BIVARIANT THEORIES IN MOTIVIC STABLE HOMOTOPY

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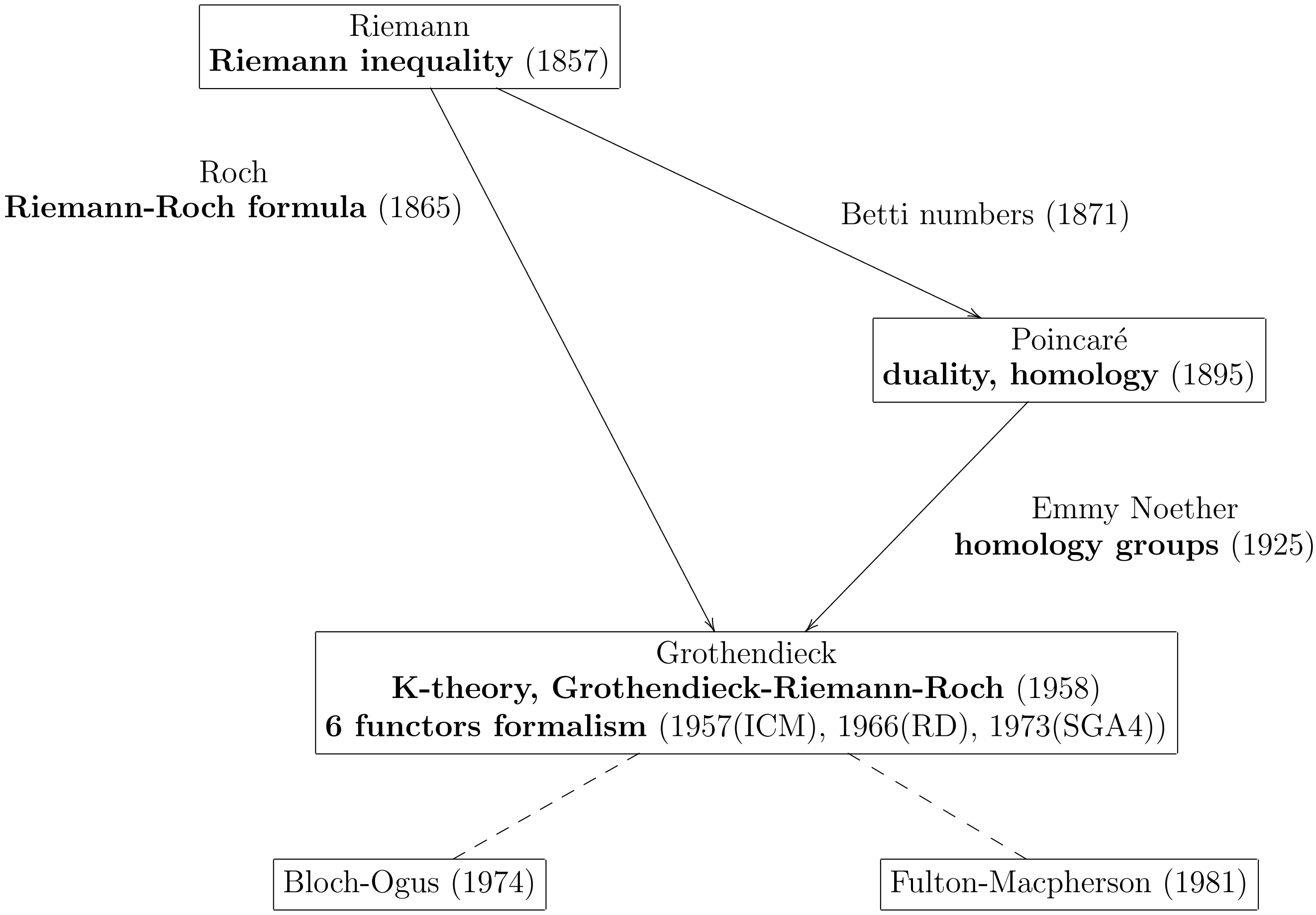
Abstract. I will explain how Grothendieck 6 functors formalism established by Ayoub in stable motivic homotopy theory together with standard orientation theory leads to a complete axiomatic in the style of bivariant cohomologies of Fulton and Macpherson.

This involves an important property of Motivic spectra, called *absolute purity*, which I will introduce and discuss in some extend. One of the main application of our constructions is a generalized Grothendieck-Riemann-Roch formula which admits many forms. I will give some new examples, such as a *residual formula*.

Date: August 2014.

I begin my talk with the following mathematical genealogy:



The aim of my talk is to show how *motivic homotopy theory* of Morel and Vœvodsky allows a synthesis of all the ideas appearing in this tree.

Contents

In all this talk, schemes will be Noetherian finite dimensional. We fix a subcategory $\mathscr{S}$ of these schemes stable by blow-up, base change and open subschemes. Unless stated otherwise, schemes and morphisms are assumed to be in $\mathscr{S}$.
Main examples:

- the category $\mathscr{R}eg$ of all regular schemes (*i.e.* its local rings are regular.)
- the category $\mathscr{S}m_S$ of smooth S-schemes, for an arbitrary base scheme S.

