ON THE HOMOTOPY HEART OF MIXED MOTIVES

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ABSTRACT. The aim of this work is to describe the heart of the perverse homotopy t-structure on the category of motivic complexes over an arbitrary base scheme S of characteristic 0. The main theorem identify this category with the category of cycle modules over S as previously defined by M. Rost, which was conjectured by Ayoub and extend the case where S is the base field previously proved in the thesis of the author.

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INTRODUCTION

The category of motivic complexes was described conjecturally by Beilinson in the form of a program almos thirty years ago. A first construction was proposed by Voevodsky in his thesis using the h-topology, which already contained in germs what he will defined later as the $\mathbf{A}^1$-homotopy theory. Though that category was defined over an arbitrary base scheme, it was too coarse to fulfil the program of Beilinson. It was realized slightly later that this theory corresponds to what is called now *étale motives*, or sometimes *Lichtenbaum motives* as the corresponding formalism was conjectured by Lichtenbaum - almost at the same as Beilinson did: see [Ayo], [CD13].

Voevodsky refined his construction to get the right integral category of motivic complexes over a perfect field: the main indication that the later category is the correct one is constituted by the fact the morphisms from the (homological) motive of a smooth scheme X to the Tate object $\mathbf{Z}(n)[2n]$ is the Chow group of n-codimensional cycles in X. The search for an extension of his definition to an arbitrary base, satisfying the program of Beilinson, has been an boiling question since then. A satisfactory answer to that question was obtained, after the fundamental works of [Ayo07a, Ayo07b] and [CD12] by M. Spitzweck in [Spi01]. More recently, a complete picture fusioning [CD12] and [Spi01] was obtained in the equi-characteristic case in [CD14]. In short, in the original construction of Voevodsky one had only to replace the Nisnevich topology by the cdh-topology if one invert the residue characteristic p.

However, one fundamental construction is still lacking on the category of motivic complexes, even if one restricts to equi-characteristics schemes: the motivic t-structure, supposed to be the t-structure which is realized to the perverse t-structure after taking l-adic realization. On the other hand, one has at our disposal a well defined t-structure on $\mathrm{DM}(k)$, for a perfect field k, whose existence was a corollary of the theory of Voevodsky. It was later extended to arbitrary base by Ayoub in [Ayo07a]. Actually, Ayoub constructed two extensions: the one we will be interested in is the so-called perverse homotopy t-structure. It has the distinctive feature to be obtained by gluing: this roughly means it is determined by restriction to the points of the base scheme. This allows the study of that category by restriction to the case of fields.

Some time ago, the heart of the homotopy t-structure over a perfect base field k was described in the thesis of the author, as the category of cycle modules over k, previously defined by Rost. This is a far-fetched generalization of the Gersten resolution of homotopy sheaf with transfers, proved by Voevodsky. Thus, it is natural to ask if this theorem extends to an arbitrary base scheme. This is what we will prove here, up to inverting the exponential characteristic of the base field.

CONVENTIONS

We will fix a base field k of characteristic 0. Unless stated otherwise, a scheme is assumed to be a separated k-scheme of finite type.

Given a k-scheme S and an extension field E/k of finite transcendance degree d, an E-point of S is a morphism $x : \operatorname{Spec} E \to S$. We will say that d is the transcendance degree of x. The set of E-points is denoted by $S(E)$.

Our convention for t-structure will be *homological*. The reason for this is that we follow definitions of Morel and Ayoub. This means concretely the following two properties for a given t-structure with truncation functors $?_{\geq 0}$ and $?_{<0}$:

- given two objects $\mathbb{E}$, $\mathbb{F}$, $\mathrm{Hom}(\mathbb{E}_{\geq 0}, \mathbb{E}_{<0}) = 0$,
- for any object $\mathbb{E}$, one has a canonical distinguished triangle

$$\mathbb{E}_{\geq 0} \to \mathbb{E} \to \mathbb{E}_{<0} \to \mathbb{E}_{\geq 0}[1].$$

1. FROM MOTIVIC SPECTRA TO CYCLE MODULES

In this section, $\mathscr{T}$ is an *oriented motivic triangulated category* in the sense of [CD12, CD13] over the category of k-schemes. Objects of this category will be refered to as *$\mathscr{T}$-motives*.

1.1. Given any $\mathscr{T}$-motive $\mathbb{E}$ over S, recall (see [Dég14]) that we have defined the Borel-Moore homology associated with $\mathbb{E}$ as follows:

$$\mathbb{E}^{BM}_{n,m}(X/S) = \mathrm{Hom}_{\mathscr{T}(S)}\left(\mathbb{1}_S(m)[n], f^!(\mathbb{E})\right),$$

where $f : X \to S$ is any separated morphism of finite type, and $(n,m) \in \mathbf{Z}^2$. It is covariant (resp. contravariant) with respect to proper (resp. étale) morphisms.

Definition 1.2. Let S be a k-scheme essentially of finite type and $\mathbb{E}$ be a $\mathscr{T}$-motive over S.

