inspired by the terminology “refined Gysin morphism” of Fulton in [Ful98]. By contrast, the commutativity of square (2) involves motivic cohomology rather than Chow groups.

2.28. Let T (resp. T') be a closed subscheme of a scheme Y with defining ideal $\mathcal{J}$ (resp. $\mathcal{J}'$). We will say that a closed immersion $i : T \to T'$ is an *exact thickening of order* r *in* Y if $\mathcal{J}' = \mathcal{J}^r$. We recall to the reader the following formulas obtained in [Dég04, 3.1, 3.3] :

Proposition 2.29. *Let* (X, Z) *and* (Y, T) *be smooth closed pairs of codimension* n *and* m *respectively. Let* $(f, g) : (Y, T) \to (X, Z)$ *be a morphism of closed pairs.*

(1) (Transversal case) *If* (f, g) *is transversal (which implies* $n = m$*) then* $(f, g)_! = g_*(n)[2n]$.

(2) (Excess intersection) *If* (f, g) *is cartesian, we put* $e = n - m$ *and* $\xi = g^* N_Z X / N_T Y$. *Then* $(f, g)_! = \mathfrak{c}_e(\xi) \boxtimes_T g_*(m)[2m]$.

(3) (Ramification case) *If* $n = m = 1$ *and the canonical closed immersion* $T \to Z \times_X Y$ *is an exact thickening of order* r *in* Y, *then* $(f, g)_! = r.g_*(1)[2]$.

Note that each case of the above proposition gives, via the commutative Diagram (2.27.a), two formulas: one involving Gysin morphisms and the other one involving the residues. When we will apply this proposition, we will always refer to one these two formulas.

Remark 2.30. In the article [Dég08a, 4.23], the case (3) has been generalized to any codimension $n = m$. In this generality, the integer r is simply the geometric multiplicity of $Z \times_X Y$ – when assumed to be connected.

Corollary 2.31. *Let (X,Z) be a smooth pair of codimension n and $i : Z \to X$ be the corresponding closed immersion.*

Then, $(1_Z \boxtimes_Z i_*) \circ i^* = i^* \boxtimes_X 1_X : M(X) \to M(Z) \otimes M(X)\,(n)[2n]$.

When the cohomology H^{**} admits a ring structure, the preceding formula gives the usual projection formula: $i_*(x.i^*(z)) = i_*(x).z$.

Proof. Just apply point (1) of the proposition to the cartesian morphism $(X,Z) \to (X \times X, Z \times X)$ induced by the diagonal embedding of X. The only thing left to check is that $(i \times 1_X)^* = i^* \otimes 1$, which was done in [Dég08b, 2.6.1]. □

Remark 2.32. In the above statement, we have loosely identified the motive $M(Z) \otimes M(X)\,(n)[2n]$ with $(M(Z)\,(n)[2n]) \otimes M(X)$ through the canonical isomorphism. This will not have any consequences in the present article. On the contrary in [Dég08b], we must be attentive to this isomorphism which may result in a change of sign (cf remark 2.6.2 of *loc. cit.*).

Another corollary of the preceding proposition is the following analog of the self-intersection formula:

Corollary 2.33. *Let (X,Z) be a smooth closed pair of codimension n with normal bundle $N_Z X$. If i denotes the corresponding closed immersion, we obtain the following equality:*

$$i^* i_* = \mathfrak{c}_n(N_Z X) \boxtimes_Z 1_{Z*}.$$

Indeed it follows from the transversal case of the preceding proposition applied to the cartesian morphism $(i, 1_Z) : (Z,Z) \to (X,Z)$ and from the commutativity of square (1) in diagram (2.27.a).

Remark 2.34. After realization, this gives the formula

$$\forall z \in H^{**}(Z), i^* i_*(z) = \mathfrak{c}_n(N_Z X).z$$

using the $H^{**}_{\mathcal{M}}(X)$ module structure on $H^{**}(X)$ given in Example 2.8. This is a generalization of the *self-intersection formula* in the Chow group (see [Ful98, Cor. 6.3]). The generalization goes in two directions. First when $H = H_{\mathcal{M}}$, we have extended it in any degree of motivic cohomology. Secondly, when $H = H_E$ (Example 1.2), we have obtain it without requiring any ring structure.

Example 2.35. Consider a vector bundle $p : E \to X$ of rank n. Let s_0 be its zero section. According to the homotopy property in $DM_{gm}(k)$, we get $s_{0*} p_* = 1$. Thus, the preceding corollary applied to s_0 implies the following formula:

$$s_0^* = \mathfrak{c}_n(p^{-1}E) \boxtimes_E p_*. \tag{2.35.a}$$

Moreover, the Gysin triangle associated with s_0 together with the isomorphism s_{0*} gives the following distinguished triangle:

$$M(E^\times) \xrightarrow{q_*} M(X) \xrightarrow{\mathfrak{c}_n(E) \boxtimes_X 1_{X*}} M(X)(n)[2n] \xrightarrow{\partial_{E,X}} M(E^\times)\,[1]$$

where $E^\times$ is the complement of the zero section and q is its canonical projection to X.

Applying the realization functor H we get:

$$\cdots \to H^{n,i}(X) \xrightarrow{q^*} H^{n,i}(E^\times) \xrightarrow{\partial_{E,X}} H^{n,i}(X) \xrightarrow{\mathfrak{c}_n(E) \cup ?} H^{n,i}(X) \to \cdots$$

This particular case of the localization exact sequence is usually called the *Gysin sequence*. It gives trivial examples where the residue is not zero. In the case $E = \mathbb{A}^n_X$, we even get that $\partial_{E,X}$ is a split epimorphism on cohomology – in fact it is a split monomorphism at the level of motives.

3. Associativity in the Gysin triangle

3.1. **Fundamental classes.**

Definition 3.1. Let (X, Z) be a smooth closed pair of codimension n and $i : Z \to X$ be the corresponding closed immersion. Let $\pi : Z \to \mathrm{Spec}(k)$ be the structural morphism of Z. We define the *motivic fundamental class* of Z in X as the following composite map:

$$\eta_X(Z) : M(X) \xrightarrow{i^*} M(Z)\,(n)[2n] \xrightarrow{\pi_*} \mathbb{Z}(n)[2n].$$

Using the correspondence between morphisms in $DM_{gm}(k)$ and classes in motivic cohomology (Par. 2.5) we deduce from that definition the familiar equality

$$\eta_X(Z) = i_*(1)$$

where i_* is the morphism induced by the Gysin morphism i^* in motivic cohomology – see Paragraph 2.23.

Remark 3.2. In [Dég11b, Prop. 3.16], we checked that the following diagram is commutative:

$$\begin{array}{ccc} CH^{n-c}(Z) & \xrightarrow{i_*} & CH^n(X) \\ \downarrow{\scriptstyle \epsilon_Z} & & \downarrow{\scriptstyle \epsilon_X} \\ H^{2n-2c,n-c}_{\mathcal{M}}(Z) & \xrightarrow{H_{\mathcal{M}}(i^*)} & H^{2n,n}_{\mathcal{M}}(X) \end{array} \tag{3.2.a}$$

where the vertical maps are the isomorphism (2.5.a) and the upper horizontal map is the usual pushforward on Chow groups. As a result, the motivic fundamental class $\eta_X(Z)$ corresponds to the usual cycle class $[Z]$ associated with Z in the Chow group of X. As natural as this seems, it is not a trivial fact. On the other hand, we only need some easy facts about fundamental classes in our treatment of the Gysin morphism, that we prove directly in the next lemmas. Section 3.3, devoted to computations and examples, will be the only occasion where we use it in this paper.

Example 3.3. Let X be a smooth scheme and $p : E \to X$ be a vector bundle of rank n. According to formula (2.35.a), the motivic fundamental class of the zero section of E/X is:

$$\eta_E(X) = \mathfrak{c}_n(p^{-1}E). \tag{3.3.a}$$

Let P/X be the projective completion of E/X. According to formula (2.21.b), the motivic fundamental class of the canonical section of P/X is:

$$\eta_P(X) = \mathfrak{t}(E). \tag{3.3.b}$$

According to the computations of the previous example, the following lemma is a generalization of formulas (2.21.b) and (2.35.a):

Lemma 3.4. *Let (X, Z) be a smooth closed pair of codimension n and $i : Z \to X$ be the corresponding closed immersion. Assume that i admits a retraction $p : X \to Z$. Then $i^* = \eta_X(Z) \boxtimes_X p_*$.*

This formula gives after realization:

$$\forall z \in H^{**}(Z),\ i_*(z) = \eta_X(Z).p^*(z)$$

using the $H^{**}_{\mathcal{M}}(X)$-module structure on $H^{**}(X)$.

