THE BOREL CHARACTER

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Abstract. The purpose of this article is first to define an explicit Borel character from Hermitian K-theory to rational MW-motivic cohomology and ordinary rational motivic cohomology, and then prove that it is an isomorphism rationally based on previous work by Fangzhou Jin, Adeel Khan and the authors.

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Introduction

The Chern character has been an essential piece of the extension of the classical Riemann-Roch formula, pioneered by Kodaira ([Kod51, Kod52]) and Serre ([Die74, VIII.12.]), and first established by Hirzebruch (ref.). By unveiling K-theory, Grothendieck discovered the true nature of the Chern character, a natural transformation from algebraic K-theory to rational Chow groups ([BS58]). Ultimately, algebraic topology established its universal property, in

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the topological setting, through the theory of formal group law associated with a (complex) oriented cohomology theory. Singular cohomology and complex K-theory are particular instances of such oriented cohomology theories; the first one has the additive formal group law while the second one has the multiplicative formal group law. Then the Chern character can be seen as the unique natural transformation of rational oriented cohomology theories, corresponding to the unique morphisms of formal group laws from the multiplicative one to the additive one.

Nowadays, algebraic topology and algebraic geometry have been intrinsically linked through the motivic homotopy theory of Morel and Voevodsky. It is natural to extend what we know in algebraic topology to this new world, a recipe that has already proved very fruitful. The first application of this recipe was the construction of motivic Steenrod operations and their application to the Bloch-Kato conjecture by Voevodsky. A second one was the introduction of orientable cohomologies together with the universal such theory, the algebraic cobordism.[1] In this context, the universality of the Chern character mentioned above in algebraic topology was extended to motivic homotopy ([Pan04], [Dég18]).

On the other hand, weaker notions of orientations have arisen in motivic homotopy theory, first concretely through Chow-Witt groups ([BM00], [Fas08]) and then theoretically with the work of Panin and Walter [PW10a, PW10c]. The latter authors introduced the conditions of Sp-orientability[2] to account for the formalism satisfied by Chow-Witt groups, and also for hermitian K-theory and Balmer's higher Witt groups. Under this weaker orientability condition, the existence of Chern classes is not guaranteed but one still gets some characteristic classes, called the Borel and Pontryagin classes.[3] This theory lead Marc Levine to set a program for enlarging the classical study and computations of characteristic classes in orientable cohomologies, mainly Chow groups, to that of Sp-orientable cohomology such as Chow-Witt groups (see e.g. [Lev17]).

In this context, the primary goal of this paper is to generalize the Chern character in the quadratic setting replacing algebraic K-theory and (higher) Chow groups by hermitian K-theory and (higher) Chow-Witt groups. The resulting map will be called the Borel character as it uses Borel classes rather than Chern classes.

In [DFJK19], using the periodicity of hermitian K-theory and higher Witt groups, we were able to get an abstract version of the Borel character under the form of an isomorphism of ring spectra:

$$\mathbf{KQ}_{\mathbb{Q}} \xrightarrow{\text{bo}} \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S+}\langle 2m\rangle \oplus \bigoplus_{m\in\mathbb{Z}} \mathbb{Q}_{S-}\langle 4m\rangle$$

where $\mathbb{Q}_{S+}$ and $\mathbb{Q}_{S-}$ denote respectively the plus-part and minus-part of the rational sphere spectrum, and we have denoted by $\langle 1\rangle$ the twist by $\mathbb{P}^1$.

In the present work, we give a more concrete construction of this isomorphism, which is better suited for computations. The main idea is to use the ideas of Riou's thesis on operations on algebraic K-theory. Secondly, we use the quadratic enrichment of the motivic cohomology

[1] Orientable cohomology theories were introduced by Panin and Smirnov, while orientable spectra were first considered by Borghesi and Vezzosi. Algebraic cobordism was first introduced, as a motivic spectrum by Voevodsky and then defined as a universal cohomology theories by Levine and Morel.

[2] as well as the other notions of SL-orientability and SL^c-orientability,

[3] The definition of these classes are recalled for Chow-Witt groups respectively in Theorem 2.0.2 and Definition 2.0.7.

ring spectrum, defined recently in [DF17], the so-called Milnor-Witt motivic cohomology ring spectrum $\mathbf{H}_{\mathrm{MW}}$. This granted, the main result of this paper is the following theorem:

Theorem 1 (see 5.0.3)**.** *Let k be a perfect field of characteristic different from 2 and 3. There exists a canonical morphism* bo *of motivic ring spectra over k which fits into the following commutative diagram:*

$$\begin{array}{ccc}
\mathbf{KQ} & \xrightarrow{\quad\mathrm{bo}\quad} & \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 4i\rangle \times \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle 2+4i\rangle \\
\downarrow & & \downarrow \\
\mathbf{KGL} & \xrightarrow[\quad\mathrm{ch}\quad]{} & \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle i\rangle
\end{array}$$

where $\langle 1\rangle$ stands for the twist by $\mathbb{P}^1$ in $\mathrm{SH}(k)$, the map ch *is Riou's lifting of the Chern character (*[Rio10, Definition 6.2.3.9]*), and the vertical maps are induced by the natural forgetful maps $\mathbf{KGL} \to \mathbf{KQ}$ and $\mathbf{H}_{\mathrm{MW}} \to \mathbf{H}_{\mathrm{M}}$.*

Moreover, the morphism bo *is an isomorphism rationally and agrees with the morphism defined in* [DFJK19] *modulo canonical isomorphisms*[4]

$$\mathbf{H}_{\mathrm{MW}}(\mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}_{S+} \times \mathbb{Q}_{S-},$$
$$\mathbf{H}_{\mathrm{M}}(\mathbb{Q}) \xrightarrow{\sim} \mathbb{Q}_{S+}.$$

As in [Rio10], one obtains operations on hermitian K-theory using its representability by symplectic Grassmannian (see recall in Section 1.2). The main point is to show that these operations stabilize. In this respect, the case of hermitian K-theory is notably more difficult than that of algebraic K-theory due to the lack of a formal group law. Following again the method of Riou, we are led to the computation of the Borel classes of the threefold product

$$(E,\psi) := (U_1,\varphi_1) \otimes (U_2,\varphi_2) \otimes (U_3,\varphi_3)$$

of bundles over $\mathrm{H}\mathbb{P}^n \times \mathrm{H}\mathbb{P}^n \times \mathrm{H}\mathbb{P}^n$. We give a complete computation of these classes in Theorem 3.3.1. This can be seen as a trace of the yet to be determined analogue of formal ternary laws for hermitian K-theory, understood as an Sp-oriented theory.[5]

The advantage of our method is that it gives an explicit formula for the Borel character bo, and especially for the projection onto the first factor. We refer the interested reader to Section 4.4. The comparison of the construction of the map bo obtained here with the one in [DFJK19] has a double advantage. First it gives a concrete computation of the abstract definition of *loc. cit.* Second, it allows us to prove that the map bo is an isomorphism of ring spectra, a property that possesses the construction done in *loc. cit.*

An important precursor of our work was Ananyevskiy's determination of stable operations on higher Witt theories [Ana15b]. The latter work was actually used in [DFJK19] via the results of [ALP17]. The advance we get here is an explicit determination of the Borel classes of a 3-fold tensor product in Chow-Witt groups, which parallels the computations of [Ana15b, Lem. 1.4], for higher Witt groups.