(1) Given any point E/S, we let $\mathscr{M}(E/S)$ be the set of sub-S-algebras A of E such that $\mathrm{Spec}(A) \to S$ if of finite type. We equipped that set with the order $A \leq B$ if B is a localization of A. This is obviously a filtering set.

Moreover, we will simply call S-*model of the point* E/S any S-scheme of the form $\operatorname{Spec} A$ for $A \in \mathscr{M}(E/S)$.

(2) Let E/S be a point of k-transcendance degree d and $r \in \mathbf{Z}$ be a integer. We put:

$$\underline{\hat{\pi}}_{0,\mathrm{r}}(\mathbb{E})(E) = \varinjlim_{A \in \mathscr{M}(E/S)} \mathbb{E}^{BM}_{2d-r,d-r}(\operatorname{Spec}(A)/S).$$

We call $\underline{\hat{\pi}}_{0,*}(\mathbb{E})$ the *Rost transform* of $\mathbb{E}$.

Example 1.3. Assume $\mathscr{T} = \mathrm{DM}$ and $S = \operatorname{Spec} k$ is the spectrum of a perfect field. Let $\mathbb{E}$ be a motivic complex over k. Let E/k be an extension field and $r \in \mathbf{Z}$. Let further $A \subset E$ be a smooth k-algebra. Then,

$$\mathbb{E}^{BM}_{2d-r,d-r}(\operatorname{Spec} A/S) \simeq \mathbb{E}^{r,r}(\operatorname{Spec} A)$$

according to [Dég14] and this isomorphism is natural with respect to localization maps.

It follows that:

$$\underline{\hat{\pi}}_{0,\mathrm{r}}(\mathbb{E})(E) = \mathbb{E}^{r,r}(E)$$

with the notations of [Dég13, 4.2.5]. According to *loc. cit.*, $\underline{\hat{\pi}}_{0,*}(E)$ as a canonical structure of a cycle module.

1.4. Given the previous example, we would like to put a cycle module structure ([Ros96]) on $\underline{\hat{\pi}}_{0,*}(\mathbb{E})$. As in [Dég13], we will describe this structure by considering a suitable version of *generic motives* over S.

In the relative context, this is dictated by the following isomorphism:

$$\operatorname{Hom}_{\mathscr{T}(S)}\left(\mathbb{1}_S(d-r)[2d-r], f^!(\mathbb{E})\right) = \operatorname{Hom}_{\mathscr{T}(S)}(f_!(\mathbb{1}_S)(d-r)[2d-r], \mathbb{E})$$

According to the general philosophy of the 6 functors formalism, we define the *BM-motive* associated with X/S (with coefficients in $\mathscr{T}$) as:

$$M^{BM}(X/S) := f_!(\mathbb{1}_X).$$

Note that it satisfies the opposite functoriality as Borel-Moore homology: covariance (resp. contravariance) with respect to étale (resp. proper) maps.

Definition 1.5. Let S be a k-scheme essentially of finite type.

(1) Let $x : \operatorname{Spec} E \to S$ be a point of S/k. We associate with E/S a pro-object of the category of separated S-schemes of finite type as follows

$$(E/S) := \text{"}\varprojlim_{A \in \mathscr{M}(E/S)}\text{"} \big(\operatorname{Spec}(A)\big).$$

(2) Given the preceding notations, assuming the point E/S has k-transcendance degree d, and given an integer $r \in \mathbf{Z}$, we define an ind-$\mathscr{T}$-motive over S as follows:

$$M^{(0)}(E/S)\{r\} := \text{"}\varprojlim_{A \in \mathscr{M}(E/S)}\text{"} \Big(M^{BM}(\operatorname{Spec}(A)/S)(d+r)[2d+r]\Big).$$

We define the category $\mathscr{T}^{(0)}(S)$ of *generic $\mathscr{T}$-motives over S* as the full sub-category of pro-objects of $\mathscr{T}(S)$ made of pro-spectra of the preceding form.

It follows from this discussion that the Rost transform of a $\mathscr{T}$-motive $\mathbb{E}$ over S is in fact a contravariant functor of the form:

$$\underline{\hat{\pi}}_{0,*}(\mathbb{E}) : \mathscr{T}^{(0)}(S)^{op} \to \mathscr{A}b.$$

with the following convention for grading: $\underline{\hat{\pi}}_{0,*}(\mathbb{E})\big(M^{(0)}(E/S)\{r\}\big) = \underline{\hat{\pi}}_{0,\text{-r}}(\mathbb{E})(E)$.

Remark 1.6. In the case $\mathscr{T} = \mathrm{DM}$ and $S = \operatorname{Spec} k$, k perfect field, the category of generic motives defined here coincides with that appearing in [Dég08b]: this follows because, whenever $f : X \to S$ is smooth of relative dimension d, $f_!(d)[2d] \simeq f_\sharp$ which implies: $M^{BM}(X/S)(d)[2] \simeq M_{gm}(X)$ where the right hand side is the geometric motive of X in the usual sense. The chosen grading on generic motives is made to give this equivalence.