Proof. Let $\pi : Z \to \mathrm{Spec}(k)$ be the structural morphism. According to formula (2.7.a), we deduce that $\pi_* \boxtimes_Z 1_{Z*} = 1_{Z*}$. The lemma follows from the following computation:

$$\begin{aligned} i^* &\overset{(1)}{=} [\pi_* \boxtimes_Z (p_* i_*)] \circ i^* = (\pi_* \otimes p_*)(1_{Z*} \boxtimes_Z i_*) \circ i^* \overset{(2)}{=} (\pi_* \otimes p_*)(i^* \boxtimes_X 1_{Z*}) \\ &= \eta_X(Z) \boxtimes_X p_* \end{aligned}$$

where Equality (1) is justified by the preceding remark and the relation $pi = 1_Z$ whereas Equality (2) is in fact Corollary 2.31. □

Lemma 3.5. *Let X be a smooth scheme and E/X be a vector bundle of rank n. Let s (resp. s_0) be a section (resp. the zero section) of E/X. Assume that s is transversal to s_0 and consider the cartesian square:*

$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{s_0} & E \end{array}$$

Then the motivic fundamental class of i is:

$$\eta_X(Z) = \mathfrak{c}_n(E).$$

Proof. Let π (resp. π') be the structural morphism of Z (resp. X). The lemma follows from the computation below:

$$\eta_X(Z) = \pi_* i^* = \pi'_* k_* i^* \overset{(1)}{=} \pi'_* s_0^* s_* \overset{(2)}{=} \mathfrak{c}_n(p^{-1}E) \circ s_* \overset{(3)}{=} \mathfrak{c}_n(E) \circ p_* \circ s_* = \mathfrak{c}_n(E).$$

Equality (1) follows from Proposition 2.29, equality (2) from Formula (3.3.a) and equality (3) from Remark 2.9. □

Example 3.6. Let E/X be a vector bundle and $p : P \to X$ be its projective completion. Let λ be the canonical dual line bundle on P. Put $F = \lambda \otimes p^{-1}(E)$ as a vector bundle over P. According to our conventions, we get a canonical embedding $\lambda^\vee \subset p^{-1}(E \oplus \mathbb{A}^1_X)$. Then the following composite map

$$\lambda^\vee \to p^{-1}(E \oplus \mathbb{A}^1_X) \to p^{-1}(E)$$

corresponds to a section σ of F/P. One can check that σ is transversal to the zero section s_0^F of F/P and that the following square is cartesian:

$$\begin{array}{ccc} X & \xrightarrow{s} & P \\ \downarrow & & \downarrow{\scriptstyle \sigma} \\ P & \xrightarrow{s_0^F} & F \end{array}$$

where s is the canonical section of P/X. Thus the preceding corollary gives the following equality: $\eta_P(X) = \mathfrak{c}_n(F)$.

3.2. **Composition of Gysin triangles.** We first establish lemmas needed for the main theorem. First of all, using the projection formula in the transversal case (cf 2.29) and the compatibility of Chern classes with pullbacks, we easily obtain the following result:

Lemma 3.7. *Let (Y, Z) be a smooth closed pair of codimension m and P/Y be a projective bundle of dimension n. We put $V = Y - Z$ and consider the following cartesian squares :*

$$\begin{array}{ccccc} P_V & \xrightarrow{\nu} & P & \xleftarrow{\iota} & P_Z \\ {\scriptstyle p_V}\downarrow & & {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_Z} \\ V & \xrightarrow{j} & Y & \xleftarrow{i} & Z \end{array}$$

Finally, we consider the canonical line bundle λ (resp. λ_V, λ_Z) on P (resp. P_V, P_Z).

Then, for any integer $r \in [0, n]$, the following diagram is commutative

$$\begin{array}{ccccccc}
M(P_V) & \xrightarrow{\nu_*} & M(P) & \xrightarrow{\iota^*} & M(P_Z)((m)) & \xrightarrow{\partial_\iota} & M(P_V)\,[1] \\
\downarrow{\scriptstyle \mathfrak{c}_1(\lambda_V)^r \boxtimes p_{V*}} & & \downarrow{\scriptstyle \mathfrak{c}_1(\lambda)^r \boxtimes p_*} & & \downarrow{\scriptstyle \mathfrak{c}_1(\lambda_Z)^r \boxtimes p_{Z*}} & & \downarrow{\scriptstyle \mathfrak{c}_1(\lambda_V)^r \boxtimes p_{V*}[1]} \\
M(V)((r)) & \xrightarrow{j_*} & M(Y)((r)) & \xrightarrow{i^*} & M(Z)((r+m)) & \xrightarrow{\partial_i} & M(V)((r))[1].
\end{array}$$

The following lemma will be in fact the crucial case in the proof of the next theorem.

Lemma 3.8. *Let X be a smooth scheme and E/X (resp. E'/X) be a vector bundle of rank n (resp. m). Let P (resp. P') be the projective completion of E/X (resp. E'/X) and i (resp. i') its canonical section.*

We put $R = P \times_X P'$ and consider the closed immersions:

$$i : X \to P, j : P \to R, k = j \circ i,$$

where $j = P \times_X i'$ and $k = (i, i')$. Then $k^* = i^* j^*$.

Proof. We consider the following canonical morphisms:

$$\begin{array}{ccc}
R & \xrightarrow{q} & P' \\
{\scriptstyle q'}\downarrow & \searrow^{\pi} & \downarrow{\scriptstyle p'} \\
P & \xrightarrow[p]{} & X
\end{array}$$

According to Lemma 3.4, we obtain

$$i^* = \eta_P(X) \boxtimes_P p_*, \quad j^* = \eta_R(P) \boxtimes_R q'_*, \quad k^* = \eta_R(X) \boxtimes_P \pi_*.$$

Applying the first case of Proposition 2.29 to the cartesian morphism of closed pairs $(q', p') : (R, P') \to (P, X)$, we obtain the relation:

$$\eta_P(X) \circ q'_* = \eta_R(P').$$

Together with the preceding computations, it implies the following equality:

$$i^* j^* = \eta_R(P) \boxtimes_P \eta_R(P') \boxtimes_P \pi_*.$$

Thus we are reduced to prove the relation:

$$\eta_R(X) = \eta_R(P) \boxtimes_R \eta_R(P'). \tag{3.8.a}$$

Consider the notations of Example 3.6 applied to the case of E/X (resp. E'/X): we get a vector bundle F/P (resp. F'/P) of rank n (resp. m) such that:

$$\eta_P(X) = \mathfrak{c}_n(F),$$
$$\text{resp. } \eta_{P'}(X) = \mathfrak{c}_m(F').$$

Let σ (resp. σ') be the section of F/P (resp. F'/P') constructed in *loc. cit.* Consider the vector bundle over R defined as:

$$G = F \times_X F' = q'^{-1}(F) \oplus q^{-1}(F').$$

We get a section $(\sigma \times_X \sigma')$ of G/P which is transversal to the zero section s_0^G and such that the following square is cartesian:

$$\begin{array}{ccc} X & \xrightarrow{i} & R \\ \downarrow & & \downarrow{\scriptstyle \sigma\times_X\sigma'} \\ R & \xrightarrow{s_0^G} & G. \end{array}$$

Thus, according to Lemma 3.5, we obtain:

$$\eta_R(X) = \mathfrak{c}_{n+m}(G).$$

The relation (3.8.a) now follows from Remark 2.9 and the equality

$$c_{n+m}(G) = q'^*(c_n(F)).q^*(c_m(F'))$$

in $CH^{n+m}(R)$. □

Theorem 3.9. *Consider a topologically cartesian square of smooth schemes*

$$\begin{array}{ccc} Z & \xrightarrow{k} & Y' \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{i} & X \end{array}$$

such that i,j,k,l are closed immersions of respective pure codimensions n, m, s, t. We put $d = n+t = m+s$ and let $i' : (Y - Z) \to (X - Y')$, $j' : (Y' - Z) \to (X - Y)$ be the closed immersion respectively induced by i, j.

Then the following diagram is commutative :

$$\begin{array}{ccccc} M(X) & \xrightarrow{j^*} & M(Y')((m)) & \xrightarrow{\partial_j} & M(X-Y')\,[1] \\ {\scriptstyle i^*}\downarrow & (1) & \downarrow{\scriptstyle k^*} & (2) & \downarrow{\scriptstyle (i')^*} \\ M(Y)((n)) & \xrightarrow[l^*]{} & M(Z)((d)) & \xrightarrow{\partial_l} & M(Y-Z)((n))[1] \\ & & \downarrow{\scriptstyle \partial_k} & (3) & \downarrow{\scriptstyle \partial_{i'}} \\ & & M(Y'-Z)((m))[1] & \xrightarrow[-\partial_{j'}]{} & M(X-Y\cup Y')\,[2] \end{array}$$

Proof. We will simply call smooth triple the data (X, Y, Y') of a triple of smooth schemes X, Y, Y' such that Y' and Y are closed subschemes of X. Such smooth triples form a category with morphisms the commutative diagrams

$$\begin{array}{ccccc} \bar{Y} & \hookrightarrow & \bar{X} & \hookleftarrow & \bar{Y}' \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \hookrightarrow & X & \hookleftarrow & Y' \end{array}$$

made of two cartesian squares. We say in addition that the morphism (f, g, g') is *transversal* if f is transversal to Y, Y' and $Y \cap Y'$.

To such a triple, we associate a geometric motive $M(X, Y, Y')$ as the cone of the canonical map of complexes of $\mathscr{S}m^{cor}(k)$

$$\begin{array}{ccccccc} \ldots & \longrightarrow & [X - Y \cup Y'] & \longrightarrow & [X - Y'] & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & [X - Y] & \longrightarrow & [X] & \longrightarrow & \ldots \end{array}$$

where $[X]$ and $[X - Y']$ are placed in degree 0. This motive is evidently functorial with respect to morphisms of smooth triples.

We will also use the notation $M\left(\frac{X/X-Y}{X-Y'/X-Y\cup Y'}\right)$ for this motive because it is more suggestive. By definition, it fits into the following diagram, with $\Omega = Y \cup Y'$:

$$\begin{array}{ccccccc}
(\mathcal{D}) : M(X-\Omega) & \longrightarrow & M(X-Y) & \longrightarrow & M\left(\frac{X-Y}{X-\Omega}\right) & \longrightarrow & M(X-\Omega)\,[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M(X-Y') & \longrightarrow & M(X) & \longrightarrow & M\left(\frac{X}{X-Y'}\right) & \longrightarrow & M(X-Y')\,[1] \\
\downarrow & & \downarrow & (1) & \downarrow & (2) & \downarrow \\
M\left(\frac{X-Y'}{X-\Omega}\right) & \longrightarrow & M\left(\frac{X}{X-Y}\right) & \longrightarrow & M\left(\frac{X/X-Y}{X-Y'/X-\Omega}\right) & \longrightarrow & M\left(\frac{X-Y'}{X-\Omega}[1]\right) \\
\downarrow & & \downarrow & & \downarrow & (3) & \downarrow \\
M(X-\Omega)\,[1] & \longrightarrow & M(X-Y)\,[1] & \longrightarrow & M\left(\frac{X-Y}{X-\Omega}\right)[1] & \longrightarrow & M(X-\Omega)\,[2].
\end{array}$$

In this diagram, every square is commutative except square (3) which is anticommutative due to the fact the permutation isomorphism on $\mathbb{Z}[1] \otimes \mathbb{Z}[1]$ is equal to -1. Moreover, any line or row of this diagram is a distinguished triangle.