1. Absolute ring spectra: the associated 4 theories

1.1. Stable motivic homotopy and 6 functors.

1.1. Let S be a scheme.

Recall the $\mathbb{A}^1$-homotopy category over S, denoted by $\mathscr{H}_\bullet(S)$.

- Objects=pointed simplicial Nisnevich sheaves over $\mathscr{S}m_S$.
  eg: $X \in \mathscr{S}m_S$, X_+=sheaf represented by X with an added base point;
- Morphisms=morphisms of pointed simplicial sheaves up to weak $\mathbb{A}^1$-equivalence: $\mathbb{A}^1_X)_+ \to X_+$ is an $\mathbb{A}^1$-equivalence (Bousfiled localization)

This is a well behaved *homotopy category* (Quillen), with tensor product $\wedge$.

The stable $\mathbb{A}^1$-homotopy category: obtained from $\mathscr{H}_\bullet(S)$ by inverting the sphere $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$.

Triangulated monoidal category,

$$\Sigma^\infty : \mathscr{H}_\bullet(S) \to S\mathscr{H}(S)$$

universal monoidal functor such that $\Sigma^\infty(\mathbb{P}^1) := S^0(1)[2]$ is $\wedge$-invertible.
Objects: spectra, $\mathbb{E} = (E_0, E_1, ...)$ collection of simplicial sheaves equiped with *suspension maps*[1]:

$$\sigma_n : \mathbb{P}^1 \wedge E_n \to E_{n+1}$$

Associated cohomology: X a smooth S-scheme, $(n, m) \in \mathbb{Z}^2$:

$$\begin{aligned}\mathbb{E}^{n,m}(X) &= \mathrm{Hom}_{S\mathscr{H}}(\Sigma^\infty X_+, \mathbb{E}(n)[m]) =: [\Sigma^\infty X_+, \mathbb{E}(n)[m]] \\ &= \begin{cases} \varinjlim_{r\to\infty} \mathrm{Hom}_{\mathscr{H}_\bullet}((\mathbb{P}^1)^r \wedge X_+, S^{m-2n} \wedge E_{r-n}), & m \geq 2n, \\ \varinjlim_{r\to\infty} \mathrm{Hom}_{\mathscr{H}_\bullet}(S^{2n-m} \wedge (\mathbb{P}^1)^r \wedge X_+, E_{r-n}), & m \leq 2n. \end{cases}\end{aligned}$$

1.2. Here, we will use *(Grothendieck) 6 functors formalism* established by Ayoub in [Ayo07], based on the theory of cross functors introduced by Vœvodsky. They are pairs of adjoint functors:

(base change, direct image)	exceptional functors	closed monoidal
$f^* : S\mathscr{H}(S) \leftrightarrows S\mathscr{H}(T) : f_*$	$p_! : S\mathscr{H}(Y) \leftrightarrows S\mathscr{H}(X) : p^!$	$(\wedge, \underline{\mathrm{Hom}}_S)$
$f : T \to S$ morphism	$p : Y \to X$ s-morphism	

Here *s-morphism* stands for *quasi-projective* originally, or *separated morphisms of finite type* if one uses the extension obtained in [CD09].[2] We will not list the complete properties of these adjoints here. At least the following ones characterize the first 4 functors:

- $f^*(\Sigma^\infty X_+(m)) = \Sigma^\infty(X \times_S T)_+(m)$,
- p proper implies: $p_! = p_*$,
- p étale implies: $p^! = p^*$.

[1]a way of understanding E_n as $\mathbb{P}^1 \wedge E_{n+1}$).

[2]By using Deligne's method to construct $p_!$ and Chow lemma for obtaining its properties.

1.2. **Absolute spectra.** In order to use effectively this formalism, we introduce the following definition:

Definition 1.3. An $\mathscr{S}$-absolute (ring) spectrum is a collection of (ring) spectra $\mathbb{E}_X$ indexed by schemes X in $\mathscr{S}$ and a collection of isomorphisms of (ring) spectra:

$$\tau_f : f^*\mathbb{E}_X \to \mathbb{E}_Y.$$

satisfying the cocyle condition.

Exemple 1.4. There are numerous examples of absolute ring spectra:

(1) $\mathscr{S}ch$-absolute: sphere spectrum S^0.