1.7. It is possible to realize contravariantly all the structural maps of a cycle (pre)module within the category of generic spectra as follows:
Corestriction, (D1).– Let $\varphi : E \to F$ be an extension of points of S with respective transcendance degree d_E and d_F over k. Then the map of pro-objects

$$(\varphi/S)^* : (E/S) \to (F/S)$$

can be written as the formal limit of morphisms

$$\text{"}\varprojlim_i\text{"}\left(Y_i \xrightarrow{f_i} X_i\right)$$

where f_i is an S-morphism between S-models, which is moreover lci of relative dimension $d = (d_F - d_E)$. Given such a morphism f_i, because it is moreover quasi-projective (in fact, affine), we get according to [Dég14] a Gysin map:

$$f_{i*} : M^{BM}(Y_i/S) \to M^{BM}(X_i/S)(d)[2d]$$

which is compatible with the étale contravariant functoriality. Twisting this maps by $\mathbb{1}_S(d_E + *)[2d_E + *]$, and by taking formal colimits with respect to the preceding presentation of $(\varphi/S)^*$, one gets a homogeneous morphism of degree 0:

$$\varphi^* : M^{BM}(F/S)\{*\} \to M^{BM}(E/S)\{*\}.$$

Restriction, (D2).– Let $\varphi : E \to F$ be an extension of finite degree of points of S. Then the map of pro-S-schemes

$$(\varphi/S)^* : (E/S) \to (F/S)$$

can be written as the formal limit of morphisms

$$\text{"}\varinjlim_i\text{"}\left(X_i \xrightarrow{f_i} Y_i\right)$$

where f_i is a finite S-morphism between respective S-models of E and F. Therefore, by contravariant functoriality of $M^{BM}(-/S)$ with respect to finite morphisms of S-schemes, we get a morphism:

$$f_i^* : M^{BM}(Y_i/S) \to M^{BM}(X_i/S).$$

This map is natural with respect to the transition morphisms with respect to indexes i (which are étale morphisms). Thus taking limits and twisting, we get a well defined morphism of generic motives:

$$\varphi_* : M^{BM}(E/S)\{*\} \to M^{BM}(F/S)\{*\}.$$

Action of K^M_*, (D3).– Recall that given any S-scheme X, we define the $\mathscr{T}$-cohomology groups of X as:

$$H^{n,m}(X,\mathscr{T}) = \mathrm{Hom}_{\mathscr{T}(X)}(\mathbb{1}_X, \mathbb{1}_X(m)[n]).$$

They are contravariant and obviously acts on $M^{BM}(X/S)(*)[*]$: indeed any element $x : \mathbb{1}_X \to \mathbb{1}_X(m)[n]$ gives, after applying $f_!$, a map:

$$M^{BM}(X/S) \to M^{BM}(X/S)(m)[n].$$

One readily deduces that, given any point E/S, the cohomology groups:

$$H^{r,r}(E,\mathscr{T}) = \varinjlim_{A\in\mathscr{M}(E/S)} H^{r,r}(\mathrm{Spec}\,A,\mathscr{T})$$

acts homogeneously on $M^{BM}(E/S)_*$, with a degree r.

Let X be a scheme. Recall that the homological motive of $\mathbf{G}_{m,X}$ is split as follows in $\mathscr{T}(X)$: $M(\mathbf{G}_{m,X}) = \mathbb{1}_X \oplus \mathbb{1}_X(1)[1]$. Thus, a global unit $u \in \mathbf{G}_m(X)$ gives a morphism of in $\mathscr{T}(X)$:

$$\{u\} : \mathbb{1}_X \to M(\mathbf{G}_{m,X}) \xrightarrow{(1)} \mathbb{1}_S(1)[1]$$

where (1) is the canonical projection. We obviously have the relation: $\{uv\} = \{u\} + \{v\}$. Thus we get a canonical morphism: $\mathbf{G}_m(X) \to H^{1,1}(X,\mathscr{T})$.

In particular we get for any point E/S a canonical morphism of abelian groups: $E^\times \to H^{1,1}(E,\mathscr{T})$. Using cup-product, we get the symbol map:

$$(E^\times)^n \to H^{n,n}(E,\mathscr{T}), (u_1,\ldots,u_n) \mapsto \{u_1,\ldots,u_n\} := \{u_1\}\cup\ldots\cup\{u_n\}.$$

According to the proof of [HK01, Prop. 1], the Steinberg relation holds for these symbols. In other words, we get a canonical morphism of $\mathbf{Z}$-graded algebras:

$$K^M_*(E) \to H^{*,*}(E,\mathscr{T})$$

Therefore, any symbol $x \in K^M_r(E)$ gives a map:

$$\gamma_x : M^{BM}(E/S)\{n\} \to M^{BM}(E/S)\{n+r\}.$$

Residues, (D4).– Let $\mathscr{O}_v$ be a discrete valuation ring containing k with fraction (resp. residue) field E (resp. κ) and maximal ideal $\mathfrak{M}_v$. Let $t : \mathrm{Spec}(\mathscr{O}_v) \to S$ be a morphism.