With the hypothesis of the theorem, the proof will consist in constructing a purity isomorphism $\mathfrak{p}_{(X,Y,Y')} : M(X,Y,Y') \to M(Z)(d)[2d]$ which satisfies the following properties :

(i) *Functoriality :* The morphism $\mathfrak{p}_{(X,Y,Y')}$ is functorial with respect to transversal morphisms of smooth triples.

(ii) *Symmetry :* The following diagram is commutative :

$$\begin{array}{ccc}
M(X,Y,Y') & \longrightarrow & M(X,Y',Y) \\
{\scriptstyle \mathfrak{p}_{(X,Y,Y')}} \searrow & & \swarrow {\scriptstyle \mathfrak{p}_{(X,Y',Y)}} \\
& M(Z)(d)[2d] &
\end{array}$$

where the horizontal map is the canonical isomorphism.

(iii) *Compatibility :* The following diagram is commutative :

$$\begin{array}{ccccccc}
M\left(\frac{X-Y'}{X-\Omega}\right) & \longrightarrow & M\left(\frac{X}{X-Y}\right) & \longrightarrow & M(X,Y,Y') & \longrightarrow & M\left(\frac{X-Y'}{X-\Omega}\right)[1] \\
\downarrow {\scriptstyle \mathfrak{p}_{(X-Y',Y-Z)}} & & \downarrow {\scriptstyle \mathfrak{p}_{(X,Y)}} & & \downarrow {\scriptstyle \mathfrak{p}_{(X,Y,Y')}} & & \downarrow {\scriptstyle \mathfrak{p}_{(X-Y',Y-Z)}[1]} \\
M(Y-Z)((n)) & \longrightarrow & M(Y)((n)) & \xrightarrow{j^*} & M(Z)((d)) & \xrightarrow{\partial_j} & M(Y-Z)((n))[1]
\end{array}$$

With this isomorphism, we can deduce the three relations of the theorem by considering squares (1), (2), (3) in the above diagram and applying the evident purity isomorphism where it belongs.

We then are reduced to construct the isomorphism and to prove the above relations. The second relation is the most difficult one because we have to show that two isomorphisms in a triangulated category are equal. This forces us to be very precise in the construction of the isomorphism.

Construction of the purity isomorphism for smooth triples :

Consider the deformation diagram (2.13.a) for the closed pair (X,Y) and put $B = B_Y(\mathbb{A}^1_X)$, $P = P_Y X$. Put also $(U,V) = (X - Y', Y - Z)$, $B_U = B \times_X U$ and $P_V = P \times_Y V$. Note that, because $Z = (Y \times_X Y')_{red}$, we get $V = Y \times_X U$; thus B_U is the deformation space of (2.13.a) for the closed pair (U,V). By functoriality of the deformation diagram and of relative motives we obtain the following morphisms of distinguished triangles :

$$\begin{array}{ccccccc}
M(U,V) & \longrightarrow & M(X,Y) & \longrightarrow & M\left(\frac{X/X-Y}{U/U-V}\right) & \xrightarrow{+1} & \\
\downarrow & & \downarrow & & \downarrow & & \\
M(B_U,\mathbb{A}^1_V) & \longrightarrow & M(B,\mathbb{A}^1_Y) & \longrightarrow & M\left(\frac{B/B-\mathbb{A}^1_Y}{B_U/B_U-\mathbb{A}^1_V}\right) & \xrightarrow{+1} & \\
\uparrow & & \uparrow & & \uparrow & & \\
M(P_V,V) & \longrightarrow & M(P,Y) & \longrightarrow & M\left(\frac{P/P-Y}{P_V/P_V-V}\right) & \xrightarrow{+1} &
\end{array}$$

According to Theorem 2.14 and homotopy invariance, the vertical maps in the first two columns are isomorphisms. As the rows in the diagram are distinguished triangles, the vertical maps in the third column also are isomorphisms.

Using Lemma 3.7 with $P = \mathbb{P}(N_Y X \oplus \mathbb{A}^1_Y)$, we can consider the following morphism of distinguished triangles :

$$\begin{array}{ccccccc}
M(P_V,V) & \longrightarrow & M(P,Y) & \longrightarrow & M\left(\frac{P/P-Y}{P_V/P_V-V}\right) & \xrightarrow{+1} & \\
\uparrow & & \uparrow & & \uparrow & & \\
M(P_V) & \longrightarrow & M(P) & \longrightarrow & M\left(\frac{P}{P_V}\right) & \xrightarrow{+1} & \\
\| & & \| & & \uparrow{\scriptstyle \mathfrak{p}^{-1}_{(P,P_Z)}} & & \\
M(P_V) & \longrightarrow & M(P) & \longrightarrow & M(P_Z)((s)) & \xrightarrow{+1} & \\
{\scriptstyle \mathfrak{l}_n(P_V)}\uparrow & & {\scriptstyle \mathfrak{l}_n(P)}\uparrow & & \uparrow{\scriptstyle \mathfrak{l}_n(P_Z)} & & \\
M(Y-Z)((n)) & \longrightarrow & M(Y)((n)) & \longrightarrow & M(Z)((d)) & \xrightarrow{+1} &
\end{array}$$

The triangle on the bottom is obtained by tensoring the Gysin triangle of the pair (Y,Z) with $\mathbb{Z}(n)[2n]$. From Theorem 2.14, the first two of the vertical composite arrows are isomorphisms, so the last one is also an isomorphism.

If we put together (vertically) the two previous diagrams, we finally obtain the following isomorphism of triangles :

$$\begin{array}{ccccccc}
M(U,V) & \longrightarrow & M(X,Y) & \longrightarrow & M(X,Y,Y') & \longrightarrow & M(U,V)\,[1] \\
\downarrow{\scriptstyle \mathfrak{p}_{(X-Y',Y-Z)}} & & \downarrow{\scriptstyle \mathfrak{p}_{(X,Y)}} & & \downarrow{\scriptstyle (*)} & & \downarrow \\
M(Y-Z)((n)) & \longrightarrow & M(Y)((n)) & \xrightarrow{j^*} & M(Z)((d)) & \xrightarrow{\partial_j} & M(Y-Z)((n))[1].
\end{array}$$

We define $\mathfrak{p}_{(X,Y,Z)}$ as the morphism labeled $(*)$ in the previous diagram so that property (iii) follows from the construction. The functoriality property (i) follows easily from the functoriality of the deformation diagram.

<u>The remaining relation</u>

To conclude it only remains to prove the symmetry property (ii). First of all, we remark that the above construction implies immediately the commutativity of the following diagram :

$$\begin{array}{ccc}
M\left(\frac{X/X-Y}{X-Y/X-Y\cup Y'}\right) & \longrightarrow & M\left(\frac{X/X-Y}{X-Z/X-Y}\right) \\
{\scriptstyle \mathfrak{p}_{(X,Y,Y')}}\searrow & & \swarrow{\scriptstyle \mathfrak{p}_{(X,Y,Z)}} \\
& M(Z)((d)), &
\end{array}$$

where the horizontal map is induced by the evident open immersions.

Thus, it will be sufficient to prove the commutativity of the following diagram :

$$\begin{array}{ccc} M\left(\frac{X}{X-Z}\right) & \xrightarrow{\alpha_{X,Y,Z}} & M\left(\frac{X/X-Y}{X-Z/X-Y}\right) \\ & {}_{\mathfrak{p}_{(X,Z)}}\searrow \quad (*) \quad \swarrow_{\mathfrak{p}_{(X,Y,Z)}} & \\ & M(Z)((n+m)), & \end{array}$$

where $\alpha_{X,Y,Z}$ denotes the canonical isomorphism.

From now on, we consider only the smooth triples (X, Y, Z) such that Z is a closed subscheme of Y. Using the functoriality of $\mathfrak{p}_{(X,Y,Z)}$, we remark that the diagram $(*)$ is natural with respect to morphisms $f : X' \to X$ which are transversal to Y and Z.

Consider the notations of the paragraph 2.13 and put $D_Z X = B_Z(\mathbb{A}^1_X)$ for short. We will expand these notations as follows :

$$D(X, Z) = D_Z X,\ B(X, Z) = B_Z X\ , P(X, Z) = P_Z X\ , N(X, Z) = N_Z X.$$

To (X, Y, Z), we associate the evident closed pair $(D_Z X, D_Z X|_Y)$ and the *double deformation space*

$$D(X, Y, Z) = D(D_Z X, D_Z X|_Y).$$

This scheme is in fact fibered over $\mathbb{A}^2_k$. The fiber over $(1, 1)$ is X and the fiber over $(0, 0)$ is $B(B_Z X \cup P_Z X, B_Z X|_Y \cup P_Z X|_Y)$. In particular, the $(0, 0)$-fiber contains the scheme $P(P_Z X, P_Z Y)$.