(2) $\mathscr{S}ch$-absolute: Vœvodsky's algebraic cobordism **MGL**:

$$\mathbf{MGL}_S := (\mathrm{Th}(\gamma_1), \dots, \mathrm{Th}(\gamma_n), \dots)$$

where γ_n is the tautological vector bundle over $\mathbf{BGL}_n$ abd Th means the Thom space. For X smooth over a field of exponential characteristic p,

$$\mathbf{MGL}_k^{2n,n}(S)[1/p] = \Omega^n(S)[1/p].$$

(3) $\mathscr{R}$eg-absolute: Over a regular scheme S, Quillen's algebraic K-theory is represented by

$$\mathbf{KGL}_S := (\mathbb{Z} \times Gr, \mathbb{Z} \times Gr, ...)$$

sequence made of $\mathbb{Z}$-self products of infinite Grassmanians at each degree.[3] X regular, (n, m) integers:

$$\mathbf{KGL}_S^{n,m}(X) = K_{m-2n}(X).$$

(4) Let S be any scheme and $\Lambda \subset \mathbb{Q}$ any ring. In general, Vœvodsky defined the Eilenberg-MacLane motivic ring spectrum $\mathbf{H}\Lambda_S$ representing motivic cohomology over S: for X smooth over S,
- $S = \mathrm{Spec}(k)$, k perfect field, $\mathbf{H}\Lambda_S^{n,m}(X) = CH^m(X, 2m - n)$ (Bloch's Higher Chow groups),
- S regular scheme, $\mathbf{H}\mathbb{Q}_S^{n,m}(X) = Gr_m^\gamma K_{2n-m}(X)_\mathbb{Q}$ (Cisinski, D. in the unequal characteristic case, [CD09, 16.1.7])

It was conjectured by Vœvodsky that the collection $\mathbf{H}\Lambda_S$ is an absolute ring spectrum:

Conjecture (Vœvodsky, [Voe98], Conj. 17)**.** *For any morphism f, the natural map: $f^*\mathbf{H}\Lambda_S \to \mathbf{H}\Lambda_T$ is an isomorphism.*

This conjecture has been proved by Cisinski, D. when T, S are geometrically unibranch and $\Lambda = \mathbb{Q}$ ([CD09, 16.1.7]).

Finally, following Riou, we will denote by $\mathbf{H}_{\mathrm{B}}$ the absolute ring spectrum corresponding to Beilinson's construction of rational motivic cohomology.

[3] To enlighten: the $\mathbb{Z} \times -$ is there because K-theory is representable by an H-space; it is the same simplicial shea at each degree because algebraic K-theory is $\mathbb{P}^1$-periodic (Bott periodicity).

(5) $\mathscr{R}$eg-absolute: Over a regular scheme S, by the recent work of Schlichting and Tripathi [ST13], hermitian K-theory (Karoubi (1973) for affine schemes, Schlichting (2000) for general schemes, is representable by a ring spectrum
$$KO_S = (\mathbb{Z} \times GrO, \mathbb{Z} \times GrO, ...)$$
where GrO is the infinite orthogonal Grassmanian.

1.3. **Beyond cohomology.**

Definition 1.5. Let $\mathbb{E}$ be an $\mathscr{S}$-absolute spectrum. Then for any s-morphism $f : X \to S$ in $\mathscr{S}$, and any integers (n, m), we define:

Cohomology	$\mathbb{E}^{n,m}(X) = [(S^0, \mathbb{E}_X(m)[n]]$
Borel-Moore homology	$\mathbb{E}^{BM}_{n,m}(X/S) = [(S^0, p^!\mathbb{E}_S(m)[n]]$
Cohomology with compact support	$\mathbb{E}^{n,m}_c(X/S) = [(S^0, p_!\mathbb{E}_X(m)[n]]$
Homology	$\mathbb{E}_{n,m}(X/S) = [(S^0, p_!p^!\mathbb{E}(-m)[-n]]$

We will say *BM-homology* (resp. *c-cohomology*) for Borel-Moore homology (resp. cohomology with compact support).

1.6. These theories enjoy a lot of properties. In fact, they satisfy (an extension of) Fulton-Macpherson bivariant formalism. We will list only the principal properties.

- *Comparison maps.–* there exist comparison maps:
$$\nu_{X/S} : \mathbb{E}^{n,m}_c(X/S) \to \mathbb{E}^{n,m}(X),$$
$$\nu^{BM}_{X/S} : \mathbb{E}_{n,m}(X/S) \to \mathbb{E}^{BM}_{n,m}(X/S)$$
which are isomorphisms whenever X/S is proper (because of 1.2).
- *Variance.–* From the adjunctions and the properties listed in 1.2, we get:

$\mathbb{E}^{n,m}(X)$	contravariant in X
$\mathbb{E}^{BM}_{n,m}(X/S)$	covariant in X/S wrt proper morphisms, contravariant in X/S wrt étale morphisms, contravariant in S (base change)
$\mathbb{E}^{n,m}_c(X/S)$	contravariant in X/S wrt proper morphisms, covariant in X/S wrt étale morphisms, contravariant in S (base change)
$\mathbb{E}_{n,m}(X/S)$	covariant in X/S contravariant in S (base change)

- *Products.–* assume $\mathbb{E}$ is an absolute ring spectrum. Then, besides the fact cohomology has a ring structure, we get for example the following refined product in the style of Fulton-Macpherson:
$$\mathbb{E}^{BM}_{**}(Y/X) \otimes \mathbb{E}^{BM}_{**}(X/S) \to \mathbb{E}^{BM}_{**}(Y/S)$$