We let $\mathscr{M}(\mathscr{O}_v/S)$ be the set of sub-S-algebra $A \subset \mathscr{O}_v$ of finite type such that $(\mathfrak{M}_v \cap A) \neq 0$, ordered by: $A \le B$ if B is a localization of A. It is obviously filtering and the funtors:

$$\mathscr{M}(O_v/S) \to \mathscr{M}(E/S), A \mapsto (A \cap E),$$
$$\mathscr{M}(O_v/S) \to \mathscr{M}(\kappa/S), A \mapsto (A \otimes_{\mathscr{O}_v} \kappa)$$

are final.

Given $A \in \mathscr{M}(\mathscr{O}_v/S)$, we put $X = \mathrm{Spec}(A)$ and let Z corresponds to the closed subscheme of X whose ideal if $A \cap \mathfrak{M}_v$.. Let $i : Z \to X$ be the corresponding closed immersion, and $j : U \to X$ be the complmentary open immersion. According to the localization property of $\mathscr{T}$, we get a canonical distinguished triangle in $\mathscr{T}(\mathscr{O}_v)$:

$$M^{BM}(Z/S)[-1] \xrightarrow{\partial_i} M^{BM}(U/S) \xrightarrow{j_*} M^{BM}(X/S) \xrightarrow{i^*} M^{BM}(Z/S).$$

Moreover, this triangle is funtorial in X with respect to the étale covariant funtoriality of $M^{BM}(-/S)$.

If d is the k-transcendance degree of E then κ has k-transcendance degee $(d-1)$. Taking the formal colimit of the morphisms

$$\partial_i(d+r)[2d+r] : M^{BM}(Z/S)(d+r)[2d+r-1] \to M^{BM}(U/S)(d+r)[2d+r]$$

over the ordered set $\mathscr{M}(\mathscr{O}_v/S)$ we finally get the desired map:

$$\partial_v : M^{(0)}(\kappa/S)\{r+1\} \to M^{(0)}(E/S)\{r\}.$$

In the following theoreom, we will use the additive category $\tilde{\mathscr{E}}_k$ introduced in [Dég13] whose objects are pairs (E,n) where E/k is a extension field and n an integer, and whose morphisms describe the functoriality of cycle premodules. We let $\tilde{\mathscr{E}}_S$ be the full subcategory of $\tilde{\mathscr{E}}_k$ whose objects are of the form (E,n) where E is a point over S.

Theorem 1.8. *Consider the preceding notations.*

Then the functoriality of generic motives over S defined above uniquely induces a contravariant functor:

$$\left(\tilde{\mathscr{E}}_S\right)^{op} \to \mathscr{T}^{(0)}(S), (E,n) \mapsto M^{(0)}(E/S)\{-n\}$$

which on morphisms respects the natural gradings on both sides.

Proof. We have only to check that the data (D1)-(D4) above satisfy the relations (R*) of [Ros96, (1.1)]. This is easily deduced from [Dég14] as in [Dég13, §3.2.10]. □

As a corollary, the Rost transform of any $\mathscr{T}$-motive $\mathbb{E}$ aquire a canonical structure of a cycle premodule which we will simply denote by $\underline{\hat{\pi}}_{0,*}(\mathbb{E})$.

Theorem 1.9. *Let $\mathbb{E}$ be a $\mathscr{T}$-motive over S.*

Then the cycle premodule $\underline{\hat{\pi}}_{0,*}(\mathbb{E})$ over S is a cycle module.

Proof. Let X be any S-scheme separated of finite type.

Let $C_*\left(X, \underline{\hat{\pi}}_{0,*}(\mathbb{E}), n\right)$ be the sequence of maps of *cycles with coefficients* in the cycle premodule $\underline{\hat{\pi}}_{0,*}(\mathbb{E})$ defined by Rost ([Ros96, §5]):

$$\text{(1.9.a)} \qquad \ldots \to \bigoplus_{x \in X_{(p)}} \underline{\hat{\pi}}_{0,\mathrm{p+n}}(\mathbb{E})(\kappa(x)) \xrightarrow{d_p} \bigoplus_{y \in X_{(p-1)}} \underline{\hat{\pi}}_{0,\mathrm{p+n\text{-}1}}(\mathbb{E})(\kappa(y)) \to \ldots$$

To conclude, we have only to show that the above sequence of maps is a complex. Recall that according to Definition 1.2, $\underline{\hat{\pi}}_{0,\mathrm{p+n}}(\mathbb{E})(\kappa(x)) = \mathbb{E}^{BM}_{p-n,-n}(E/S)$, the transcendance degree of $\kappa(x)/k$ being p. Thus, replacing $\mathbb{E}$ by $\mathbb{E}(n)[n]$, we can assume that $n=0$.