We now put $\left\{ \begin{array}{ll} D = D(X, Y, Z), & R = P(R_Z X, R_Z Y) \\ D' = D(Y, Y, Z), & P = R_Z Y. \end{array} \right.$

Remark also that $D(Z, Z, Z) = \mathbb{A}^2_Z$ and that $R = P \times_Z P'$ where $P' = P_Y X|_Z$.[13] From the description of the fibers of D given above, we obtain a deformation diagram of smooth triples :

$$(X, Y, Z) \to (D, D', \mathbb{A}^2_Z) \leftarrow (R, P, Z).$$

Note that these morphisms are on the smaller closed subscheme the $(0, 0)$-section and $(1, 1)$-section of $\mathbb{A}^2_Z$ over Z, denoted respectively by s_0 and s_1. Now we apply these morphisms to the diagram $(*)$ in order to obtain the following commutative diagram :

$$\begin{array}{ccccc} M_Z(X) & \longrightarrow & M_{\mathbb{A}^2_Z}(D) & \longleftarrow & M_Z(R) \\ \downarrow{\scriptstyle \mathfrak{p}_{(X,Z)}} \ \searrow^{\alpha_{X,Y,Z}} & & \downarrow{\scriptstyle \mathfrak{p}_{(D,\mathbb{A}^2_Z)}} \ \searrow & & \downarrow{\scriptstyle \mathfrak{p}_{(R,Z)}} \ \searrow^{\alpha_{R,P,Z}} \\ & M(X, Y, Z) \longrightarrow & & M(D, D', \mathbb{A}^2_Z) \longleftarrow & M(R, P, Z) \\ \swarrow_{\mathfrak{p}_{(X,Y,Z)}} & & \swarrow_{\mathfrak{p}_{(D,D',Z)}} & & \swarrow_{\mathfrak{p}_{(R,P,Z)}} \\ M(Z)((n+m)) & \xrightarrow[s_{1*}]{} & M(\mathbb{A}^2_Z)((n+m)) & \xleftarrow[s_{0*}]{} & M(Z)((n+m)). \end{array}$$

One knows that every part of this diagram save the triangle ones are commutative. As the morphisms s_{1*} and s_{0*} are isomorphisms, the commutativity of the left triangle is equivalent to the commutativity of the right one.

Thus, we are reduced to the case of the smooth triple (R, P, Z). Now, using the canonical split epimorphism $M(R) \to M_Z(R)$, we are reduced to prove the

[13] The last property is equivalent to the identification: $N(N_Z X, N_Z Y) = N_Z Y \oplus N_Y X|_Z$.

commutativity of the diagram :

$$\begin{array}{ccc} M(R) & \longrightarrow & M\left(\frac{R/R-P}{R-Z/R-P}\right) \\ {\scriptstyle i^*}\downarrow & & \\ M(Z)((d)) & \xleftarrow{\mathfrak{p}_{(R,P,Z)}} & \end{array}$$

where $i : Z \to R$ denotes the canonical closed immersion.

Using the property (iii) of the isomorphism $\mathfrak{p}_{(R,P,Z)}$, we are finally reduced to prove the commutativity of the triangle

$$\begin{array}{ccc} M(R) & \xrightarrow{j^*} & M(P)((n)) \\ {\scriptstyle i^*}\downarrow & & \\ M(Z)((d)) & \xleftarrow{k^*} & \end{array}$$

where j and k are the evident closed embeddings. This is Lemma 3.8. □

As a corollary (take $j = i \circ l$, $k = 1_Z$), we get the functoriality of the Gysin morphism of a closed immersion :

Corollary 3.10. *Let $Z \xrightarrow{l} Y \xrightarrow{i} X$ be closed immersion between smooth schemes such that i is of pure codimension n.*

Then, $l^* \circ i^* = (i \circ l)^*$.

3.3. Fundamental classes and residues.

3.11. Let us assume that H is the realization functor associated with a mixed Weil theory – Example 1.1(2). Recall that the twist on the cohomology H^{**} is given by the tensor product with the K-vector space $K(1) := H^1(\mathbb{G}_m)$. According to the axioms of a mixed Weil cohomology, it has rank one. In the remaining of this section, we will fix a trivialization $K(1) \simeq K$ and forget about the twists on H-cohomology.

Consider a smooth closed pair (X, Z) of codimension $n > 0$, and let i (resp. j) be the immersion of Z (resp. $U = X - Z$) in X. The beginning of the localization long exact sequence in motivic cohomology and H-cohomology is:

$$\begin{array}{ccccccccccccc} 0 \to & H^{2n-1,n-1}_{\mathcal{M}}(X) & \xrightarrow{j^*} & H^{2n-1,n-1}_{\mathcal{M}}(U) & \xrightarrow{\partial_{X,Z}} & CH^0(Z) & \xrightarrow{i_*} & CH^n(X) & \xrightarrow{j^*} & CH^n(U) & \longrightarrow 0 & & \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\ 0 \longrightarrow & H^{2n-1}(X) & \xrightarrow{j^*} & H^{2n-1}(U) & \xrightarrow{\partial_{X,Z}} & H^0(Z) & \xrightarrow{i_*} & H^{2n}(X) & \xrightarrow{j^*} & H^{2n}(U) & \xrightarrow{\partial_{X,Z}} & H^1(Z) & \end{array}$$

where the vertical maps are given by the regulator map. In the following examples, we will also use Remark 3.2 to identify the map $i_* : CH^0(Z) \to CH^n(X)$ with the usual pushforward of cycles.

The exactness of the long exact sequence in the above diagram gives immediately the following:

Proposition 3.12. *Consider the above assumptions and suppose Z is connected. Then the following properties are equivalent:*

(i) *The fundamental class $\eta_X(Z)$ has no torsion in $CH^n(X)$ (resp. is not homologically equivalent to zero with respect to H).*

(ii) *The residue map $\partial_{X,Z} : H^{2n-1,n-1}_{\mathcal{M}}(U) \to \mathbb{Z}$ (resp. $\partial_{X,Z} : H^{2n-1}(U) \to K$) is zero.*

(iii) *The map $j^* : H^{2n-1,n-1}_{\mathcal{M}}(X) \to H^{2n-1,n-1}_{\mathcal{M}}(U)$ (resp. $j^* : H^{2n-1}(X) \to H^{2n-1}(U)$) is an isomorphism.*

Example 3.13. We consider the assumptions of the above proposition.

(1) Assume that X is proper. Then, for any integer r, the degree of $r.\eta_X(Z)$ is r so that $\eta_X(Z)$ has no torsion. Thus $\partial_{X,Z} = 0$ and j^* is an isomorphism both in motivic cohomology and in H-cohomology.

(2) When $CH^n(X) = 0$ (for example $X = \mathbb{A}^r_k$), then $i_* = 0$ so that $\partial_{X,Z}$ is an epimorphism both for $H_{\mathcal{M}}$ and H (this generalizes Example 2.35).

(3) Assume k is algebraically closed and $\bar{X}$ is a smooth projective connected n-dimensional scheme. Then the group $CH_0(\bar{X})_0$ of 0-cycles of degree 0 usually contain torsion elements: according to a celebrated theorem of Roĭtman, its l-torsion part is isomorphic to $(\mathbb{Z}/l)^{2g}$ where g is the dimension of the Albanese variety associated with $\bar{X}$. Let α be a non zero l-torsion 0-cycle in $\bar{X}$. Let T be the support of α and choose an element $x \in T$. Put $X = \bar{X} - (T - \{x\})$, $Z = \{x\}$. Then $\eta_X(Z) = x$ is a torsion element of $CH^n(X)$. Moreover the torsion order m of x is characterized by the relation
$$\mathrm{Im}\Big(\partial_{X,Z} : H^{2n-1,n-1}_{\mathcal{M}}(X - Z) \to \mathbb{Z}\Big) = m.\mathbb{Z}.$$
In particular, this gives examples of non trivial elements, with residues $m > 0$, in the motivic cohomology group $H^{2n-1,n-1}_{\mathcal{M}}(X - Z)$ which is rather mysterious when $n > 1$ in the current state of our knowledge.

Remark 3.14. One can remark that given a cohomological class $u \in H^{2n}(U)$, the element $\partial_{X,Z}(u) \in H^1(Z)$ is an obstruction for u to be algebraic. In fact, if this residue is non zero, u cannot lies in the image of j^* according to the above diagram.

3.15. In the particular case $n = 1$, *i.e.* Z is a divisor in X, the situation becomes much more familiar Indeed, the preceding diagram becomes:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathbb{G}_m(X) & \xrightarrow{j^*} & \mathbb{G}_m(U) & \xrightarrow{\partial_{X,Z}} & \mathbb{Z}^{\pi_0(Z)} & \xrightarrow{i_*} & \mathrm{Pic}(X) & \xrightarrow{j^*} & \mathrm{Pic}(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(X) & \xrightarrow{j^*} & H^1(U) & \xrightarrow{\partial_{X,Z}} & K^{\pi_0(Z)} & \xrightarrow{i_*} & H^2(X) & \xrightarrow{j^*} & H^2(U) & \xrightarrow{\partial_{X,Z}} & H^1(Z)
\end{array}$$

Moreover, the morphism $\partial_{X,Z} : \mathbb{G}_m(U) \to \mathbb{Z}^{\pi_0(Z)}$ is the usual divisor map, which to a unit on U, considered as a meromorphic function on X with support in Z, associates its canonical divisor.

Example 3.16. (1) Assume k is a number field and C is an elliptic curve defined over k with base k-point x_0. The Jacobian $J = C(k)$ might contains a torsion point x different from x_0.[14] Then, according to the isomorphism

$$C(k) \to \mathrm{Pic}(X)^0, x \mapsto x - x_0, \tag{3.16.a}$$

the cycle $(x - x_0)$ is torsion. The situation is then analogue to that of Example 3.13(3): if we put $X = C - \{x_0\}$ and $Z = \{x\}$, then $\eta_X(Z)$ is a torsion element in $\mathrm{Pic}(X)$. The fact that the torsion order m of x in $C(k)$ is characterized by the relation:
$$\mathrm{Im}\Big(\partial_{X,Z} : \mathbb{G}_m(X - Z) \to \mathbb{Z}\Big) = m.\mathbb{Z}$$

[14]For example one can consider the Fermat curve over $\mathbb{Q}$
$$C \subset \mathbb{P}^2 : x^3 + y^3 - z^3 = 0$$
with base point $(1, -1, 0)$. It has exactly two $\mathbb{Q}$-points of order 3: $(1, 0, 1)$ and $(0, 1, 1)$ (see [Sil09] for many other examples).

is now a tautology – given the isomorphism (3.16.a).