The same thing works for cohomology with compact support but for homology, we only get an exterior product. It is worth to note a last product, generalization of the cap-product:

$$\mathbb{E}_c^{**}(X/S) \otimes \mathbb{E}_{**}^{BM}(X/S) \to \mathbb{E}_{**}(X/S)$$

1.7. *Link with Bloch-Ogus axioms.–* in fact, the concept of (relative) Borel-Moore homology hides another kind of cohomology. Let $i : Z \to S$ be a closed immersion. Then:

$$\mathbb{E}_{n,m}^{BM}(Z/X) = \mathbb{E}_Z^{-n,-m}(X), \text{ cohomology of } X \text{ with support in } Z.$$

In fact, one gets by adjuntion: $[S^0, i^!(\mathbb{E}_X)] = [i_!(S^0), \mathbb{E}_X]$.
But from the localization property in $S\mathscr{H}$, we also get: $i_!(S^0) = \Sigma^\infty(X/X - Z)$.

It is worth to spell out the refined product in that case: $T \subset Z \subset X$,

$$\mathbb{E}_T^{**}(Z) \otimes \mathbb{E}_Z^{**}(X) \to \mathbb{E}_T^{**}(X).$$

2. Absolute purity and orientations

2.1. **Absolute purity.**

2.1. *Absolute purity in étale cohomology:*

Conjectured (Grothendieck, SGA5), proved partially (Thomason, 1984), proved in general (Gabber, 2000).

In motivic stable homotopy, let $i : Z \to X$ be a regular closed immersion. Deformation to the normal cone diagram (extensively used by Fulton):

$$\begin{tikzcd}
Z \arrow[r, "s_1"] \arrow[d, hook, "i"'] & \mathbb{A}^1_Z \arrow[d, hook] & Z \arrow[l, "s_0"'] \arrow[d, hook, "s"] \\
X \arrow[r, "d_1"] \arrow[d] & D_Z X \arrow[d] & N_Z X \arrow[l, "d_0"'] \arrow[d] \\
\{1\} \arrow[r] & \mathbb{A}^1 & \{0\} \arrow[l]
\end{tikzcd}$$

where:

- Deformation space: $D_Z X := B_{Z\times\{0\}}(\mathbb{A}^1_X) - B_Z X$, difference of the indicated blow-ups. It is flat over $\mathbb{A}^1$. $D_Z Z = \mathbb{A}^1_Z$.
- Normal bundle of Z in X: $N_Z X$=fiber of $D_Z X$ over 0.

Note that, because of homotopy invariance: $s_0^* = (p^*)^{-1} = s_1^*$ where $p : \mathbb{A}^1_X \to X$.

Definition 2.2. Let $\mathbb{E}$ be an $\mathscr{S}$-absolute (ring) spectrum.

For any regular closed immersion i as above, we say i is $\mathbb{E}$-pure if the morphisms

$$\mathbb{E}_Z^{**}(X) \xleftarrow{d_1^*} \mathbb{E}^{**}_{\mathbb{A}^1_Z}(D_Z X) \xrightarrow{d_0^*} \mathbb{E}_Z^{**}(N_Z X)$$

are isomorphisms.

We say $\mathbb{E}$ is *$\mathscr{S}$-absolutely pure* if any regular closed immersion i in $\mathscr{S}$ is $\mathbb{E}$-pure.

Exemple 2.3. (1) for any scheme S, any $\mathscr{S}m_S$-absolute ring spectrum is absolutely pure according to the relative purity theorem of Morel-Voevodsky.
(2) **KGL** (resp. $H\mathbb{Q}$) is $\mathscr{R}$eg-absolutely pure (by Cisinski, D., [CD09, 13.6.1, 14.4.1], essentially an application of Quillen purity property of algebraic K-theory).
(3) According to [NSØ09], $\mathbf{MGL} \otimes \mathbb{Q} = \mathbf{H}_{\mathbb{B}}[b_1, \ldots, b_n, \ldots]$. Thus $\mathbf{MGL} \otimes \mathbb{Q}$ is $\mathscr{R}$eg-absolutely pure.
(4) It seems that the method of proof of Cisinski-Deglise gives that KO is $\mathscr{R}$eg-absolutely pure.

In view of these examples, I think it is natural to expect the following:

Conjecture. (1) *The absolute ring spectrum* **MGL** *is absolutely pure.*
(2) *The absolute ring spectrum* S^0 *is absolutely pure.*

2.2. **Orientation.**

2.4. There are many contributors to this theory in motivic stable homotopy, mainly because it is obtained by analogy with the topological case. It would be pointless to mention them all but certainly Morel-Voevodsky are the initiators and Panin is a lead contributor.