It is well known that the niveau filtration on X induces a spectral sequence for the Borel-Moore homology $\mathbb{E}^{BM}_{*,0}$ as follows:

$$E^1_{p,q} = \bigoplus_{x \in X_{(p)}} \mathbb{E}^{BM}_{p+q,0}(\kappa(x)/S) \Rightarrow \mathbb{E}^{BM}_{p+q,0}(X/S).$$

From what we have just said, we get a canonical isomorphism: $C_p\left(X, \underline{\hat{\pi}}_{0,*}(\mathbb{E}), 0\right) \simeq E^1_{p,0}$. Moreover, according to the proof of [Dég12, Prop. 2.7] (see also the proof of [Dég13, Prop. 3.3.4]), this isomorphism is compatible with the differentials on both sides giving an isomorphism of towers:

$$C_*\left(X, \underline{\hat{\pi}}_{0,*}(\mathbb{E}), 0\right) \simeq E^1_{*,0}.$$

Because $E^1_{*,0}$ is a complex, we are done. □

2. The perverse homotopy t-structure

2.1. The perverse homotopy t-structure was introduced by Ayoub in [Ayo07a]. More precisely, he studied after Morel t-structure which satisfies the gluing formalism on stable homotopy functors.

Let us recall the definition of this t-structure. According to [Ayo07a], it is possible to describe t-structures on a compactly generated triangulated category[1] $\mathscr{T}$ as follows.

First, given a set of compact objects $\mathscr{G}$ in $\mathscr{T}$, we let $\langle\langle\mathscr{G}\rangle\rangle_+$ be the smallest full subcategory $\mathscr{T}_0$ of $\mathscr{T}$ such that:

- for any $A \in \mathscr{G}$, A is an object of $\mathscr{T}_0$;
- given any distinguished triangle in T,
$$A' \to A \to A'' \to A'[1]$$
if A' and A'' belongs to $\mathscr{T}_0$, then A belongs to $\mathscr{T}_0$;
- $\mathscr{T}_0$ is closed under small sums;
- $\mathscr{T}_0$ is closed under suspension: A belongs to $\mathscr{T}_0$ implies $A[1]$ belongs to $\mathscr{T}_0$.

Then Ayoub proved the following result (see [Ayo07a, 2.1.70]) which he attributes to Morel:

Proposition 2.2. *Consider the above assumptions and put:*
$\mathscr{T}_{\geq 0} = \langle\langle\mathscr{G}\rangle\rangle_+$, $\mathscr{T}_{<0} = (\mathscr{T}_{\geq 0})^{\perp}$.[2]
Then the pair $(\mathscr{T}_{\geq 0}, \mathscr{T}_{<0})$ *defines a t-structure on* $\mathscr{T}$.

In the assumptions of the proposition, the pair $(\mathscr{T}_{\geq 0}, \mathscr{T}_{<0})$ is called the *t-structure generated by* $\mathscr{G}$.

Remark 2.3. This result is best described using the notion of *aisle* introduced in [KV88, §1]: it can be restated by saying that the inclusion functor
$$\langle\langle\mathscr{G}\rangle\rangle_+ \to \mathscr{T}$$
admits a right adjoint. Acording to this remark, we can relax the assumption that $\mathscr{T}$ is compactly generated. For example, if $\mathscr{T}$ is the homotopy category of a left proper combinatorial model category, the theory of Bousfield localization provides the right adjoint for any set of objects $\mathscr{G}$. The same is true using results of Lurie if $\mathscr{T}$ is the homotopy category of a presentable stable ∞-category. Finally that one should be able to replace compactly generated by well generated in the sense of Neeman but we do not know any reference.

Definition 2.4. Let $\mathscr{T}$ be a premotivic triangulated category which is compactly generated by Tate twists (cf [CD12, 1.3.16]).[3]

Given any k-scheme S, we define the *homotopy t-structure* on $\mathscr{T}(S)$ as the t-structure generated by the following objects:
$$M_S(X)(n)[n]$$
for a smooth scheme X/S and an integer $n \in \mathbf{Z}$.

[1] *i.e.* a triangulated category which admits small sums and such that there exists a set of generators which are compact.

[2] As usual, given a full subcategory $\mathscr{U}$ of $\mathscr{T}$, we denote by $\mathscr{U}^{\perp}$ the full additive subcategory consisting of the objects $B \in \mathscr{T}$ satisfying $\mathrm{Hom}_{\mathscr{T}}(A,B) = 0$ for all $A \in \mathscr{U}$.

[3] *i.e.* given any k-scheme S, $\mathscr{T}(S)$ is generated by objects $M_S(X)(n)$ for X/S smooth and $n \in \mathbf{Z}$, and these objects are compact.

We denote simply by t this t-structure.

Example 2.5. The homotopy t-structure on DM(k) coincides with Voevodsky's homotopy t-structure (see [Dég08a]). The homotopy t-structure on SH(k) coincides with Morel's homotopy t-structure (see [Dég13, Rem. 1.1.5]).

2.6. We will need a finer t-structure defined by Ayoub. We will define it on any We will define it within a motivic triangulated category $\mathscr{T}$ over k-schemes which is generated by Tate twists.

Recall this assumption guarantees that a $\mathscr{T}$-motives over S is constructible if and only if it is compact (see [CD12, 1.4.11]). Let $\mathscr{T}_c(S)$ be the full subcategory made by constructible $\mathscr{T}$-motives over S. According to [Ayo07a, 2.2.37], we get that $\mathscr{T}_c(S)$ is stable under the 6 operations (here we use that k is of characteristic 0).

Definition 2.7. Let $\mathscr{T}$ be a motivic triangulated category satisfying the above assumptions.