[4] The second isomorphism is that obtained in [CD09]. The first one was first obtained by Garkusha in [Gar17]. We provide another construction of these isomorphisms in the proof of Theorem 5.0.3. Note however that the central ingredient, the results of [ALP17], is the same in both constructions.

[5] The only text we are aware of concerning these questions are the unpublished notes of Charles Walter [Wal12].

Plan. In Section 1, we recall the definition of the ring spectra that will be our main object of study, the one representing Milnor-Witt motivic cohomology and hermitian K-theory.

In Section 2, we recall the construction of the Borel classes in Chow-Witt groups. This material is based on the theory of Sp-orientable ring spectra due to Panin and Walter. We include here a direct treatment for the comfort of the reader, specialized to Chow-Witt groups.

Section 3 is the technical heart of the paper. By reducing to Chow groups and Witt cohomology, we compute the Borel classes of a 3-fold tensor product of vector bundles in Chow-Witt groups.

In Section 4, we deduce from the preceding computation the stable operations corresponding to the projection of the Borel character. As in the case of the classical Chern character, defined only up to inverting the integers, the stable operations constructed involve inverting certain classes of quadratic forms. We make this explicit in Section 4.3.

In the end, Section 5 contains the final definition of the (concrete) Borel character as well as the comparison isomorphism with the (abstract) Borel character of [DFJK19].

Conventions. We work over a perfect field k of characteristic different from $2, 3$. For any integer n, we will denote by $\langle n\rangle$ the twist in the stable motivic homotopy category by $(\mathbb{P}^1)^{\wedge n}$.

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1. The spectra

1.1. **Milnor-Witt motivic cohomology.** In this section, we quickly review the basic material needed to understand the spectra relevant for this article. On the one hand, we have the $\mathbb{P}^1$-spectrum $\mathbf{H}_{\mathrm{MW}}$ representing MW-motivic cohomology. This spectrum was constructed in [DF16]. In degree n, it is of the form $\mathrm{K}(\tilde{\mathbb{Z}}(n), 2n)$ and the morphisms

$$\mathbb{P}^1 \wedge \mathrm{K}(\tilde{\mathbb{Z}}(n), 2n) \to \mathrm{K}(\tilde{\mathbb{Z}}(n+1), 2n+2)$$

are the adjoints of the weak-equivalences $\mathrm{K}(\tilde{\mathbb{Z}}(n), 2n) \to \Omega_{\mathbb{P}^1} \mathrm{K}(\tilde{\mathbb{Z}}(n+1), 2n+2)$. Equivalently, they are induced by the fact that the functor from $\mathrm{SH}(k)_s$ to the effective category of MW-motives is lax-monoidal and the tensor product

$$\tilde{\mathbb{Z}}\langle 1\rangle \otimes \tilde{\mathbb{Z}}\langle n\rangle \to \tilde{\mathbb{Z}}\langle n+1\rangle.$$

For the sake of completeness, recall from [DF16] that for any smooth scheme X, we have an isomorphism

$$[X, \mathrm{K}(\tilde{\mathbb{Z}}(n), 2n)] = \widetilde{\mathrm{CH}}^n(X).$$

functorial in X.

1.2. **Hermitian K-theory.** Recall from [PW10b] that Hermitian K-theory is represented by an explicit spectrum in the stable homotopy category. In this article, it will be convenient to consider a slightly different model, in the form of a $(\mathbb{P}^1)^{\wedge 4}$-spectrum that we now describe. Recall first that Panin and Walter defined a smooth affine scheme $\mathrm{H}\mathbb{P}^n$ for any $n \in \mathbb{N}$ ([PW10c]). On $\mathrm{H}\mathbb{P}^n$, there is a canonical bundle U which is symplectic of rank 2. We'll denote by φ the symplectic form in the sequel. For any $n \in \mathbb{N}$, there are morphisms

$$i_n : \mathrm{H}\mathbb{P}^n \to \mathrm{H}\mathbb{P}^{n+1}$$

such that $i_n^*(U,\varphi) = (U,\varphi)$ and the colimit (say in the category of sheaves of sets) is denoted by $\mathrm{H}\mathbb{P}^\infty$. As $\mathrm{H}\mathbb{P}^0 = \operatorname{Spec} k$, we consider all these schemes as pointed by x_0. It is a model of BSp_2. Recall moreover that $\mathrm{H}\mathbb{P}^1$ is weak-equivalent to $(\mathbb{P}^1)^{\wedge 2}$. In fact $\mathrm{H}\mathbb{P}^1 = Q_4$, where the latter is the affine scheme defined in [ADF16].

Notation 1.2.1. We set $S := \mathrm{H}\mathbb{P}^1$. We also denote by Ω_S the adjoint of $S \wedge \_$.

Next, let HGr be the hyperbolic Grassmannian of Panin and Walter ([PW10c]). It represents symplectic K-theory in the unstable homotopy category, i.e. we have

$$[X, \mathbb{Z} \times \mathrm{HGr}] = \mathrm{K\,Sp}(X) = \mathrm{GW}^2(X).$$

By Schlichting and Tripathi ([ST15]), we have identifications

$$\Omega_S^n \mathrm{HGr} \simeq \begin{cases} \mathrm{OGr} & \text{if } n \text{ is odd.} \\ \mathrm{HGr} & \text{if } n \text{ is even.} \end{cases}$$

Here, OGr is the orthogonal Grassmannian constructed in [ST15], which has the property to represent orthogonal K-theory, i.e. $[X, \mathbb{Z} \times \mathrm{OGr}] = \mathrm{GW}^0(X)$. This periodicity allows us to define a $S^{\wedge 2}$-spectrum with term HGr in degree n and bonding map

$$S^{\wedge 2} \wedge \mathrm{HGr} \to \mathrm{HGr}$$

adjoint to the identification $\mathrm{HGr} \simeq \Omega_S^2 \mathrm{HGr}$ given above. For any smooth scheme X, this identification reads as

$$\mathrm{GW}^2(X) = [X_+, \mathrm{HGr}] \to [X_+, \Omega_S^2 \mathrm{HGr}] = [X_+ \wedge S^{\wedge 2}, \mathrm{HGr}] = \mathrm{GW}^2(X_+ \wedge S^{\wedge 2}).$$

If we write (U_1, φ_1) for the canonical bundle of the first factor S and (U_2, φ_2) for the canonical bundle of the second term, then the above composite is just the multiplication by

$$((U_1, \varphi_1) - H)((U_2, \varphi_2) - H)$$

where H is the usual symplectic form on $\mathcal{O}^2$.