We recall the main definition and steps: let $\mathbb{P}^\infty_S$ be the pointed space obtained by taking colimit of the tower:

$$\mathbb{P}^1_S \to \ldots \to \mathbb{P}^n_S \to \ldots$$

Definition 2.5. An orientation of a ring spectrum $\mathbb{E}$ over S with unit η_S is a class $c \in \mathbb{E}^{2,1}(\mathbb{P}^\infty_S)$ such that

- $c|_{\mathbb{P}^0_S} = 0$,
- $c|_{\mathbb{P}^1_S} = \eta_S$ through the suspension isomorphism $\tilde{\mathbb{E}}^{2,1}(\mathbb{P}^1_S) \simeq \mathbb{E}^{0,0}(S)$,

Similarly, an orientation of an absolute ring $\mathbb{E}$ spectrum is an orientation c_S of each $\mathbb{E}_S$ stable by pullbacks induced by the structural isomorphisms τ_f.

Exemple 2.6. The examples (2)-(3) are all absolute oriented ring spectra.

2.7. Recall from [MV99] that $\mathbb{P}^\infty = B\mathbb{G}_m$, in particular, there is a map:

$$\mathrm{Pic}(S) = H^1_{\mathrm{Nis}}(S, \mathbb{G}_m) \to [S^0, \mathbb{P}^\infty_S]^{unst}$$

which is an isomorphism whenever S is regular (in fact: semi-normal).

Thus an orientation immediately yields the first Chern classes:

$$c_1 : \mathrm{Pic}(S) \to [S^0, \mathbb{P}^\infty_S]^{unst} \xrightarrow{\Sigma^\infty} [S^0, \Sigma^\infty \mathbb{P}^\infty_S] \xrightarrow{(c_S)_*} [S^0, \mathbb{E}_S(1)[2]] = \mathbb{E}^{2,1}(S).$$

They satisfy all the expected properties and can be extended to a theory of higher Chern classes.

However, one should be careful that c_1 is *not an isomorphism of groups* in general. Instead, one has:

There exists a (commutative) formal group law $F(x,y)$ with coefficients in $\mathbb{E}^{**}(S)$:

$$F(x,y) = x + y + \sum_{i,j\geq 1} a_{ij}^S.x^i, y^j$$

(recall $a_{ij} = a_{ji}$ of bidegree $(2(i+j), i+j)$) such that for any line bundles L, L' over S,

$$c_1(L \otimes L') = F(c_1(L), c_1(L')).$$

Note that this formula makes sense as $c_1(L)$ is nilpotent for any L (because we restricted to Noetherian schemes).

Exemple 2.8. As in topology:

(1) for $\mathbf{H}_{\mathrm{B}}$ (or $\mathbf{H}\mathbb{Z}_S$), $F(x,y) = x + y$: additive formal group law.
(2) for $\mathbf{KGL}$, $F(x,y) = x+y-\beta.xy$: multiplicative formal group law (β=Bott element).
(3) for $\mathbf{MGL}_k[1/p]$, $F(x,y)$ is the Lazard universal formal group law.

2.3. **Riemann-Roch: change of orientation.**

2.9. Let $(\mathbb{E}, c)$ and $(\mathbb{F}, d)$ be oriented ring spectra with respective associated formal group law F_c and F_d. Let $\varphi : \mathbb{E} \to \mathbb{F}$ be a morphism of ring spectra, which induces:

$$\varphi_* : \mathbb{E}^{**} \to \mathbb{F}^{**}.$$

Recall:

$$\mathbb{F}^{**}(\mathbb{P}^\infty_S) \simeq \mathbb{F}^{**}(S)[[d]]$$

Thus, one gets:

$$\varphi_*(c) = \Phi(d)$$

for a power series $\Phi(t)$ with coefficients in $\mathbb{F}^{**}(S)$. On the other hand, $\varphi_*(c)$, of degree $(2,1)$, is an orientation of $\mathbb{F}$. Thus, necessarily,

$$\Phi(t) = t + \text{terms of deg. } > 1.$$

By definition, we formally get the equality of power series in (x,y):

$$F_c(\Phi(x), \Phi(y)) = \Phi(F_d(x,y))$$

Thus, Φ is a (strict) isomorphism of formal group law from F_c to F_d.

Note finally that in this context, we can uniquely associated with φ the Todd class $\mathrm{Td}_\varphi(E)$ of any virtual vector bundle E/S characterized by the relation:

$$\varphi_*(c_1(L)) = \mathrm{Td}_\varphi(-L).d_1(L).$$

Exemple 2.10. According to Riou, it is possible to lift the (higher) Chern character in a (iso)morphism of ring spectra:

$$\mathrm{ch}_t : \mathbf{KGL}_{\mathbb{Q}} \to \oplus_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{B}}(i)[2i].$$

Thus corresponding power series Φ is an isomorphism from a multiplicative formal group law to an additive formal group law. It is well known that such an isomorphism is unique. It is define by the usual exponential series. Thus we get bakc the usual formula for the classical Todd class.