(2) Assume $k = \mathbb{Q}$ and X is a smooth projective curve. Let I be a finite set of prime numbers. For any $p \in I$, one can find a closed point x_p of X whose degree is p. Then, for degree reasons, the cycles $\{x_p, p \in I\}$ in $CH_0(X)$ form a $\mathbb{Z}$-free family. Thus, if we put:
$$Z = \{x_p, p \in I\},$$
we are in the case where $i_* : \mathbb{Z}^{\pi_0(X)} \to \mathrm{Pic}(X)$ is injective, the residue map $\partial_{X,Z} : \mathbb{G}_m(U) \to \mathbb{Z}^{\pi_0(X)}$ is zero and $j^* : \mathbb{G}_m(X) \to \mathbb{G}_m(U)$ is an isomorphism.

4. Gysin morphism

In this section, motives are considered in the category $DM_{gm}(k)$.

4.1. **Construction.**

4.1.1. *Preliminaries.*

Lemma 4.1. *Let X be a smooth scheme, P/X and Q/X be projective bundles of respective dimensions n and m. We consider λ_P (resp. λ_Q) the canonical dual line bundle on P (resp. Q) and λ'_P (resp. λ'_Q) its pullback on $P \times_X Q$. Let $p : P \times_X Q \to X$ be the canonical projection.*

Then, the morphism $\sigma : M(P \times_X Q) \longrightarrow \bigoplus_{i,j} M(X)(i+j)[2(i+j)]$ given by the formula
$$\sigma = \sum_{0 \leq i \leq n,\, 0 \leq j \leq m} \mathfrak{c}_1(\lambda'_P)^i \boxtimes \mathfrak{c}_1(\lambda'_Q)^j \boxtimes p_*$$
is an isomorphism.

Proof. As σ is compatible with pullback, we can assume using property **(MV)** of Proposition 2.4 that P and Q are trivializable projective bundles. Using the invariance of σ under automorphisms of P or Q, we can assume that P and Q are trivial projective bundles. From the definition of σ, we are reduced to the case $X = \mathrm{Spec}(k)$. Then, σ is just the tensor product of the two projective bundle isomorphisms (cf paragraph 2.10) for P and Q. □

The following proposition is the key point in the definition of the Gysin morphism for a projective morphism.

Proposition 4.2. *Let X be a smooth scheme, $p : P \to X$ be a projective bundle of rank n and $s : X \to P$ a section of p.*

Then, the composite map $M(X)((n)) \xrightarrow{\mathfrak{l}_n(P)} M(P) \xrightarrow{s^*} M(X)((n))$ is the identity.[15]

Proof. In this proof, we work in the category $DM_{gm}^{eff}(k)$.

Let $\eta_P(X)$ be the motivic fundamental class associated with s (see Definition 3.1). According to Lemma 3.4, we obtain: $s^* = \eta_P(X) \boxtimes_P p_*$.

Let E/X be the vector bundle on X such that $P = \mathbb{P}(E)$. Let λ be the canonical dual line bundle on P. If we consider the line bundle $L = s^{-1}(\lambda^\vee)$ on X, the section s corresponds uniquely to a monomorphism $L \to E$ of vector bundles on P. We consider the following vector bundle on P:
$$F = \lambda \otimes p^{-1}(E/L).$$

[15] In fact, this result holds in the effective category $DM_{gm}^{eff}(k)$ as the proof will show.

Then the canonical morphism:

$$\lambda^\vee \to p^{-1}(E) \to p^{-1}(E/L)$$

made by the canonical inclusion and the canonical projection induces a section σ of F/P which is transversal to the zero section s_0^F of F/P and such that the following square is cartesian:

$$\begin{array}{ccc} X & \xrightarrow{s} & P \\ \downarrow & & \downarrow{\scriptstyle \sigma} \\ P & \xrightarrow{s_0^F} & F. \end{array}$$

Thus, according to Lemma 3.5, we get: $\eta_P(X) = \mathfrak{c}_n(F)$.

According to the projective bundle theorem, $CH^*(P)$ is a free $CH^*(X)$-module with basis $1, ..., c_1(\lambda)^n$: using the definition of F, we easily get that the coefficient of $c_1(\lambda)^n$ in $c_n(F)$ relative to this basis is 1. Given the definition of $\mathfrak{l}_n(P)$ and the equality $s^* = \mathfrak{c}_n(F) \boxtimes_P p_*$, this proves the proposition. □

Remark 4.3. As a corollary, we obtain the following reinforcement of Theorem 2.14, more precisely of the normalization condition for the purity isomorphism :

Let X be a smooth scheme, P/X be a projective bundle of rank n, and $s : X \to P$ be a section of P/X. Then, the purity isomorphism $\mathfrak{p}_{(P,s(X))}$ is the inverse isomorphism of the composition

$$M(X)((n)) \xrightarrow{\mathfrak{l}_n(P)} M(P) \xrightarrow{(1)} M_{s(X)}(P)$$

where (1) is the canonical map.

4.1.2. *Gysin morphism of a projection.* The following definition will be a particular case of Definition 4.8.

Definition 4.4. Let X be a smooth scheme, P be a projective bundle of rank n over X and $p : P \to X$ be the canonical projection.

Using the notation of (2.10.b), we put:

$$p^* = \mathfrak{l}_n(P)(-n)[-2n] : M(X) \to M(P)\,(-n)[-2n]$$

and call it the Gysin morphism of p.

Example 4.5. The Gysin morphism p^* defined above induces pushforward on cohomology:

$$p_* : H^{a,b}(P) \to H^{a-2n,b-n}(X).$$

According to the projection formula, this morphism is uniquely characterized by the following properties:

(1) $p_*(y.p^*(x)) = p_*(y)$.

(2) $p_*(c_1(\lambda)^i) = \begin{cases} 1 & \text{if i=n} \\ 0 & \text{otherwise.} \end{cases}$

As a result, we easily get that the morphism

$$H_{\mathcal{M}}(p^*) : H^{2i,i}_{\mathcal{M}}(P) \to H^{2(i-n),i-n}_{\mathcal{M}}(X)$$

coincides with the usual pushforward on Chow groups through the isomorphism (2.5.a).

Lemma 4.6. *Let P, Q be projective bundles over a smooth scheme X of respective ranks n, m. Consider the following projections :*

$$\begin{array}{ccccc} & & P & & \\ & \overset{q'}{\nearrow} & & \overset{p}{\searrow} & \\ P\times_X Q & & & & X \\ & \underset{p'}{\searrow} & & \underset{q}{\nearrow} & \\ & & Q & & \end{array}$$

Then, the following diagram is commutative :

$$\begin{array}{ccccc} & & M(P)((-m)) & & \\ & \overset{p^*}{\nearrow} & & \overset{q'^*}{\searrow} & \\ M(X) & & & & M(P\times_X Q)((-n-m)) \\ & \underset{q^*}{\searrow} & & \underset{p'^*}{\nearrow} & \\ & & M(Q)((-n)) & & \end{array}$$

Proof. Indeed, using the compatibility of the motivic Chern class with pullback (cf 2.5), we see that both composite morphisms q'^*p^* and p'^*q^* are equal (up to twist and suspension) to the composite

$$M(X)((n+m)) \to \bigoplus_{i\leq n, j\leq m} M(X)((i+j)) \to M(P\times_X Q)\,,$$

where the first arrow is the obvious split monomorphism and the second arrow is the inverse isomorphism to the one constructed in Lemma 4.1. □

4.1.3. *General case.* The following lemma is all we need to finish the construction of the Gysin morphism of a projective morphism :

Lemma 4.7. *Consider a commutative diagram*

$$\begin{array}{ccccc} & & P & & \\ & \overset{i}{\nearrow} & & \overset{p}{\searrow} & \\ Y & & & & X \\ & \underset{j}{\searrow} & & \underset{q}{\nearrow} & \\ & & Q & & \end{array}$$

where X and Y are smooth schemes, i (resp. j) is a closed immersion of codimension $n+d$ (resp. $m+d$), P (resp. Q) is a projective bundle over X of dimension n (resp. m) with projection p (resp. q).

Then, the following diagram is commutative

$$\begin{array}{ccccc} & & M(P))((m)) & & \\ & \overset{p^*}{\nearrow} & & \overset{i^*}{\searrow} & \\ M(X)((n+m)) & & & & M(Y)((n+m+d)). \\ & \underset{q^*}{\searrow} & & \underset{j^*}{\nearrow} & \\ & & M(Q)((n)) & & \end{array} \tag{4.7.a}$$

Proof. Considering the diagonal embedding $Y \xrightarrow{(i,j)} P\times_X Q$, we divide diagram (4.7.a) into three parts:

$$\begin{array}{ccccc} & \overset{p^*}{\nearrow} & M(P)((m)) & \overset{i^*}{\searrow} & \\ & & \downarrow{\scriptstyle p'^*} \quad {\scriptstyle (2)} & & \\ M(X)((n+m)) & {\scriptstyle (1)} & M(P\times_X Q) & \xrightarrow{(i,j)^*} & M(Y)((n+m+d)). \\ & & \uparrow{\scriptstyle q'^*} \quad {\scriptstyle (3)} & & \\ & \underset{q^*}{\searrow} & M(Q)((n)) & \underset{j^*}{\nearrow} & \end{array}$$

The commutativity of part (1) is Lemma 4.6. The commutativity of part (2) and that of part (3) are equivalent to the case $X = Q$, $q = 1_X$ – and thus $m = 0$.