Definition 1.2.2. We denote by $\mathbf{HGr}$ the $S^{\wedge 2}$-spectrum defined above.

Finally, let us introduce another $S^{\wedge 2}$-spectrum derived from the $\mathbb{P}^1$-spectrum $\mathbf{H}_{\mathrm{MW}}$. Its term in degree n is $\mathrm{K}(\mathbb{Z}(2+4n), 4+8n)$ and the bonding maps are alternatively given by the adjoints of the weak-equivalences

$$\mathrm{K}(\mathbb{Z}(2+4n), 4+8n) \simeq \Omega_S^2 \mathrm{K}(\mathbb{Z}(6+4n), 12+8n)$$

or by the tensor products as above. These correspond on smooth schemes to the maps

$$\widetilde{\mathrm{CH}}^{4n+2}(X) \to \widetilde{\mathrm{CH}}^{4n+6}(X_+ \wedge S^{\wedge 2})$$

obtained by multiplication by the product of the Euler classes u_1 and u_2 associated respectively to (U_1, φ_1) and (U_2, φ_2).

Definition 1.2.3. We denote by $\mathbf{H}_{\mathrm{MW}}\langle 2 \rangle$ the S^2-spectrum defined above.

2. Borel and Pontryagin classes

The construction of the Borel classes is based on the following computation that can be found in either [Yan17] or [HW18].

Lemma 2.0.1. *Let X be a smooth scheme. For any $n \geq 0$, we have*

$$\widetilde{\mathrm{CH}}^*(\mathrm{H}\mathbb{P}^n \times X) = \widetilde{\mathrm{CH}}^*(X)[b_1]/\langle b_1^{n+1}\rangle$$

where $b_1 \in \widetilde{\mathrm{CH}}^2(\mathrm{H}\mathbb{P}^n)$ is the Euler class of the canonical bundle U of $\mathrm{H}\mathbb{P}^n$.

Theorem 2.0.2. *Let X be a smooth scheme and let (E,φ) be a rank $2n$ symplectic bundle on X. Then,*

$$\widetilde{\mathrm{CH}}^*(\mathrm{H}\mathbb{P}(E)) = \widetilde{\mathrm{CH}}^*(X)[b_1]/f(b_1)$$

where $b_1 \in \widetilde{\mathrm{CH}}^2(\mathrm{H}\mathbb{P}(E))$ is the Euler class of the tautological bundle U of $\mathrm{H}\mathbb{P}(E)$ and

$$f(b_1) = b_1^n - b_1(E,\varphi)b_1^{n-1} + b_2(E,\varphi)b_1^{n-2} + \ldots + (-1)^i b_i(E,\varphi)b_1^{n-i} + \ldots + (-1)^n b_n(E,\varphi).$$

The classe $b_i(E,\varphi)$ is called the i-th Borel class of (E,φ).

Remark 2.0.3. The Borel class b_i is of weight $2i$, i.e. belongs to $\widetilde{\mathrm{CH}}^{2i}(X)$.

We can define the Borel classes for other cohomology theories, for instance for the cohomology of the (unramified) Witt sheaf or for ordinary Chow groups. We will use the same notations for the Borel classes in different cohomology theories when no confusion can arise. In some cases, the Borel classes for Chow-Witt groups can be computed using their analogues in Chow groups and the cohomology of the sheaves $\mathbf{I}^*$.

Lemma 2.0.4. *Let X be a smooth scheme such that the homomorphism*

$$\widetilde{\mathrm{CH}}^*(X) \to \mathrm{CH}^*(X) \oplus \mathrm{H}^*(X, \mathbf{I}^*)$$

induced from the homomorphism of sheaves

$$\mathbf{K}_*^{\mathrm{MW}} \to \mathbf{K}_*^{\mathrm{M}} \oplus \mathbf{I}^*$$

is injective. Then, the homomorphism of graded rings

$$\widetilde{\mathrm{CH}}^*(\mathrm{H}\mathbb{P}(E)) \to \mathrm{CH}^*(\mathrm{H}\mathbb{P}(E)) \oplus \mathrm{H}^*(\mathrm{H}\mathbb{P}(E), \mathbf{I}^*)$$

is injective for any symplectic bundle E over X.

Proof. The proof is obvious by observing that the Euler class (in Chow-Witt theory) is mapped to the corresponding Euler classes in the relevant theories. □

This result implies the following corollary that will be fundamental in our computations.

Corollary 2.0.5. *Let $X = \mathrm{H}\mathbb{P}^{n_1} \times \mathrm{H}\mathbb{P}^{n_2} \times \mathrm{H}\mathbb{P}^{n_3} \times \ldots \times \mathrm{H}\mathbb{P}^{n_m}$ for some integers $n_1, \ldots, n_m \in \mathbb{N}$. Then, the homomorphism of graded rings*

$$\widetilde{\mathrm{CH}}^*(X) \to \mathrm{CH}^*(X) \oplus \mathrm{H}^*(X, \mathbf{I}^*)$$

is injective. In particular, the Borel classes of any symplectic bundle on X is uniquely determined by the corresponding classes in Chow groups and $\mathbf{I}^*$-cohomology.

We may further simplify the above corollary using the following lemma.

Lemma 2.0.6. *For X as above, the morphism of sheaves $\mathbf{I}^* \to \mathbf{W}$ induces a canonical isomorphism of graded rings*

$$\oplus_{i\in\mathbb{N}} \mathrm{H}^i(X, \mathbf{I}^i) = \oplus_{i\in\mathbb{N}} \mathrm{H}^i(X, \mathbf{W})$$

Proof. Both terms are free $W(k)$-algebras and the generators are sent to each other under the map. □

We may extend the definition of Borel classes to non-symplectic bundles as follows.

Definition 2.0.7. Let X be a smooth scheme and let E be a vector bundle of rank n over X. Let $H(E)$ be the hyperbolic (symplectic) space on E. The Pontryagin classes of E are the Borel classes of $H(E)$, i.e.

$$p_i(E) = b_i(H(E)).$$

Remark 2.0.8. Our notation agrees with the notation in [Ana15a] and [Lev18] only up to sign. We'll make this clear when this is relevant in the computations.

In the next few results, we are going to prove that the odd Pontryagin classes, i.e. the classes b_{2i+1} of a vector bundle E are hyperbolic, showing that the odd Pontryagin classes of E are trivial in the cohomology of the Witt sheaf. This result is well-known, but we add it for the sake of completeness.