Remarque 2.11. As it is clear from the preceding discussion, the Todd class, and later on the GRR-formula can be reduced to the effect of changing the orientation of a ring spectrum.

3. Fundamental classes and duality

3.1. **Main theorem.**

3.1. Orientation theory yields a whole family of *characteristic classes* (first example: Chern classes). Recall the *Thom classes*:

Let $(\mathbb{E}, c)$ be an oriented ring spectrum and E/S be a vector bundle of rank n, with zero setion s, canonical projection p, $\mathbb{P}(E)$ is projectivization and $\mathbb{P}(E \oplus 1)$ is projective completion.

Then one computes:

$$\mathbb{E}^{**}(\mathrm{Th}(E)) := \mathbb{E}^{**}_S(E) \simeq \mathbb{E}^{**}(\mathbb{P}(E \oplus 1))/\mathbb{E}^{**}(\mathbb{P}(E)).$$

Thus, $\mathbb{E}^{**}(\mathrm{Th}(E))$ is a free $\mathbb{E}^{**}(X)$-module of rank 1. The *Thom class*,

$$\mathfrak{t}(E) = \sum_{i=0}^{n} p^*(c_i(E)).c^{n-i} \in \mathbb{E}^{2n,n}(\mathbb{P}(E \oplus 1))$$

where $c = -c_1(\mathcal{O}(-1))$, yields a well defined element $\bar{\mathfrak{t}}(E)$ of $\mathbb{E}^{2n,n}(\mathrm{Th}(E))$ which is a base of $\mathbb{E}^{**}(\mathrm{Th}(E))/\mathbb{E}^{**}(X)$.

Exemple 3.2. Let $i : Z \to X$ be an $\mathbb{E}$-pure regular closed immersion of codimension c.

Then, using the notation of Definition 2.2, one defines the refined (or local) fundamental class of i (or Z in X) as:

$$\bar{\eta}_i = d_1^*(d_0^*)^{-1}(\bar{\mathfrak{t}}(N_Z X)).$$

This is a base of the $\mathbb{E}^{**}(Z)$-module $\mathbb{E}^{**}_Z(X)$ – which is therefore free of rank 1.

In the following theorem as well as in the rest of this talk, $\mathscr{S}$ is assumed to be either $\mathscr{R}$eg or $\mathscr{S}m_S$ in the applications.

Theorem 3.3. *Let $(\mathbb{E}, c)$ be an $\mathscr{S}$-absolutely pure oriented ring spectrum.*

There exists a family of fundamental classes $\bar{\eta}_f \in \mathbb{E}^{BM}_{**}(X/S)$ *indexed by lci qp-morphisms f uniquely characterized by the following properties:*

1. *If $i : Z \to X$ is a regular closed immersion in $\mathscr{S}$, $\bar{\eta}_i$ coincides with the fundamental class in $\mathbb{E}^{**}_Z(X)$ defined in the previous example.*

(2) *if $j: U \to S$ is an open immersion, $\bar{\eta}_j$ is the image of* 1 *by the isomorphism:*

$$\mathbb{E}^{BM}_{0,0}(U/S) \simeq \mathbb{E}^{0,0}(U).$$

(3) *for composable quasi-projective lci morphisms $Y \xrightarrow{g} X \xrightarrow{f} S$, one has:*

$$\bar{\eta}_g \cdot \bar{\eta}_f = \bar{\eta}_{fg} \in \mathbb{E}^{BM}_{**}(Y/S).$$

(4) *$\bar{\eta}_f$ is stable by smooth base change.*

(Indication). In fact, we use Ayoub's purity result: given a smooth qp-projective morphism $f: X \to S$, with tangent bundle T_f,

$$f^! \simeq \mathrm{Th}(T_f) \otimes f^*.$$

This implies:

$$\mathbb{E}^{BM}_{**}(X/S) \simeq \mathbb{E}^{**}(\mathrm{Th}(-T_f))$$

so that we can define $\bar{\eta}_f$ as the preimage of the (inverse) Thom class $\bar{\mathfrak{t}}(-T_f)$. This gives a definition of fundamental classes in general using quasi-projectivity. The stability by pullback is formal. Roughly, the independance on the chosen factorization is proved by the unicity statement.

On the other hand, when $f = p$ is the projection of projective bundle, $\bar{\eta}_p$ is uniquely defined by the relation:

$$\bar{\eta}_{p'} \,.\, \bar{\eta}_\delta = 1$$

where $p': P \times_X P \to P$, is (one of) the canonical projection and δ the diagonal embedding of P/X.