Given any separated k-scheme S of finite type, we define the *perverse homotopy t-structure on $\mathscr{T}(S)$* as the t-structure generated by the following objects:

$$f_!p^!(\mathbb{1}_k)(n)[n]$$

where $f : X \to S$ is a k-morphism for a separated k-scheme of finite type X, p is the structural morphism of X/k, and $n \in \mathbf{Z}$ is any integer.

We denote by t^p this t-structure and by ${}^{\mathrm{p}}\mathrm{H}_0\,\mathscr{T}(S)$ its heart.

Example 2.8. According to [Ayo07a, 2.2.86, 2.2.82], the homotopy and perverse homotopy t-structures on $\mathscr{T}(k)$ coincides (here we use that k is of characteristic 0). The same is true for $\mathscr{T}(E)$ here E/k is any finite field extension.

We will need the following proposition due to Ayoub:

Proposition 2.9. *Let $\mathscr{T}$ be as in the above definition.*

Let $i : Z \to S$ be a closed immersion with complementary open immersion $j : U \to S$. Then the following assertions hold:

(1) *i^* is t^p-positive and i_* is t^p-exact;*
(2) *$j_!$ is t^p-positive and j^* is t^p-exact;*
(3) *a $\mathscr{T}$-motive M is t^p-positive if and only if $j^*(M)$ and $i^*(M)$ are so.*

In fact, refering to *loc. cit.*, Point (1) is proved in 2.2.77 and 2.2.78, Point (2) in 2.2.62 and Point (3) is a formal consequence of (1)+(2) and the localization property of $\mathscr{T}$.

We formally deduce from that theorem the following corollary:

Corollary 2.10. *Using the notations of the previous corollary, the abelian category ${}^{\mathrm{p}}\mathrm{H}_0\,\mathscr{T}(S)$ is the glueing of the abelian categories ${}^{\mathrm{p}}\mathrm{H}_0\,\mathscr{T}(Z)$ and ${}^{\mathrm{p}}\mathrm{H}_0\,\mathscr{T}(U)$ along the following exact functors:*

$${}^{\mathrm{p}}\mathrm{H}_0\,\mathscr{T}(Z) \xrightarrow{i_*} {}^{\mathrm{p}}\mathrm{H}_0\,\mathscr{T}(X) \xrightarrow{j^*} {}^{\mathrm{p}}\mathrm{H}_0\,\mathscr{T}(U).$$

Moreover, the left (resp. right) adjoint functor to i_* is ${}^{\mathrm{p}}\tau^p_{\leq 0}(i^*)$ (resp. ${}^{\mathrm{p}}\tau^p_{\geq 0}(i^*)$) and the left (resp. right) adjoint functor tp j^* is ${}^{\mathrm{p}}\tau^p_{\leq 0}(j_!)$ (resp. ${}^{\mathrm{p}}\tau^p_{\geq 0}(j_*)$).

The following result is obtain using the same proof as [CD12, 4.4.25] but using Hironaka resolution of singularities as we have restricted to characteristic 0:

Proposition 2.11. *Let $\varphi^* : \mathscr{T} \leftrightarrows \mathscr{T}' : \varphi_*$ be an adjunction of motivic triangulated categories over k-schemes of finite type. Assume $\mathscr{T}$ and $\mathscr{T}'$ are generated by Tate twists.*

Then the induced functor $\varphi^* : \mathscr{T}_c \to \mathscr{T}'_c$ between the constructible parts commutes with the 6 operations.

Corollary 2.12. *Consider the assumptions of the previous proposition and assume $\mathscr{T}$, $\mathscr{T}'$ are compactly generated by their Tate twist.*

Then φ^* (resp. φ_*) is t^p-positive (resp. t^p-negative).

Indeed, according to the previous proposition, φ^* preserves the generators of the perverse t-structure. The assertion for φ_* is a formal consequence.

2.13. We will consider the following adjuntions of motivic triangulated categories:

$$\mathrm{SH} \underset{N}{\overset{K}{\rightleftarrows}} \mathrm{D}_{\mathbf{A}^1} \underset{\gamma_*}{\overset{\gamma^*}{\rightleftarrows}} \mathrm{DM}_{\mathrm{cdh}}$$

where (K,N) is induced by the Dold-Kan equivalence (see [Ayo07b, §4]) and (γ^*,γ_*) if induced by the functor adding transfers at the level of sheaves (see [CD13]).

According to its construction, the functor N commutes with $j_\sharp$ for j an open immersion. The same is true for γ_* according to [CD12, 6.3.11] (more precisely its variant for the cdh-topology). We easily deduce our first lemma:

Lemma 2.14. *The functors N and γ_* above are t^p-exact.*

Proof. Indeed, these two functors formally commutes with f_* and $p^!$ (see [CD12, 2.4.53]). According to the construction of $p_!$ in [CD12] and the remark preceding the lemma, they also commutes with $p_!$. Thus they preserve the generators of the perverse homotopy t-structure, so they are t^p-positive. The preceding corollary allows to conclude. □

2.15. Following the terminology of [Dég13], we will say that a spectrum $\mathbb{E}$ in $\mathrm{SH}(S)$ is *weakly orientable* if $\eta \wedge \mathbb{E} = 0$ where $\eta : S^0 \to S^0$ is the Hopf map in $\mathrm{SH}(S)$.