Lemma 2.0.9. *Let U be the tautological bundle on $\mathrm{H}\mathbb{P}^n$ for some $n \geq 1$ and let $u \in k^\times$. Then $b_1(U, u\varphi) = \langle u\rangle p_1(U, \varphi)$. In other terms, the first Borel class is $\mathrm{GW}(k)$-linear.*

Proof. The symplectic form φ corresponds to a trivialization of the determinant of the form

$$U \wedge U \to \mathcal{O}_X$$

given by $x \wedge y \to \varphi(x, y)$. Changing φ by a unit u changes correspondingly the orientation above. The computation of the first Borel class (i.e. the Euler class) depends on the choice of a trivialization, and we then find that $p_1(U, u\varphi) = \langle u\rangle p_1(U, \varphi)$. For the linearity, it suffices to observe that $\langle u\rangle \cdot (U, \varphi) = (U, u\varphi)$ by definition. □

Corollary 2.0.10. *Let X be a smooth variety and let (E, φ) be a symplectic form on X. Let $u \in \mathcal{O}(X)^\times$ be a unit. Then, the Borel classes of $(E, u\varphi)$ are of the form*

$$b_i(E, u\varphi) = \langle u^i\rangle b_i(E, \varphi).$$

Proof. Let (U, ψ) be the tautological rank 2 bundle over the hyperbolic Grassmannian of (E, φ). A straightforward computation shows that the tautological rank 2 bundle on $\mathrm{HGr}(E, u\varphi)$ is precisely $(U, u\psi)$. The first Borel class of the latter is $\langle u\rangle b_1(U, \psi)$. Now, the defining polynomial of the Borel classes are

$$b_1(U, u\psi)^n = b_1(E, u\varphi) b_1(U, u\psi)^{n-1} - \ldots - (-1)^n b_n(E, u\varphi).$$

and the same applies for (E, φ) (replacing $(U, u\psi)$ by (U, ψ)). We get

$$\langle u^n\rangle b_1(U, \psi)^n = \langle u^{n-1}\rangle b_1(E, u\varphi) b_1(U, \psi)^{n-1} - \ldots - (-1)^n b_n(E, u\varphi).$$

Multiplying by $\langle u^n\rangle$ on both sides and using $\langle u^{2i}\rangle = 1$ for $i \in \mathbb{N}$, we obtain the result. □

See also [HW18, Proposition 7.8] for the next corollary.

Corollary 2.0.11. *Let V be a vector bundle of rank r. Then, the odd Pontryagin classes of V are fixed under the action of $\mathcal{O}(X)^\times$.*

Proof. It suffices to observe that $\langle u\rangle H(V)\simeq H(V)$ in that case. □

Corollary 2.0.12. *Let V be a rank r vector bundle over HP^n for some $n\in\mathbb{N}$. Then, the odd Pontryagin classes of V are hyperbolic.*

Proof. Consider $\pi:\mathrm{HP}^n\times\mathbb{G}_m\to\mathrm{HP}^n$ and the pull-back of V to this scheme. By the fundamental property of the Borel classes, the pull-back of the Borel classes to this scheme are the Borel classes of the pull-back. It follows that

$$\langle t\rangle\pi^*p_{2i+1}(H(V))=\pi^*p_{2i+1}(H(V))$$

for each $i\in\mathbb{N}$, where t is a coordinate of $\mathbb{G}_m$. On the other hand,

$$\widetilde{\mathrm{CH}}^{4i+2}(\mathrm{HP}^n\times\mathbb{G}_m)=\widetilde{\mathrm{CH}}^0(\mathbb{G}_m)\pi^*b_1^{2i+1}$$

where b_i is the class of the tautological bundle on HP^n. Thus, $\pi^*b_{2i+1}(H(V))=\pi^*(\alpha)\cdot\pi^*b_1^{2i+1}$, with $\alpha\in\mathrm{GW}(k)$. It follows that $\langle t\rangle\alpha=\alpha$ and using the residue homomorphism associated to the t-adic valuation, we obtain that the class of α in $\mathrm{W}(k)$ is trivial. Therefore, α is hyperbolic. □

2.1. **The symplectic splitting principle.** One of the main properties of Sp-oriented theories is the splitting principle. For any smooth scheme X and any symplectic bundle (E,φ) of rank $2n$ over X, there exists a smooth scheme $p:Y\to X$ such that p is smooth (and affine) and the pull-back p^* is injective on the relevant cohomology theory. Moreover, the original bundle splits as

$$p^*E=(E_1,\varphi_1)\perp\ldots\perp(E_n,\varphi_n)$$

with $E_1,\ldots,E_n$ symplectic of rank 2.

In view of the Whitney formula, the Borel classes of (E,φ) are all expressed in terms of the first Borel classes ξ_i of (E_i,φ_i), which are called *Borel roots of* (E,φ). In general, it is difficult to find Borel roots explicitly but crucial for computations. In fact, most of our efforts in the sequel will be devoted to the computations of Borel roots of a threefold product of rank 2 symplectic bundles ([PW10c, Theorem 10.2]).

3. The threefold product

Let $n\geq 1$ be an integer and let HP^n be as in Section 1. We denote as usual the canonical bundle of HP^n by U and its symplectic form by φ. Consider the threefold product $(\mathrm{HP}^n)^{\times 3}$ and let (U_i,φ_i) be the pull-back of (U,φ) along the respective projections $\pi_i:(\mathrm{HP}^n)^{\times 3}\to\mathrm{HP}^n$. Our aim in this section is to compute the Borel classes of the product $(U_1,\varphi_1)\otimes(U_2,\varphi_2)\otimes(U_2,\varphi_2)$. We first recall that the Chow-Witt ring of $(\mathrm{HP}^n)^{\times 3}$ is of the form

$$\widetilde{\mathrm{CH}}^*((\mathrm{HP}^n)^{\times 3})=\widetilde{\mathrm{CH}}^*[u_1,u_2,u_3]/\langle u_1^{n+1},u_2^{n+1},u_3^{n+1}\rangle$$

where $u_i=b_1(U_i,\varphi_i)$ sits in degree 2. In the rest of the section, we assume that n is large enough so that we don't have to bother about the relations $u_i^{n+1}=0$. In fact, we may work in the Chow-Witt ring of $(\mathrm{HP}^\infty)^{\times 3}$ which is actually a power series in the variables u_1,u_2,u_3. Let now

$$(E,\psi):=(U_1,\varphi_1)\otimes(U_2,\varphi_2)\otimes(U_3,\varphi_3)$$

on $(\mathrm{HP}^n)^{\times 3}$. Our aim in this section is to compute the Borel classes of (E,ψ). We start with the Chow-valued Borel classes as a warm up.