The most important point in this theorem is the relation:

$$\bar{\eta}_T(Z).\,\bar{\eta}_Z(X) = \bar{\eta}_Z(X)$$

for regular closed immersions $T \subset Z \subset X$. This is proved using a double deformation space to reduce to the analog property of Thom classes, which is obvious by the Whitney sum formula. ☐

3.4. These classes enjoy the following properties:

- *Excess intersection formula.*– Given a cartesian square

$$\begin{tikzcd} X' \arrow[r, "p'"] \arrow[d, "f"'] & S' \arrow[d, "f"] \\ X \arrow[r, "p"] & S \end{tikzcd}$$

of squares in $\mathscr{S}$ such that p is qp and lci. Let ξ be the associated excess intersection bundle, of rank e (this is 0 is the square is transversal). Then:

$$f^*(\bar{\eta}_p) = c_e(\xi).\,\bar{\eta}_{p'}\,.$$

- *GRR formula.*– Let $(\mathbb{E}, c)$ and $(\mathbb{F}, d)$ be absolutely pure oriented ring spectra and $\varphi : \mathbb{E} \to \mathbb{F}$ be a morphism of ring spectra. Let Td_φ the associated Todd class and $\varphi_* : \mathbb{E}^{BM}_{**} \to \mathbb{F}^{BM}_{**}$ the induced natural transformation.

  Let $f : X \to S$ be an lci qp-morphism with virtual tangent bundle τ_f, the generalized GRR-formula reads:

$$\boxed{\varphi_*(\bar{\eta}^{\mathbb{E}}_f) = \mathrm{Td}_\varphi(-\tau_f).\,\bar{\eta}^{\mathbb{F}}_f\,.} \tag{3.4.a}$$

Remarque 3.5. According to a work in progress of A. Navarro, it is possible to extend this construction to singular schemes and to avoid the assumption of absolute purity (this is not the case however for the duality results of the next section).

3.2. **Duality.**

3.6. Consider the assumptions of the preceding theorem. Let $\mu : \mathbb{E}_S \wedge \mathbb{E}_S \to \mathbb{E}_S$ be the multiplication of the ring spectrum $\mathbb{E}_S$.

By definition, the fundamental class of an lci qp-morphism f in $\mathscr{S}$, say of dimension d, is a morphism

$$S^0(d)[2d] \to f^!(\mathbb{E}_S).$$

It induces a well defined morphism:

$$\tilde{\eta}_f : f^*(\mathbb{E}_S)(d)[2d] \xrightarrow{1\wedge\bar{\eta}_f} f^*(\mathbb{E}_S) \wedge f^!(\mathbb{E}_S) \xrightarrow{(*)} f^!(\mathbb{E}_S \wedge \mathbb{E}_S) \xrightarrow{\mu} f^!(\mathbb{E}_S).$$

where the pairing $(*)$ exists from the 6 functors formalism. A formal corollary of the main theorem is the following result:

Corollaire 3.7. *With the assumptions and notations above, the map $\tilde{\eta}_f$ is an isomorphism.*

3.8. This corollary as many applications, and in fact contains the most interesting part of both Bloch-Ogus and Fulton-Macpherson formalisms.

- *Classical duality*: f lci and qp of relative dimension d, $\tilde{\eta}_f$ induces isomorphisms:

$$\mathbb{E}^{n,i}(X) \to \mathbb{E}^{BM}_{2d-n,d-i}(X/S), x \mapsto x \cdot \bar{\eta}_f,$$
$$\mathbb{E}^{n,i}_c(X/S) \to \mathbb{E}_{2d-n,d-i}(X/S), x \mapsto x \cap \bar{\eta}_f\,.$$

  Thus, when f is in addition proper, the 4 theories are isomorphic.
- *Bloch-Ogus style duality*: Consider a commutative diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr] & & X \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

made of qp and lci morphism, X/S of dimension d. Then there is an isomorphism:

$$\mathbb{E}^{BM}_{n,i}(Y/X) \to \mathbb{E}^{BM}_{n-2d,i-d}(Y/S), y \mapsto y \cdot \bar{\eta}_p$$

In particular, if $f = i$ is a closed immersion and $S = \mathrm{Spec}(k)$, one recovers the isomorphism of Bloch-Ogus:

$$\mathbb{E}^{r,s}_Y(X) \simeq \mathbb{E}^{BM}_{2d-r,d-s}(X/S).$$

- *Exceptional variance*: according to the above duality, the four theories aquire new variance as follows, where morphisms are assumed to belong in $\mathscr{S}$:

$\mathbb{E}^{**}(X)$	covariant in X wrt projective lci morphisms,
$\mathbb{E}^{BM}_{**}(X/S)$	contravariant in X/S wrt lci qp-morphisms,
$\mathbb{E}^{**}_c(X/S)$	covariant in X/S wrt lci qp-morphisms,
$\mathbb{E}_{**}(X/S)$	contravariant in X/S wrt projective lci morphisms.

In each case, the functoriality morphism is homogenous of bidegree $(-2d, -d)$.