Assume we are in this case. We introduce the following morphisms where the square (*) is cartesian and γ is the graph of the X-morphism i:

$$\begin{array}{ccccc} & \overset{\gamma}{\nearrow} & P_Y & \xrightarrow{p'} & Y \\ Y & & {\scriptstyle j'}\downarrow & (*) & \downarrow{\scriptstyle j} \\ & \underset{i}{\searrow} & P & \xrightarrow{p} & X \end{array}$$

Note that γ is a section of p'. Thus, Proposition 4.2 gives: $\gamma^* p'^* = 1$, and we reduce the commutativity of the diagram (4.7.a) to that of the following one:

$$\begin{array}{ccccc} & \overset{\gamma^*}{\swarrow} & M(P_Y)((d)) & \xleftarrow{p'^*} & M(Y)((n+d)) \\ M(Y)((n+d)) & (4) & \uparrow{\scriptstyle j'^*} & (5) & \uparrow{\scriptstyle j^*} \\ & \underset{i^*}{\nwarrow} & M(P) & \xleftarrow{p^*} & M(X)((n)) \end{array}$$

Then commutativity of part (4) is Corollary 3.10 and that of part (5) follows from Lemma 3.7. □

Let $f : Y \to X$ be a projective morphism between smooth schemes. Following the terminology of Fulton (see [Ful98, §6.6]), we say that f has codimension d if it can be factored into a closed immersion $Y \to P$ of codimension e followed by the projection $P \to X$ of a projective bundle of dimension $e - d$. In fact, the integer d is uniquely determined (cf *loc.cit.* appendix B.7.6). Using the preceding lemma, we can finally introduce the general definition :

Definition 4.8. Let X, Y be smooth schemes and $f : Y \to X$ be a projective morphism of codimension d.

We define the Gysin morphism associated with f in $DM_{gm}(k)$

$$f^* : M(X) \to M(Y)((d))$$

by choosing a factorisation of f into $Y \xrightarrow{i} P \xrightarrow{p} X$ where i is a closed immersion of pure codimension $n + d$ and p is the projection of a projective bundle of rank n, and putting :

$$f^* = \Big[M(X)((n)) \xrightarrow{\mathfrak{l}_n(P)} M(P) \xrightarrow{i^*} M(Y)((n+d)) \Big]((-n)),$$

definition which does not depend upon the choices made according to the previous lemma.

The map induced by the previous Gysin morphism on motivic cohomology does extend the usual pushfoward on Chow groups:

Proposition 4.9. *Let X, Y be smooth schemes and $f : Y \to X$ be a projective morphism of codimension d.*

Then for any integer n, the following diagram is commutative:

$$\begin{array}{ccc} CH^n(Y) & \xrightarrow{f_*} & CH^{n-d}(X) \\ {\scriptstyle \epsilon_Y}\downarrow & & \downarrow{\scriptstyle \epsilon_X} \\ H^{2n,n}_{\mathcal{M}}(Y) & \xrightarrow{H_{\mathcal{M}}(f^*)} & H^{2(n-d),n-d}_{\mathcal{M}}(X). \end{array}$$

In fact, this follows from Remark 3.2 and Example 4.5.

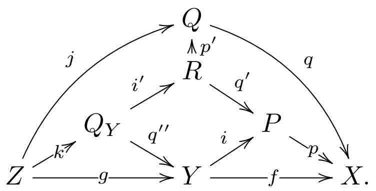
4.2. **Properties.**

4.2.1. *Functoriality.*

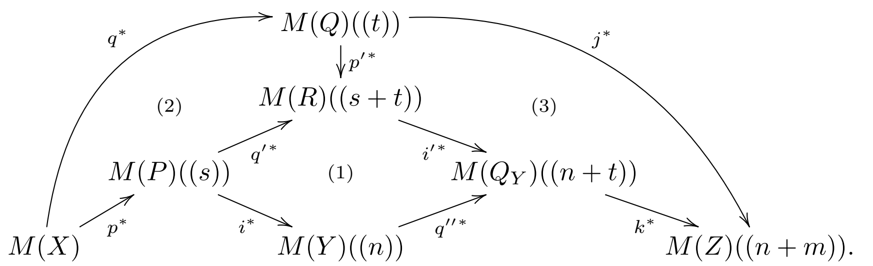
Proposition 4.10. *Let X, Y, Z be smooth schemes and $Z \xrightarrow{g} Y \xrightarrow{f} X$ be projective morphisms of respective codimensions m and n.*

Then, in $DM_{gm}(k)$, we get the equality : $g^* \circ f^* = (fg)^*$.

Proof. We first choose projective bundles P, Q over X, of respective dimensions s and t, fitting into the following diagram with $R = P \times_X Q$ and $Q_Y = Q \times_X Y$:



The prime exponent of a symbol indicates that the morphism is deduced by base change from the morphism with the same symbol. We then have to prove that the following diagram of $DM_{gm}(k)$ commutes :



The commutativity of part (1) is a corollary of Lemma 3.7, that of part (2) is Lemma 4.6 and that of part (3) follows from Lemma 4.7 and Corollary 3.10. □

Example 4.11. Let $i : Z \to X$ be a closed immersion of codimension n between smooth schemes. Assume i admits a proper retraction p – note that p is then projective. According to the above result, one gets $p^*i^* = 1$ so that i^* becomes a split epimorphism. As a result, the Gysin triangle associated with i is split and one get: $M(X) = M(X - Z) \oplus M(Z)(n)[2n]$.

Equivalently, the residue $\partial_{X,Z}$ is zero: on any realization, there is no obstruction to extend a cohomology class $u \in H^{**}(X - Z)$ to $H^{**}(X)$.

Of course, this extends the fact already noted in Lemma 2.12.

Remark 4.12. Note it is important to require in the preceding example that p is proper. Indeed, when i is the zero section of a trivial vector bundle of rank n, $\partial_{X,Z}$ is a non zero epimorphism.

This shows it is not reasonable to look for a theory of Gysin morphisms without properness assumption.

4.2.2. *Additivity.* Using Proposition 2.26, we readily deduce from our definition The following result:

Proposition 4.13. *Let X, Y be smooth schemes and $f : Y \to X$ be a projective morphism of codimension d.*

Let $(X_i)_{i \in I}$ (resp. $(Y_j)_{j \in J}$) be the connected components of X (resp. Y). For any $j \in J$, we let $\phi(j)$ be the unique element of I such that $f(Y_j) \subset X_{\phi(j)}$.

Using additivity of motives, one can write uniquely $f^* = f^*_{ij}$ where f^*_{ij} is a morphism $M(X_i) \to M(Y_j)(d)[2d]$. Then for any couple (i,j), one has:

$$f^*_{ij} = \begin{cases} \left(f|^{X_i}_{Y_j}\right)^* & \text{if } i = \phi(j), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 4.14. As already observed in Remark 2.19 for the purity isomorphism, this proposition allows one to extend the definition of the Gysin morphism to the case where the codimension of the projective morphism f is non constant.

4.2.3. *Projection formula and excess of intersection.* From Definition 4.8 and Proposition 2.29 we directly obtain the following proposition :

Proposition 4.15. *Consider a cartesian square of smooth schemes*

$$\begin{array}{ccc} T & \xrightarrow{g} & Z \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{f} & X \end{array} \tag{4.15.a}$$

such that f and g are projective morphisms of the same codimensions.

Then, the relation $f^*p_* = q_*g^*$ holds in $DM_{gm}(k)$.

4.16. Consider now a cartesian square of shape (4.15.a) such that f (resp. g) is a projective morphism of codimension m (resp. m). Then $m \leq n$ and we call $e = n - m$ the *excess of dimension* attached with (4.15.a).

We can also associate with the above square a vector bundle ξ of rank e, called the *excess bundle*. Choose $Y \xrightarrow{i} P \xrightarrow{\pi} X$ a factorisation of f such that i is a closed immersion of codimension r and π is the projection of a projective bundle of dimension s. We consider the following cartesian squares:

$$\begin{array}{ccccc} T & \xrightarrow{i'} & Q & \xrightarrow{\pi'} & Z \\ {\scriptstyle q}\downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{i} & P & \xrightarrow{\pi} & X \end{array}$$

Then N_TQ is a sub-vector bundle of $q^{-1}N_YP$ and we put $\xi = q^{-1}N_YP/N_TQ$. This definition is independent of the choice of P (see [Ful98], proof of prop. 6.6).

The following proposition is now a straightforward consequence of Definition 4.8 and the second case of Proposition 2.29 :

Proposition 4.17. *Consider the above notations.*

Then, the relation $f^*p_* = \big(\mathfrak{c}_e(\xi) \boxtimes q_*((m))\big) \circ g^*$ holds in $DM_{gm}(k)$.

4.2.4. *Compatibility with the Gysin triangle.*

Proposition 4.18. *Consider a topologically cartesian square of smooth schemes*

$$\begin{array}{ccc} T & \xrightarrow{j} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{i} & X \end{array}$$

such that f and g are projective morphisms, i and j are closed immersions. Put $U = X - Z$, $V = Y - T$ and let $h : V \to U$ be the projective morphism induced by f. Let n, m, p, q be respectively the relative codimensions of i, j, f, g.

Then the following diagram is commutative

$$\begin{array}{ccccccc} M(V)((p)) & \longrightarrow & M(Y)((p)) & \xrightarrow{j^*} & M(T)((m+p)) & \xrightarrow{\partial_{Y,T}} & M(V)((p))[1] \\ {\scriptstyle h^*}\uparrow & & {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle g^*((n))} & & \uparrow{\scriptstyle h^*} \\ M(U) & \longrightarrow & M(X) & \xrightarrow{i^*} & M(Z)((n)) & \xrightarrow{\partial_{X,Z}} & M(U)\,[1] \end{array}$$

where the two lines are the obvious Gysin triangles.

Proof. Use the definition of the Gysin morphism and apply Lemma 3.7, Theorem 3.9. □

4.2.5. *Gysin morphisms and transfers.*

4.19. Recall that Voevodsky's motives are built in with transfers. Given a finite surjective morphism $f : Y \to X$ between smooth schemes, the transpose of the graph of f defines a cycle ${}^tf$ in $X \times Y$ which is finite and dominant over any component of X ; in other words, a finite correspondence from X to Y. Therefore one gets a morphism of geometric motives:

$$ {}^tf_* : M(X) \to M(Y). $$

Theorem 4.20. *Given the above notations, one gets:* $f^* = {}^tf_*$.

The proof relies on the detailed study of the Gersten resolution done in [Dég11a]. However, the case where f is étale is much more elementary and in the proof below, we first give a direct argument in this case.