3.1. **The Chow-valued Borel classes.** In this section, we compute the Chow-valued Borel classes of the threefold product (E,ψ) above. To avoid confusion with Borel classes in other cohomology theories, we denote by b_i^{CH} the Chow-valued Borel classes. We start by observing that we may split each U_i using the splitting principle for Chow groups. We find Chern roots of the form $\xi_i, -\xi_i$ for each bundle with the property that $\xi_i^2 = b_1^{\mathrm{CH}}(U_i) = -c_2(U_i)$. We then obtain Chern roots of the form $\pm\xi_1 \pm \xi_2 \pm \xi_3$ for E and an elementary (but cumbersome) computation yields Chern classes of the form

$$\begin{aligned}
b_1^{\mathrm{CH}}(E,\psi) &= 4(u_1+u_2+u_3)\\
b_2^{\mathrm{CH}}(E,\psi) &= 6(u_1^2+u_2^2+u_3^2)+4(u_1u_2+u_1u_3+u_2u_3)\\
b_3^{\mathrm{CH}}(E,\psi) &= 4(u_1^3+u_2^3+u_3^3)-4(u_1^2u_2+u_1^2u_3+u_2^2u_1+u_2^2u_3+u_3^2u_1+u_3^2u_2)+40u_1u_2u_3\\
b_4^{\mathrm{CH}}(E,\psi) &= u_1^4+u_2^4+u_3^4-4(u_1^3u_2+u_1^3u_3+u_2^3u_1+u_2^3u_3+u_3^3u_1+u_3^3u_2)+\\
&\quad +6(u_1^2u_2^2+u_1^2u_3^2+u_2^2u_3^2)+4(u_1^2u_2u_3+u_2^2u_1u_3+u_3^2u_1u_2).
\end{aligned}$$

3.2. **The Witt-valued Borel classes.** We pass to the computation of the Witt-valued Borel classes of (E,ψ) that we denote b_i^{W} (accordingly, we denote by p_i^{W} the Pontryagin classes). Luckily, we can rely on the following two theorems of Marc Levine ([Lev18, Theorem 7.1, Proposition 8.1]) which compute the Pontryagin classes of a tensor product of bundles. We know that the odd Pontryagin classes are trivial (Corollary 2.0.12) and we focus on the even classes. Before stating the results, we introduce the following notation. For any positive integer m, set

$$m!! = \prod_{i=0}^{[\frac{m}{2}]} m-2i.$$

Theorem 3.2.1. *Let $E \to X$ and $E' \to Y$ be rank two bundles on smooth schemes X and Y. Let $\pi_X : X\times Y \to X$ and $\pi_Y : X\times Y \to Y$ be the respective projections. Then the Pontryagin classes of $\pi_X^*E\otimes\pi_Y^*E'$ are as follows:*

$$\begin{aligned}
p_2^{\mathrm{W}}(\pi_X^*E\otimes\pi_Y^*E') &= -2(\pi_X^*e(E)^2+\pi_Y^*e(E')^2)\\
p_4^{\mathrm{W}}(\pi_X^*E\otimes\pi_Y^*E') &= (\pi_X^*e(E)^2-\pi_Y^*e(E')^2)^2.
\end{aligned}$$

while $p_{2n}^{\mathrm{W}}(\pi_X^*E\otimes\pi_Y^*E') = 0$ for $n \geq 3$. Here, $e(E)$ (and $e(E')$) are the Euler classes of respectively E and E'. They coincide with $b_1^{\mathrm{W}}(E)$ and $b_1^{\mathrm{W}}(E')$ in case E and E' are symplectic.

Remark 3.2.2. The reader may have observed that the sign of p_2 is different from the one in [Lev18, Proposition 8.1]. This is due to the fact that our definition of Pontryagin classes differ from the one used in *loc. cit.* by a sign in case of p_2.

Theorem 3.2.3. *Let X be a smooth scheme over a perfect field k and let $E \to X$ be a rank two bundle. Let m be a positive integer be such that $2m$ is prime to the residual characteristic of k. Then, we have*

$$e(\mathrm{Sym}^m E) = \begin{cases} m!!e(E)^{l+1} & \text{if } m = 2l+1.\\ 0 & \text{if } m = 2l.\end{cases}$$

In view of [Lev18, Remark 8.2], these two theorems are sufficient to compute characteristic classes of bundles in Witt cohomology. For now, we compute the first two Borel classes of

$$(E,\psi) := (U_1,\varphi_1)\otimes(U_2,\varphi_2)\otimes(U_3,\varphi_3).$$

Proposition 3.2.4. *We have* $b_1^{\mathrm{W}}(E,\psi)=0$ *and* $b_2^{\mathrm{W}}(E,\psi)=-2(u_1^2+u_2^2+u_3^2)$.

Proof. We start with the computation of b_1^{W}. The presentation of $\mathrm{H}^*((\mathrm{HP}^n)^{\times 3},\mathbf{W})$ shows that $b_1^{\mathrm{W}}(E,\psi)=\sum \pi_i^*(\alpha_i)$ for some $\alpha_i\in \mathrm{H}^2(\mathrm{HP}^n,\mathbf{W})$. We may find α_1 by choosing the base point x_0 on HP^n and pulling-back along $\mathrm{HP}^n\to(\mathrm{HP}^n)^{\times 3}$ given by $x\mapsto(x,x_0,x_0)$. The pull-back of (E,ψ) is of the form $(U_1,\varphi_1)\otimes(\mathcal{O}^2,h)\otimes(\mathcal{O}^2,h)$ which is hyperbolic. The result follows from Corollary 2.0.12.

Now, we can write

$$b_2^{\mathrm{W}}(E,\psi)=\sum_{1\leq i\leq j\leq 3}\alpha_{ij}u_iu_j$$

with $\alpha_{ij}\in \mathrm{W}(k)$. To identify α_{ii}, it is sufficient to consider the embeddings of the form

$$\mathrm{HP}^n\to(\mathrm{HP}^n)^{\times 3}$$

given by $x\mapsto(x,x_0,x_0)$ (or a permutation). It follows that we have to compute $b_2^{\mathrm{W}}((U,\varphi)\otimes(\mathcal{O}^4,h))$. Now, $(U,\varphi)\otimes(\mathcal{O}^4,h)$ is of the form

$$(U,\varphi)\oplus(U,\varphi)\oplus(U,-\varphi)\oplus(U,-\varphi)$$

and the Borel polynomial of (U,φ) is $(1+u_1t)$. Then, the Borel polynomial of $(U,\varphi)\otimes(\mathcal{O}^4,h)$ is

$$(1+u_1t)^2(1-u_1t)^2=(1-u_1^2t^2)^2$$

Thus $b_2=-2u_1^2$.