The first step to our main theorem is the following result:

Theorem 2.16. *Consider the notations of the preceding lemma.*

(1) *The adjunction of abelian categories induced by (K,N)*

$$^{\mathrm{p}}\tau_{\leq 0}(K) : {}^{\mathrm{p}}\mathrm{H}_0\,\mathrm{SH}(S) \leftrightarrows {}^{\mathrm{p}}\mathrm{H}_0\,\mathrm{D}_{\mathbf{A}^1}(S) : N$$

is an equivalence of categories.

(2) *The functor induced by γ_**

$$\gamma_* : {}^{\mathrm{p}}\mathrm{H}_0\,\mathrm{DM}_{\mathrm{cdh}}(S) \to {}^{\mathrm{p}}\mathrm{H}_0\,\mathrm{SH}(S)$$

is fully faithful and its essential image is made of the weakly orientable spectra.

Proof. The proof is based on the fact that these results are known when $S = \mathrm{Spec}\,E$ for E a finite extension of k: in this case, the homotopy perverse t-structure coincides with the homotopy t-structure (Example 2.8); then Point (1) follows from the explicit description of the heart of the homotopy t-structure (cf [Mor12]) and Point (2) is the main result of [Dég13].

The proof will use noetherian induction with the help of the following lemma:

Lemma 2.17. *Let S be an integral k-scheme of finite type with funtion field E. Let $\mathscr{T}$ be one of the motivic triangulated categories* SH, $\mathrm{D}_{\mathbf{A}^1}$, $\mathrm{DM}_{\mathrm{cdh}}$.

Then the canonical map

$$^{\mathrm{p}}\mathrm{H}_0\,\mathscr{T}(\mathrm{Spec}(E)) \to \varinjlim_{U\subset S} \left(^{\mathrm{p}}\mathrm{H}_0\,\mathscr{T}(U)\right)$$

where the left hand side refers to the perverse homotopy t-structure with respect to E as a base field, and the limits runs over the non empty open subscheme of S, is an equivalence of abelian categories.

Proof. Note the map is well defined as for any open immersion j, j^* is t^p-exact (Prop. 2.9). The lemma now follows formally from the fact $\mathscr{T}$ is continuous (*cf* [CD12, Def. 4.3.2]) and the perverse homotopy t-structure is left complete. □

We will prove this by induction on the dimension of S, the case of dimension 0 being known from the above remark. As a first step, we can assume S is integral, up to removing the intersection of its irreducible components.

Point (1) follows easily from this lemma and the remark of the beginning of the proof, by noetherian induction.

For Point (2), we first prove that γ_* is fully faithul. In other words, for any object K of the heart of DM(S), the adjunction map

$$ad : {}^{\mathrm{p}}\tau_{\leq 0}(\gamma^*)\gamma_*(K) \to K$$

is an isomorphism.

According to the preceding lemma and the initial remark, there exists an open subscheme $j : U \to S$ such that the result is known in DM(U) for $j^*(K)$. Let $i : Z \to X$ be the complementary closed immersion. According to Corollary 2.10, to prove that ad is an isomorphism, is suffices to check that ${}^{\mathrm{p}}\tau_{\leq 0}\, i^*(ad)$ is an isomorphism.

We have the following computation:

$${}^{\mathrm{p}}\tau_{\leq 0}\, i^*\, {}^{\mathrm{p}}\tau_{\leq 0}(\gamma^*)\gamma_* \simeq {}^{\mathrm{p}}\tau_{\leq 0}(i^*\gamma^*)\gamma_* \simeq {}^{\mathrm{p}}\tau_{\leq 0}\,\gamma^*(i^*\gamma_*) \underset{(1)}{\simeq} {}^{\mathrm{p}}\tau_{\leq 0}\,\gamma^*(\gamma_* i^*) \underset{(2)}{\simeq} {}^{\mathrm{p}}\tau_{\leq 0}\,\gamma^*\gamma_*({}^{\mathrm{p}}\tau_{\leq 0}\, i^*)$$

where the isomorphism (1) follows from the fact γ_* commutes with $j_\sharp$ and the localization property for DM and $\mathrm{D}_{\mathbf{A}^1}$ and isomorphism (2) because γ_* preserves t^p-negative objects. Thus, we are reduced to prove the assertion to ${}^{\mathrm{p}}\tau_{\leq 0}\, i^*(K)$ which follows by the induction hypothesis.

The last assertion follows similarly by induction, by another application of Corollary 2.10 and the fact that condition of being weakly orientable is compatible with localization. □

3. From cycle modules to homotopy modules

3.1. Let M be a cycle module over a scheme S.

For any smooth S-scheme X, we let $C_*(X,M)$ be the (graded) complex of cycles with coefficients in M defined in [Ros96].

This complexe enjoys the following functoriality property: let $f : Y \to X$ be a morphism of smooth S-scheme.

We consider the factorization:

$$Y \xrightarrow{\gamma} Y \times_S X \xrightarrow{\pi} X$$

where γ is the graph of f and π the natural projection.