Proof. 1) Assume f is an étale cover:

Consider the cartesian square of smooth schemes

$$\begin{array}{ccc} Y \times_X Y & \xrightarrow{g} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & X. \end{array}$$

We first prove that ${}^tf'_* f^* = g^* \, {}^tf_*$. Choose a factorisation $Y \xrightarrow{i} P \xrightarrow{\pi} X$ of f into a closed immersion and the projection of a projective bundle. The preceding square can be divided into two squares

$$\begin{array}{ccccc} Y \times_X Y & \xrightarrow{j} & P \times_X Y & \xrightarrow{q} & Y \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f''}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & P & \xrightarrow{\pi} & X. \end{array}$$

The assertion then follows from the commutativity of the following diagram.

$$\begin{array}{ccccc} M(Y \times_X Y) & \xleftarrow{j^*} & M(P \times_X Y) & \xleftarrow{q^*} & M(Y) \\ {\scriptstyle {}^tf'_*}\uparrow & (1) & {\scriptstyle {}^tf''_*}\uparrow & (2) & \uparrow{\scriptstyle {}^tf_*} \\ M(Y) & \xleftarrow[i^*]{} & M(P) & \xleftarrow[p^*]{} & M(X) \end{array}$$

The commutativity of part (1) follows from [Dég08b], prop. 2.5.2 (case 1) and that of part (2) from [Dég08b], prop. 2.2.15 (case 3).

Then, considering the diagonal immersion $Y \xrightarrow{\delta} Y \times_X Y$, it suffices to prove in view of Proposition 4.10 that $\delta^* \circ {}^tf'_* = 1$. As Y/X is étale, Y is a connected component of $Y \times_X Y$. Thus, $M(Y)$ is a direct factor of $M(Y \times_X Y)$. Using Proposition 2.29, we get that δ^* is the canonical projection on this direct factor. On the other hand, ${}^tf'_*$ is the canonical inclusion so that we are done.

2) The general case:

We use the setting of [Dég11b]. The triangulated category $DM_{gm}(k)$ can be embedded in a larger triangulated category $DM(k)$ (see [Dég11b, 4.7 and 4.11]). Let us put $\alpha = {}^tf_* - f^*$. It suffices to prove that for any object $\mathbb{E}$ of $DM(k)$, the induced map:

$$ \alpha^* : \mathrm{Hom}_{DM}(M(Y), \mathbb{E}) \to \mathrm{Hom}_{DM}(M(X), \mathbb{E}) $$

is zero (in fact, the case $\mathbb{E} = M(Y)$ is sufficient). Let $\mathcal{T}$ be the full triangulated subcategory made by the object $\mathbb{E}$ such that $\alpha^* = 0$. *A priori* it is a thick triangulated subcategory. Moreover, as $M(X)$ and $M(Y)$ are compact objects, it is stable by direct sums.

The category $DM(k)$ admits a non degenerated t-structure – the *homotopy t-structure*, [Dég11b, 5.6]. Therefore, it is sufficient to prove that $\mathcal{T}$ contains any object of the heart of the homotopy t-structure. Therefore, according to Theorem 5.11 of [Dég11b], we are reduced to prove $\alpha^* = 0$ when $\mathbb{E}$ is a homotopy module. This has been checked in [Dég11b, 3.16] – taking into account [Dég11b, (2.5.a)]. □

Example 4.21. Let X (resp. Y) be a smooth connected scheme with function field K (resp. L) and $f : Y \to X$ be a finite dominant morphism. The morphism f induces a finite extension $K \to L$ whose Gysin morphism induces on $H^{1,1}_{\mathcal{M}}$ a morphism of the form:

$$f_* : \mathbb{G}_m(Y) \simeq H^{1,1}_{\mathcal{M}}(Y) \to H^{1,1}_{\mathcal{M}}(X) \simeq \mathbb{G}_m(X).$$

According to the previous theorem and [Dég08b, 2.2.4], this morphism is obtained by restriction of the norm morphism of L/K with respect to the inclusions $\mathbb{G}_m(X) \subset K^\times$ and $\mathbb{G}_m(Y) \subset L^\times$.

Given an integer $n > 1$, we also obtain that the following diagram is commutative:

$$\begin{array}{ccc} H^{n,n}_{\mathcal{M}}(Y) & \xrightarrow{f_*} & H^{n,n}_{\mathcal{M}}(X) \\ \downarrow & & \downarrow \\ K^M_n(L) & \xrightarrow{N_{L/K}} & K^M_n(K) \end{array}$$

where K^M_* is the Milnor K-theory functor, $N_{L/K}$ is the Bass-Tate transfer on Milnor K-theory and the vertical maps are obtained using the isomorphism:

$$K^M_n(K) = \varinjlim_{U \subset X, U \neq \varnothing} H^{n,n}_{\mathcal{M}}(U).$$

We refer the reader to [SV00]: Theorem 3.4 for this isomorphism and Lemma 3.4.4 for the commutativity of the above diagram.

4.3. **Duality pairings, motive with compact support.**

4.22. We first recall the abstract definition of duality in monoidal categories. Let $\mathscr{C}$ be a symmetric monoidal category with product $\otimes$ and unit $\mathbf{1}$. An object X of $\mathscr{C}$ is said to be *strongly dualizable* if there exists an object X^* of $\mathscr{C}$ and two maps

$$\eta : \mathbf{1} \to X^* \otimes X, \quad \epsilon : X \otimes X^* \to \mathbf{1}$$

such that the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{X \otimes \eta} & X \otimes X^* \otimes X \\ & {\scriptstyle 1_X} \searrow & \downarrow {\scriptstyle \epsilon \otimes X} \\ & & X \end{array} \qquad \begin{array}{ccc} X^* & \xrightarrow{\eta \otimes X^*} & X^* \otimes X \otimes X^* \\ & {\scriptstyle 1_{X^*}} \searrow & \downarrow {\scriptstyle X^* \otimes \epsilon} \\ & & X^* \end{array}$$

The object X^* is called a *strong dual* of X. For any objects Y and Z of $\mathscr{C}$, we then have a canonical bijection

$$\mathrm{Hom}_{\mathscr{C}}(Z \otimes X, Y) \simeq \mathrm{Hom}_{\mathscr{C}}(Z, X^* \otimes Y).$$

In other words, $X^* \otimes Y$ is the internal Hom of the pair (X, Y) for any Y. In particular, such a dual is unique up to a canonical isomorphism. If X^* is a strong dual of X, then X is a strong dual of X^*.

Suppose $\mathscr{C}$ is a closed symmetric monoidal triangulated category. Denote by $\underline{\mathrm{Hom}}$ its internal Hom. For any objects X and Y of $\mathscr{C}$ the evaluation map

$$X \otimes \underline{\mathrm{Hom}}(X, \mathbf{1}) \to \mathbf{1}$$

tensored with the identity of Y defines by adjunction a map

$$\underline{\mathrm{Hom}}(X, \mathbf{1}) \otimes Y \to \underline{\mathrm{Hom}}(X, Y).$$

The object X is strongly dualizable if and only if this map is an isomorphism for all objects Y in $\mathscr{C}$. In this case indeed, $X^* = \underline{\mathrm{Hom}}(X, \mathbf{1})$.

4.23. Let X be a smooth projective k-scheme of pure dimension n and denote by $p : X \to \mathrm{Spec}(k)$ the canonical projection, $\delta : X \to X \times_k X$ the diagonal embedding.

Then we can define morphisms

$$\eta : \mathbb{Z} \xrightarrow{p^*} M(X)(-n)[-2n] \xrightarrow{\delta_*} M(X)(-n)[-2n] \otimes M(X)$$
$$\epsilon : M(X) \otimes M(X)(-n)[-2n] \xrightarrow{\delta^*} M(X) \xrightarrow{p_*} \mathbb{Z}.$$

One checks easily using the properties of the Gysin morphism these maps turn $M(X)(-n)[-2n]$ into the dual of $M(X)$. We thus have obtained :

Theorem 4.24. *Let X/k be a smooth projective scheme.*

Then the couple of morphisms (η, ϵ) defined above is a duality pairing. Thus $M(X)$ is strongly dualizable with dual $M(X)(-n)[-2n]$.

Example 4.25. (1) Using the duality obtained previously in conjunction with the isomorphism (2.5.a), we obtain for smooth projective schemes X and Y, d being the dimension of Y, a canonical map:

$$\begin{aligned} CH^d(X \times Y) &\simeq \mathrm{Hom}_{DM^{eff}_{gm}(k)}(M(X) \otimes M(Y), \mathbb{Z}(d)[2d]) \\ &\xrightarrow{(*)} \mathrm{Hom}_{DM_{gm}(k)}(M(X) \otimes M(Y), \mathbb{Z}(d)[2d]) \\ &= \mathrm{Hom}_{DM_{gm}(k)}(M(X), M(Y)). \end{aligned}$$

As the isomorphism (2.5.a) is compatible with products and pullbacks, we check easily this defines a monoidal functor from Chow motives to mixed motives obtaining a new construction of the stable version of the functor which appears in [FSV00, chap. 5, 2.1.4].

Recall finally that the cancellation theorem of Voevodsky [Voe10] says precisely the map $(*)$ is an isomorphism. In particular, the functor from Chow motives to mixed motives is a full embedding.

(2) The Gysin morphism $p^* : \mathbb{Z}(n)[2n] \to M(X)$ defines indeed a homological class η_X in $H^{\mathcal{M}}_{2n,n}(X) = \mathrm{Hom}_{DM_{gm}(k)}(\mathbb{Z}(n)[2n], M(X))$.

The duality above induces an isomorphism

$$H^{p,q}_{\mathcal{M}}(X) \to H^{\mathcal{M}}_{p-2n,q-n}(X)$$

which is by definition the cap-product by η_X. This is one of the usual form of Poincaré duality between cohomology and homology, the class η_X being the fundamental class of X.