For α_{ij} with $i\neq j$, we consider embeddings of the form

$$(\mathrm{HP}^n)^{\times 2}\to(\mathrm{HP}^n)^{\times 3}$$

given by $(x,y)\mapsto(x,y,x_0)$ (or a permutation). The pull-back of $\sum_{1\leq i\leq j\leq 3}\alpha_{ij}u_uu_j$ along this morphism is

$$\alpha_{11}u_1^2+\alpha_{12}u_1u_2+\alpha_{22}u_2^2$$

with $\alpha_{11}=\alpha_{22}=-2$. Thus, it suffices to compute $b_2^{\mathrm{W}}((U_1,\varphi_1)\otimes(U_2,\varphi_2)\otimes(\mathcal{O}^2,h))$ to deduce the value of α_{12}. By definition, we have

$$b_2^{\mathrm{W}}((U_1,\varphi_1)\otimes(U_2,\varphi_2)\otimes(\mathcal{O}^2,h))=p_2^{\mathrm{W}}((U_1,\varphi_1)\otimes(U_2,\varphi_2))$$

and Theorem 3.2.1 shows that the class of α_{12} is trivial in the Witt group. □

The computation of the remaining two Borel classes will be more involved. We start with the computation of the Borel classes of $(U^{\otimes 3},\varphi^{\otimes 3})$ on HP^n. This requires to reduce to symmetric powers of U and a direct computation (using the fact that $6\in k^\times$) shows that:

(3.2.4.a) $$(U^{\otimes 3},\varphi^{\otimes 3})\simeq(U,\langle 2\rangle\varphi)\perp(U,\langle 6\rangle\varphi)\perp(\mathrm{Sym}^3U,\mathrm{Sym}^3\varphi).$$

Corollary 3.2.5. *The Borel classes of* $(\mathrm{Sym}^3U,\mathrm{Sym}^3\varphi)$ *are as follows:*

$$\begin{aligned}b_1^{\mathrm{W}}(\mathrm{Sym}^3U,\mathrm{Sym}^3\varphi)&=(-3+\langle 3\rangle)u\\ b_2^{\mathrm{W}}(\mathrm{Sym}^3U,\mathrm{Sym}^3\varphi)&=(-4+\langle 3\rangle)u^2\end{aligned}$$

while $b_i(\mathrm{Sym}^3U,\psi)=0$ *for* $i\geq 3$.

Proof. We use the decomposition (3.2.4.a) above. Let α et β be Borel roots of $(\mathrm{Sym}^3 U, \psi)$. Corollary 2.0.10 yields $b_1^{\mathrm{W}}(U, \langle 2\rangle\varphi) = \langle 2\rangle b_1^{\mathrm{W}}(U,\varphi)$ and $b_1^{\mathrm{W}}(U, \langle 6\rangle\varphi) = \langle 6\rangle b_1^{\mathrm{W}}(U,\varphi)$ and it follows that the Borel roots of $(U^{\otimes 3}, \varphi^{\otimes 3})$ are $\langle 2\rangle u, \langle 6\rangle u, \alpha, \beta$. The Borel polynomial is of the form

$$b_t^{\mathrm{W}}(U^{\otimes 3}, \varphi^{\otimes 3}) = (1+\langle 2\rangle ut)(1+\langle 6\rangle ut)(1+\alpha t)(1+\beta t)$$

Its term of degrees 1 and 2 are respectively given by the previous proposition (using the diagonal embedding $\mathrm{H}\mathbb{P}^n \to (\mathrm{H}\mathbb{P}^n)^{\times 3}$) and we therefore obtain equalities

$$\begin{aligned} \alpha+\beta+\langle 2\rangle u + \langle 6\rangle u &= 0 \\ \alpha\beta + (\alpha+\beta)(\langle 2\rangle+\langle 6\rangle)u + \langle 3\rangle u^2 &= -6u^2. \end{aligned}$$

A simple computation yields $b_1^{\mathrm{W}} = (\langle -2\rangle + \langle -6\rangle)u$ and $b_2^{\mathrm{W}} = (-4+\langle 3\rangle)u^2$. To conclude, it suffices to show that

$$\langle -2\rangle + \langle -6\rangle = -3 + \langle 3\rangle$$

It suffices to prove it either for a finite field (if k is of positive characteristic), either for $\mathbb{Q}$ in case k is of characteristic zero. The first case is obvious as both forms have the same rank and same discriminant, the second case is obtained via a comparison of residues.

The formula for $b_i^{\mathrm{W}}(\mathrm{Sym}^3 U, \psi) = 0$ for $i \geq 3$ is obvious since $\mathrm{Sym}^3 U$ is of rank 4. □

Corollary 3.2.6. *The Borel classes of $(U^{\otimes 3}, \varphi^{\otimes 3})$ are as follows:*

$$\begin{aligned} b_1^{\mathrm{W}}(U^{\otimes 3}, \varphi^{\otimes 3}) &= 0 \\ b_2^{\mathrm{W}}(U^{\otimes 3}, \varphi^{\otimes 3}) &= -6u^2 \\ b_3^{\mathrm{W}}(U^{\otimes 3}, \varphi^{\otimes 3}) &= -8u^3 \\ b_4^{\mathrm{W}}(U^{\otimes 3}, \varphi^{\otimes 3}) &= -3u^4 \end{aligned}$$

while $b_i^{\mathrm{W}}(U^{\otimes 3}, \varphi^{\otimes 3}) = 0$ for $i \geq 5$.

Proof. We can use the decomposition

$$(U^{\otimes 3}, \varphi^{\otimes 3}) \simeq (U, \langle 2\rangle\varphi) \perp (U, \langle 6\rangle\varphi) \perp (\mathrm{Sym}^3 U, \mathrm{Sym}^3\varphi)$$

above, as well as the complete knowledge of the Borel classes of the right-hand terms. The Borel polynomials of the first two terms are $(1+\langle 2\rangle ut)$ and $(1+\langle 6\rangle ut)$. The Borel polynomial of the last term is

$$(1+(-3+\langle 3\rangle)ut + (-4+\langle 3\rangle)ut^2)$$

by the above corollary. We then compute the Borel polynomial and find the relevant terms using $4\langle 3\rangle = 4$. □

Proposition 3.2.7. *Let $(E,\psi) := (U_1,\varphi_1)\otimes(U_2,\varphi_2)\otimes(U_3,\varphi_3)$ on $(\mathrm{H}\mathbb{P}^n)^{\times 3}$. Then,*

$$b_3^{\mathrm{W}}(E,\psi) = -8u_1u_2u_3$$

Proof. We may write

$$b_3^{\mathrm{W}}(E,\psi) = \alpha\sum_{i=1}^{3} u_i^3 + \beta \sum_{1\leq i\neq j\leq 3}^{3} u_i^2 u_j + \gamma u_1u_2u_3$$

for some coefficients in the Witt group $\mathrm{W}(k)$. To determine α, we again pull-back along the morphism

$$\mathrm{H}\mathbb{P}^n \to (\mathrm{H}\mathbb{P}^n)^{\times 3}$$

given by $x \mapsto (x, x_0, x_0)$. The relevant bundle is hyperbolic, and we conclude using Corollary 2.0.12. For β, we pull-back along

$$(\mathrm{HP}^n)^{\times 2} \to (\mathrm{HP}^n)^{\times 3}$$

given by $(x, y) \mapsto (x, y, x_0)$ and we compute the third Borel class of $(U_1, \varphi_1) \otimes (U_2, \varphi_2) \otimes (\mathcal{O}^2, h)$ and the same argument as above shows that $\beta = 0$ as well. To find γ, we pull-back along the diagonal morphism

$$\mathrm{HP}^n \to (\mathrm{HP}^n)^{\times 3}$$

By Corollary 3.2.6, we have

$$3\alpha + 6\beta + \gamma = -8$$

and the result follows. □

Proposition 3.2.8. *Let* $(E, \psi) := (U_1, \varphi_1) \otimes (U_2, \varphi_2) \otimes (U_3, \varphi_3)$ *on* $(\mathrm{HP}^n)^{\times 3}$. *Then,*

$$b_4^{\mathrm{W}}(E, \psi) = \sum_{i=1}^{3} u_i^4 - 2\Big(\sum_{1 \le i < j \le 3} u_i^2 u_j^2\Big)$$