(1) the morphism π being smooth, thus flat, there exists a canonical pullback map:
$$\pi^* : C_*(X,M) \to C_*(Y \times_S X, M)$$
(2) the immersion γ being regular, with normal bundle $p : T_f \to Y$, we have a natural deformation map:
$$\sigma_\gamma : C_*(Y \times_S X, M) \to C_*(T_f, M)$$
(3) by the homotopy property of the Chow groups with coefficients, the pullback map:
$$p^* : C_*(Y,M) \to C_*(T_f, M)$$
is a quasi-isomorphism (in fact a chain homotopy equivalence).

Thus, in the derived category $\mathrm{D}(\mathscr{A}b)$, we get a well defined map
$$f^* : C_*(X,M) \to C_*(Y,M).$$
According to [Ros96], they satisfy the usual cocycle condition.

Note in particular that the contravariant funtorality with respect flat morphisms gives, for any (smooth) S-scheme X, a sheaf of complexes on the small Nisnevich site of X:
$$C_*(-,M)_X : V/X \mapsto C_*(V,X).$$
Using the technique of [Lev06, 7.3, 7.4], it is possible to rectify these maps and to obtain the following result:

Proposition 3.2. *Given the assumptions above, there exists a canonical complex $C_*(M)$ of Nisnevich sheaves of graded abelian groups over Sm_S such that, for any smooth S-scheme X, there exists a natural isomorphism:*
$$C_*(M)|_{X_{\mathrm{Nis}}} \xrightarrow{\epsilon_X} C_*(-,M)_X$$

Corollary 3.3. *With the notations of the preceding theorem, the complex $C_*(M)$ is Nisnevich and $\mathbf{A}^1$-local (*i.e.* fibrant with respect to the $\mathbf{A}^1$-local model category on the category of complexes of Nisnevich sheaves on Sm_S).*

Indeed, this follows from the fact that for any scheme X, the complexes $C_*(-,M)_X$ satisfies the Brown-Gersten property ([Dég06, 6.10]) and its cohomology as a presheaf is $\mathbf{A}^1$-invariant (*cf* [Ros96, 8.6]).

Using the localization sequence for Chow group with coefficients, we can further deduce from the preceding construction the following result:

Proposition 3.4. *Consider the notations of the previous proposition.*

(1) *Then there exists a canonical Tate spectrum $\mathbf{H}(M)$ associated with M and a quasi-isomorphism of $\mathbf{Z}$-graded complexes:*
$$\Omega^\infty(\mathbf{H}(M)(*)[*]) \to C_*(M).$$
The association $M \mapsto \mathbf{H}(M)$ is functorial in M.

(2) *If $k : A \to S$ is the inclusion of a closed or open subscheme of S, then there is a canonical isomorphism:*
$$k^!(\mathbf{H}(M)) \simeq \mathbf{H}(M|_A),$$
where $M|_A$ is the naive restriction of the cycle module M to points of A.

Point (1) is obtained using the isomorphism $A^1(\mathbf{G}_m, M) = A^1(S,M) \oplus A^0(S,M)$. Point (2) is obvious in the case of an open immersion. The case where k is a closed immersion follows from the existence of push-forward, notably: $k_* : C_*(A,M) \to C_*(S,M)$.

3.5. It is not difficult to deduce from the spectral sequence [Ros96, §8] that the Chow group of $\mathbf{P}^n_S$ with coefficients in a cycle module splits. This implies that $\mathbf{H}(M)$ is weakly orientable in the sense of 2.15.

Finally, we have defined a functor:

$$\mathbf{H} : \mathscr{M}Cycl(k)_S \to {}^{\mathrm{p}}\mathrm{H}_0\,\mathrm{DM}_{\mathrm{cdh}}(S). \tag{3.5.a}$$

4. The main theorem

Theorem 4.1. *Consider the notations introduced above.*

Then the functors:

$$\underline{\hat{\pi}}_{0,*} : {}^{\mathrm{p}}\mathrm{H}_0\,\mathrm{DM}_{\mathrm{cdh}}(S) \leftrightarrows \mathscr{M}Cycl(k)_S : \mathbf{H}$$

defines mutually inverse equivalences of categories.

Proof. We prove this assertion by noetherian induction on S. The case where $\dim(S) = 0$ follows from [Dég11].

Then, we can argue by induction on the dimension. Indeed, $\mathscr{M}Cycl(k)_S$ is obviously a glueing of $\mathscr{M}Cycl(k)_Z$ and $\mathscr{M}Cycl(k)_U$. Moreover, if S is irreducible with function field E,

$$\mathscr{M}Cycl(k)_E = \varinjlim_{U \subset S} \mathscr{M}Cycl(k)_S.$$

Finally, $\underline{\hat{\pi}}_{0,*}$ (resp. $\mathbf{H}$) commutes with $k^!$ where k is an open or closed immersion, $k^!$ on cycle modules being the obvious restriction. In the first case, this follows by definition of $\underline{\hat{\pi}}_{0,*}$ and in the second case, this is a consequence of 3.4. □

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