Note this formula can be extended to the case of the cohomology with coefficients in an object $\mathcal{E}$ of $DM(k)$ (Example 1.2). We left the formulation to the reader as an exercice.

(3) Assume the realization functor H is associated with a mixed Weil theory (case (2) of Example 1.1). The Künneth formula implies the following functor,
$$H^* : DM_{gm}(k)^{op} \to (K - ev)^{\mathbb{Z}}, M \mapsto \big(H^i(M)\big)_{i\in\mathbb{Z}}$$
is monoidal where we put the usual tensor structure on $\mathbb{Z}$-graded vector spaces. Applying this functor to the duality pairing ϵ of the previous Theorem, we get a perfect pairing:
$$\langle .,. \rangle : H^i(X) \otimes H^{2d-i}(X)(d) \to K.$$
We obtain from the definition:
$$\langle x, y\rangle = p_*(x.y)$$
where $p_* : H^{2d,d}(X) \to H^{0,0}(k) = K$ is induced by the Gysin morphism p^* on motives. This is another form of Poincaré duality (using the Künneth formula), and p_* is usually called the *trace map* of X/k.

4.26. The last application of this section uses the stable version of the category of motivic complexes as defined in [CD09a, 7.15] and denoted by $DM(k)$. Remember it is a triangulated symmetric monoidal category. Moreover, there is a canonical monoidal fully faithful functor $DM_{gm}(k) \to DM(k)$ (see [CD09b, 10.1.4]). The idea of the following definition comes from [CD07, 2.6.3]:

Definition 4.27. Let X be a smooth scheme of dimension d.

We define the motive with compact support of X as the object of $DM(k)$
$$M^c(X) = \mathrm{R}\underline{\mathrm{Hom}}_{DM(k)}(M(X), \mathbb{Z}(d)[2d]).$$

This motive with compact support satisfies the following properties:

(i) For any morphism $f : Y \to X$ of relative dimension n between smooth schemes, the usual functoriality of motives induces:
$$f^* : M^c(X)(n)[2n] \to M^c(Y).$$

(ii) For any projective morphism $f : Y \to X$ between smooth schemes, the Gysin morphism of f induces:
$$f_* : M^c(Y) \to M^c(X).$$

(iii) Let $i : Z \to X$ be a closed immersion between smooth schemes, and j the complementary open immersion. Then the Gysin triangle associated with (X, Z) induces a distinguished triangle:
$$M^c(Z) \xrightarrow{i_*} M^c(X) \xrightarrow{j^*} M^c(U) \xrightarrow{\partial'_{X,Z}} M^c(Z)[1].$$

(iv) If X is a smooth k-scheme of relative dimension d, p its structural morphism and δ its diagonal embedding, the composite morphism
$$M(X) \otimes M(X) \xrightarrow{\delta^*} M(X)(d)[2d] \xrightarrow{p_*} \mathbb{Z}(d)[2d]$$
induces a map
$$\phi_X : M(X) \to M^c(X)$$
which is an isomorphism when X is projective (cf 4.24). Moreover, for any open immersion $j : U \to X$, $j^* \circ \phi_X \circ j_* = \phi_U$ (this follows easily from 4.15).

Remark 4.28. Note also that the formulas we have proved for the Gysin morphism or the Gysin triangle correspond to formulas involving the data (i), (ii) or (iii) of motives with compact support.

4.29. Consider a smooth scheme X of pure dimension d. According to Definition 4.27, as soon as $M(X)$ admits a strong dual $M(X)^\vee$ in $DM(k)$, we get a canonical isomorphism:

$$M^c(X) = M(X)^\vee(d)[2d]. \tag{4.29.a}$$

The same remark can be applied if we work in $DM(k) \otimes \mathbb{Q}$. Recall that duality is known in the following cases (it follows for example from the main theorem of [Rio05]):

Proposition 4.30. *Let X be a smooth scheme of dimension d.*

1. *Assume k admits resolution of singularities.*
   Then $M(X)$ is strongly dualizable in $DM_{gm}(k)$.
2. *In any case, $M(X) \otimes \mathbb{Q}$ is strongly dualizable in $DM_{gm}(k) \otimes \mathbb{Q}$.*

Recall that Voevodsky has defined a motive with compact support (even without the smoothness assumption). It satisfies all the properties listed above except that (i) and (iii) requires resolution of singularities. Then according to the preceding proposition and formula (4.29.a), our definition agrees with that of Voevodsky if resolution of singularities holds over k (apply [FSV00, chap. 5, th. 4.3.7]). This implies in particular that $M^c(X)$ is in $DM_{gm}(k)$ or, in the words of Voevodsky, it is *geometric*. Moreover, we know from the second case of the preceding proposition that $M^c(X) \otimes \mathbb{Q}$ is always geometric.

References

[Ayo07] J. Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. *Astérisque*, (314):x+466 pp. (2008), 2007.

[BVK08] L. Barbieri-Viale and B. Kahn. A note on relative duality for Voevodsky motives. *Tohoku Math. J. (2)*, 60(3):349–356, 2008.

[CD07] D.-C. Cisinski and F. Déglise. Mixed Weil cohomologies. arXiv:0712.3291, 2007.

[CD09a] D.-C. Cisinski and F. Déglise. Local and stable homological algebra in Grothendieck abelian categories. *HHA*, 11(1):219–260, 2009.

[CD09b] D.-C. Cisinski and F. Déglise. Triangulated categories of mixed motives. arXiv:0912.2110, 2009.

[Dég02] F. Déglise. *Modules homotopiques avec transferts et motifs génériques*. PhD thesis, Université Paris VII, 2002.

[Dég04] F. Déglise. Interprétation motivique de la formule d'excès d'intersection. *C. R. Math. Acad. Sci. Paris*, 338(1):41–46, 2004. Presented by J.P. Serre.

[Dég08a] F. Déglise. Around the Gysin triangle II. *Doc. Math.*, 13:613–675, 2008.

[Dég08b] F. Déglise. Motifs génériques. *Rendiconti Sem. Mat. Univ. Padova*, 119, 2008.

[Dég11a] F. Déglise. Coniveau filtration and motives. 2011.

[Dég11b] F. Déglise. Modules homotopiques. *Doc. Math.*, 16:411–455, 2011.

[Del77] P. Deligne. *Cohomologie étale*. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA $4\frac{1}{2}$, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

[FL98] E. M. Friedlander and H. B. Lawson. Moving algebraic cycles of bounded degree. *Invent. Math.*, 132(1):91–119, 1998.

[FSV00] E.M. Friedlander, A. Suslin, and V. Voevodsky. *Cycles, Transfers and Motivic homology theories.* Princeton Univ. Press, 2000.

[Ful98] W. Fulton. *Intersection theory.* Springer, second edition, 1998.

[HK06] A. Huber and B. Kahn. The slice filtration and mixed Tate motives. *Compos. Math.*, 142(4):907–936, 2006.

[Hub00] A. Huber. Realization of Voevodsky's motives. *J. Algebraic Geom.*, 9(4):755–799, 2000.

[Hub04] A. Huber. Corrigendum to: “Realization of Voevodsky’s motives” [J. Algebraic Geom. **9** (2000), no. 4, 755–799; mr1775312]. *J. Algebraic Geom.*, 13(1):195–207, 2004.

[Ivo10] F. Ivorra. Réalisation ℓ-adique des motifs triangulés géométriques. II. *Math. Z.*, 265(1):221–247, 2010.

[Lev98] M. Levine. *Mixed motives*, volume 57 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.

[MV99] F. Morel and V. Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[Nen06] A. Nenashev. Gysin maps in oriented theories. *J. Algebra*, 302(1):200–213, 2006.

[PY09] I. A. Panin and S. A. Yagunov. A duality theorem for motives (*russian*). *Algebra i Analiz*, 21(2):205–213, 2009. transl. in St. Petersburg Math. J. 21 (2010), no. 2, 309–315.

[Rio05] J. Riou. Dualité de Spanier-Whitehead en géométrie algébrique. *C. R. Math. Acad. Sci. Paris*, 340(6):431–436, 2005.

[RØ08] O. Röndigs and P. A. Østvær. Modules over motivic cohomology. *Adv. Math.*, 219(2):689–727, 2008.

[Ros96] M. Rost. Chow groups with coefficients. *Doc. Math. J.*, pages 319–393, 1996.

[Ser84] J.-P. Serre. *Groupes algébriques et corps de classes*. Publications de l’Institut Mathématique de l’Université de Nancago [Publications of the Mathematical Institute of the University of Nancago], 7. Hermann, Paris, second edition, 1984. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], 1264.

[Sil09] J. H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer, Dordrecht, second edition, 2009.

[SV00] A. Suslin and V. Voevodsky. Bloch-Kato conjecture and motivic cohomology with finite coefficients. In *The arithmetic and geometry of algebraic cycles (Banff, AB, 1998)*, volume 548 of *NATO Sci. Ser. C Math. Phys. Sci.*, pages 117–189. Kluwer Acad. Publ., Dordrecht, 2000.

[Voe96] V. Voevodsky. La conjecture de Milnor. preprint, 1996.

[Voe02] V. Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. *Int. Math. Res. Not.*, (7):351–355, 2002.

[Voe10] V. Voevodsky. Cancellation theorem. *Doc. Math.*, (Extra volume: Andrei A. Suslin sixtieth birthday):671–685, 2010.

[EGA2] A. Grothendieck. Éléments de géométrie algébrique. *Rédigés avec la collaboration de J. Dieudonné*. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.

[SGA4] M. Artin, A. Grothendieck, and J.-L. Verdier. *Théorie des topos et cohomologie étale des schémas*, volume 269, 270, 305 of *Lecture Notes in Mathematics*. Springer-Verlag, 1972–1973. Séminaire de Géométrie Algébrique du Bois–Marie 1963–64 (SGA 4).

LAGA, CNRS (UMR 7539), Université Paris 13, Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

E-mail address: deglise@math.univ-paris13.fr

URL: http://www.math.univ-paris13.fr/~deglise/