Proof. As before, we write

$$\begin{aligned} b_4^{\mathrm{W}}(E, \psi) \;=\; & \alpha\Big(\sum_{i=1}^{3} u_i^4\Big) + \beta\Big(\sum_{1 \le i \neq j \le 3}^{3} u_i^3 u_j\Big) + \gamma\Big(\sum_{1 \le i < j \le 3} u_i^2 u_j^2\Big) + \\ & + \delta(u_1^2 u_2 u_3 + u_1 u_2^2 u_3 + u_1 u_2 u_3^2) \end{aligned}$$

To find α, we can as usual pull-back along the morphism

$$\mathrm{HP}^n \to (\mathrm{HP}^n)^{\times 3}$$

given by $x \mapsto (x, x_0, x_0)$. The relevant bundle is of the form $(U, \varphi) \otimes (\mathcal{O}^4, h)$ and we can use Theorem 3.2.1 to find $\alpha = 1$.

Next, we use the embedding

$$(\mathrm{HP}^n)^{\times 2} \to (\mathrm{HP}^n)^{\times 3}$$

given by $(x, y) \mapsto (x, y, x_0)$. The pull-back of our expression is

$$u_1^4 + u_2^4 + \beta(u_1^3 u_2 + u_1 u_2^3) + \gamma u_1^2 u_2^2$$

On the other hand, the pull-back of (E, ψ) is $(U_1, \varphi_1) \otimes (U_2, \varphi_2) \otimes (\mathcal{O}^2, h)$ and the fourth Borel class is the fourth Pontryagin class of $U_1 \otimes U_2$. We deduce from Theorem 3.2.1 that $\gamma = -2$ and $\beta = 0$.

To conclude, we use the diagonal embedding. Using Corollary 3.2.6, we find that

$$3 - 6 + 3\delta = -3$$

i.e. that $\delta = 0$. □

3.3. **The Borel classes.** We can now assemble the pieces obtained in Sections 3.1 and 3.2 to obtain the Borel classes using Corollary 2.0.5 and Lemma 2.0.6.

Theorem 3.3.1. *Let $n \geq 5$ and let*

$$(E,\psi) := (U_1,\varphi_1) \otimes (U_2,\varphi_2) \otimes (U_3,\varphi_3)$$

be the threefold product of tautological bundles on $X = (\mathrm{H}\mathbb{P}^n)^{\times 3}$. Let $u_1, u_2, u_3 \in \widetilde{\mathrm{CH}}^2(X)$ be the respective Euler classes of the bundles and let $h := \langle 1, -1\rangle$. Then, we have

$$\begin{aligned}
b_1(E,\psi) &= 2h(u_1+u_2+u_3).\\
b_2(E,\psi) &= (2\langle -1\rangle + 2h)(u_1^2+u_2^2+u_3^2) + 2h(u_1u_2+u_1u_3+u_2u_3).\\
b_3(E,\psi) &= 2h(u_1^3+u_2^3+u_3^3) - 2h(u_1^2u_2+u_1^2u_3+u_2^2u_1+u_2^2u_3+u_3^2u_1+u_3^2u_2)+\\
&\quad +(8\langle -1\rangle + 16h)u_1u_2u_3.\\
b_4(E,\psi) &= u_1^4+u_2^4+u_3^4 - 2h(u_1^3u_2+u_1^3u_3+u_2^3u_1+u_2^3u_3+u_3^3u_1+u_3^3u_2)+\\
&\quad +(2\langle -1\rangle + 2h)(u_1^2u_2^2+u_1^2u_3^2+u_2^2u_3^2) + 2h(u_1^2u_2u_3+u_2^2u_1u_3+u_3^2u_1u_2).
\end{aligned}$$

4. Stable operations

In this section, we build morphisms of spectra

$$\mathrm{B}_{2n} : \mathbf{HGr} \to \mathbf{H}_{\mathrm{MW}}\langle 2+4n\rangle.$$

for any $n \in \mathbb{N}$. We start with the description of the possible morphisms

$$\mathrm{HGr} \to \mathrm{K}(\tilde{\mathbb{Z}}(4n+2), 8n+4)$$

in degree n of our spectra. The following theorem is analogous to [Rio10, Theorem 6.2.1.2]:

Theorem 4.0.1. *Let k be a perfect field and let $n \geq 0$. Then, the functor π_0 induces a bijection*

$$[\mathbb{Z}\times\mathrm{HGr}, \mathrm{K}(\tilde{\mathbb{Z}}(2n), 4n))]_{\mathcal{H}(k)} \simeq \mathrm{Hom}_{Fun((Sm/S)^{op},Sets)}(\mathrm{GW}^2(\cdot), \widetilde{\mathrm{CH}}^{2n}(\cdot)) \tag{4.0.1.a}$$

Moreover, the right-hand term is isomorphic to the graded piece of degree $2n$ in the polynomial algebra $\mathrm{GW}(k)[b_1, b_2, \ldots]$ where b_i is of degree i.

Proof. The first claim follows from Proposition 4.0.4 and Proposition 4.0.5 below, and the second claim follows from [PW10c, Theorem 11.4]. □

Definition 4.0.2. Let S be a noetherian scheme and let E be a group object in $\mathcal{H}(S)$. For $X \in Sm/S$ and $n \in \mathbb{N}$, let $\pi_n E \in Fun((Sm/S)^{op}, Sets)$ be the presheaf $X \mapsto [X, R\Omega^n E]_{\mathcal{H}(S)}$. We say that E satisfies property (HK) if the following vanishing condition holds:

$$R^1 \lim_{(r,d)\in\mathbb{N}^2} \pi_1 E(\mathrm{HGr}(r,d)) = 0. \tag{4.0.2.a}$$

The following lemma follows from [PW10b, Theorem 8.1]:

Lemma 4.0.3. *Let S be a regular noetherian scheme of finite Krull dimension. If $\mathcal{X} = \mathbb{Z}\times\mathrm{HGr}$ as a presheaf in $Fun((Sm/S)^{op}, Sets)$, then for any affine scheme $U \in Sm/S$, the following canonical map is surjective:*

$$\mathcal{X}(U) \to \pi_0\mathcal{X}(U). \tag{4.0.3.a}$$

Proposition 4.0.4. *Let S be a regular noetherian scheme of finite Krull dimension and let E be a group object in $\mathcal{H}_\bullet(S)$ which satisfies property (HK). Then the following canonical map is an isomorphism:*

$$[\mathbb{Z} \times \mathrm{HGr}, E]_{\mathcal{H}(S)} \xrightarrow{\sim} Hom_{Fun((Sm/S)^{op},Sets)}(\mathrm{GW}^2(\cdot), \pi_0 E). \tag{4.0.4.a}$$

This set of morphisms can also be identified with an infinite product indexed by $\mathbb{Z}$ of copies of the limit $\lim_{(r,d)\in\mathbb{N}^2}[\mathrm{HGr}_{r,d}, E]_{\mathcal{H}(S)}$.