3.9. *Generalized Riemann-Roch formulas.–* Let $\varphi : \mathbb{E} \to \mathbb{F}$ be a morphism of absolutely pure ring spectra, each equiped with an orientation c and d. Then the GRR-formula (3.4.a) induces an analogous formula for all the morphisms defined in the previous paragraph:

- Applied to the Chern character $\mathrm{ch}_t : \mathbf{KGL}_{\mathbb{Q}} \to \mathbf{H}_{\text{Б}}$, and to the corresponding cohomology theories, we get a higher GRR-formula in the style of Gillet (which nevertheless was still unproved for arbitrary regular schemes).
- The case of BM-homology is a generalization of formulas of Fulton-Macpherson. This is particularly relevant in the case of ch_t.

3.10. *Residue maps.–* Let $i : Z \to X$ be a closed immersion. A particular case of duality is:

$$\mathbb{E}^{n,m}(Z) \to \mathbb{E}^{n+2c,m+c}_Z(X), z \mapsto \bar{\eta}_i\, .z$$

is an isomorphism. Applied to the boundary of the localization long exact sequence:

$$\mathbb{E}^{n-1,m}(X - Z) \to \mathbb{E}^{n,m}_Z(X)$$

we get a canonical *residue* map:

$$\partial^{\mathbb{E}}_{X,Z} : \mathbb{E}^{n,m}(X - Z) \to \mathbb{E}^{n-2c+1,m-c}(Z).$$

Exemple 3.11. (1) when $\mathbb{E}$ represents De Rham cohomology (in characteristic 0), X is a regular proper curve and Z a point, this residue map coincide with Tate residue of differential forms (thus with Leray residues).
(2) When A is a DVR, $X = \mathrm{Spec}(A)$, Z the closed point, $\mathbb{E} = \mathbf{H}_{\text{Б}}$ (or $\mathbb{E} = \mathbf{H}\mathbb{Z}_k$), and $n = m$, this map coincide with Milnor residue symbols in Milnor K-theory.

It is worth to mention that there is GRR-formula for this residue. Let us state it in the case of the usual Chern character:

Proposition 3.12. *Consider a closed immersion $Z \to X$ of regular schemes, and let c be its codimension. Let $N_Z X$ be the normal bundle of Z in X and put $U = X - Z$.*

Then the following diagram is commutative:

$$\begin{tikzcd} K_r(U) \arrow[r, "\partial_{X,Z}"] \arrow[d, "\mathrm{ch}^U_{r,n}"'] & K_{r-1}(Z) \arrow[d, "\sum_{i+j=n-c} \mathrm{Td}_i(-N_Z X).\, \mathrm{ch}^Z_{r-1,j}"] \\ \mathbf{H}^{2n-r,n}_{\text{B}}(U) \arrow[r, "\partial_{X,Z}"] & \mathbf{H}^{2(n-c)-r+1,n-c}_{\text{B}}(U) \end{tikzcd}$$

Exemple 3.13. (case $r = 1$) Then one gets an explicit description of the residue morphism for K-theory, when $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$, A and A/I being regular rings.

Indeed, one knows that $K_1(A_I) = \mathrm{GL}(A_I)^{ab}$, the abelianization of the group of invertible matrices of arbitrary dimensions. Assume we are given an endomorphism $u : A^r \to A^r$ such that $u \otimes_A A_I$ is an automorphism of A^r_I. We will denote by $[u]$ the class of this isomorphism is $K_1(A)$. By assumption, u is a monomorphism whose cokernel if supported on I. We denote by $[\mathrm{coKer}(u)]$ the class of the corresponding (finitely presented) A/I-module in $K_0(A/I)$. With these notations, one has the following formula:

$$\partial_{X,Z}([u]) = [\mathrm{coKer}(u)].$$

Assume further that $n = c = 1$. Recall that the Chern component $\mathrm{ch}_{1,1}$: $K_1(A) \to \mathbf{H}^{1,1}_{\text{B}}(A) = A^\times \otimes \mathbb{Q}$ sends any matrix of $GL(A)$ to its determinant.

Assume Z is connected. By assumption, $I = (\pi)$ for a prime divisor π: A_I is a discrete valuation ring. We let v_π denote its valuation. Then, giving the above notations, the residual Riemann-Roch formula lands in $\mathbf{H}^{0,0}_{\text{B}}(Z) = \mathbb{Q}$ and reads:

$$v_\pi\big(\det(u)\big) = \mathrm{rk}_{A/I}\big([\mathrm{coKer}(u)]\big).$$

Note that in fact, it is an integral formula as all the members are integers.

References

[Ayo07] J. Ayoub. *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique (I)*, volume 314 of *Astérisque*. Soc. Math. France, 2007.

[CD09] D.-C. Cisinski and F. Déglise. Triangulated categories of mixed motives. arXiv:0912.2110, 2009.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[NSØ09] Niko Naumann, Markus Spitzweck, and Paul Arne Østvær. Motivic Landweber exactness. *Doc. Math.*, 14:551–593, 2009.

[ST13] M. Schlichting and G. S. Tripathi. Geometric models for higher grothendieck-witt groups in A1-homotopy theory. arXiv:1309.5818, 2013.

[Voe98] Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory. In *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, number Extra Vol. I, pages 579–604 (electronic), 1998.