Proof. Follows from Lemma 4.0.3, [PW10b, Theorem 8.2] and [Rio10, Proposition 1.2.9]. □

Proposition 4.0.5. *Let k be a perfect field and let $n \geqslant 0$. Then the object $\mathrm{K}(\tilde{\mathbb{Z}}(2n), 4n)$ in $\mathcal{H}_\bullet(S)$, as well as its loop spaces, satisfy property (HK).*

Proof. It suffices to show that for all $r' \geqslant r$, $d' \geqslant d + r' - r$ and $i \geqslant 0$, the following canonical map is surjective:

$$H^{4n-i}_{MW}(\mathrm{HGr}(r', d'), \tilde{\mathbb{Z}}(2n)) \to H^{4n-i}_{MW}(\mathrm{HGr}(r, d), \tilde{\mathbb{Z}}(2n)). \tag{4.0.5.a}$$

This follows from [PW10c, Theorem 11.4] and the fact that Milnor-Witt motivic cohomology is Sp-oriented. □

For additive morphisms, the analogue of [Rio10, Proposition 6.2.2.1] reads as follows:

Proposition 4.0.6. *For any $n \geq 0$, the map given by evaluation at $[U, \varphi]$ in $\mathrm{GW}^2(\mathrm{H}\mathbb{P}^n)$ induces an isomorphism of $\mathrm{GW}(k)$-modules*

$$\begin{aligned} \mathrm{Hom}_{Fun((Sm/S)^{op},Ab)}(\mathrm{GW}^2(\cdot), \widetilde{\mathrm{CH}}^{2n}(\cdot)) &\xrightarrow{\sim} \lim_{r\in\mathbb{N}} \widetilde{\mathrm{CH}}^{2n}(\mathrm{H}\mathbb{P}^r) \\ &\simeq \widetilde{\mathrm{CH}}^{2n}(\mathrm{H}\mathbb{P}^n) \simeq \mathrm{GW}(k). \end{aligned} \tag{4.0.6.a}$$

Proof. The injectivity follows from the symplectic splitting principle in [PW10c, Theorem 10.2] (recalled in Section 2.1 above). For the surjectivity, the result is clear for $n = 0$. For $n \geqslant 1$, by [Rio10, Lemma 6.2.2.2], there exists a unique group homomorphism

$$\chi_n : (1 + A[[t]]^+, \times) \to (A, +) \tag{4.0.6.b}$$

functorial for all commutative rings A (where $1 + A[[t]]^+$ is the set of formal series with coefficients in A with constant term 1 endowed with the multiplication of formal series), such that for any $x \in A$ we have

$$\chi_n(1 + xt) = x^n \tag{4.0.6.c}$$

and χ_n vanishes on the subgroup $1 + t^{n+1}A[[t]]$. For $X \in Sm/k$ and (E, ψ) a symplectic bundle over X, consider the Borel polynomial

$$b_t(E, \psi) := \sum_{i\geqslant 0} b_i(E, \psi)t^i \in \widetilde{\mathrm{CH}}^{2*}(X)[[t]]. \tag{4.0.6.d}$$

Denote $\tilde{\chi}_{2n}(E, \psi) := \chi_n(b_t(E, \psi)) \in \widetilde{\mathrm{CH}}^{2n}(X)$. Then the map $(E, \psi) \mapsto \tilde{\chi}_{2n}(E, \psi)$ defines a natural transformation $\mathrm{GW}^2(\cdot) \to \widetilde{\mathrm{CH}}^{2n}(\cdot)$ as required. □

Definition 4.0.7. We denote by $\tilde{\chi}_{2n}$ the generator of the left-hand side in the isomorphism (4.0.6.a), as constructed explicitly in the proof above. It is characterized by the fact that $\tilde{\chi}_{2n}(E, \psi) = b_1(E, \psi)^n$ for any rank 2 symplectic bundle over X.

There is a more useful characterization of $\tilde{\chi}_{2n}$. Indeed, a rank 2 symplectic bundle over X corresponds to (the homotopy class of) a map $X \to \mathrm{HP}^m$ for some m large enough. Thus, the operation $\tilde{\chi}_{2n}$ is characterized by the fact that $\tilde{\chi}_{2n}(U,\psi) = b_1(U,\psi)^n$ for the universal bundle (U,ψ) on HP^m for $m \geq n+1$.

As in [Rio10, Remark 6.2.2.3], we may observe that the operation $\tilde{\chi}_{2n}$ can be computed inductively using the Newton relations

$$\tilde{\chi}_{2n} - b_1\tilde{\chi}_{2n-2} + b_2\tilde{\chi}_{2n-4} + \ldots + (-1)^{n-1}b_{n-1}\tilde{\chi}_2 + (-1)^n n b_n = 0. \tag{4.0.7.a}$$

We have

$$\tilde{\chi_2} = b_1,\ \tilde{\chi_4} = b_1^2 - 2b_2,\ \tilde{\chi_6} = b_1^3 - 3b_1b_2 + 3b_3.$$

The analogue of [Rio10, Corollary 6.2.2.4] holds in this context.

Now, suppose that we have for $n \geq 0$ an additive transformation

$$\tau : \mathrm{HGr} \to \mathrm{K}(\tilde{\mathbb{Z}}(2n+4), 4n+8).$$

In view of the diagram

$$\begin{array}{ccc} \mathrm{HGr} & \longrightarrow & \Omega^2_S\,\mathrm{HGr} \\ \downarrow & & \downarrow \\ \mathrm{K}(\tilde{\mathbb{Z}}(2n), 4n) & \longrightarrow & \Omega^2_S\,\mathrm{K}(\tilde{\mathbb{Z}}(2n+4), 4n+8) \end{array}$$

in which the horizontal arrows are weak-equivalences, the operation τ induces an operation $\Omega^2_S\tau$ on the left-hand side. It is actually characterized on smooth schemes by the following diagram.

Definition 4.0.8. Let $n \geq 0$ and let $\tau : \mathrm{GW}^2(\_) \to \widetilde{\mathrm{CH}}^{2n+4}(\_)$ be an additive transformation. We define an additive transformation $\Omega^2_S(\tau)$ by the following commutative diagram

$$\begin{array}{ccc} \mathrm{GW}^2(X) & \xrightarrow{(U_1-H)(U_2-H)} & \mathrm{GW}^2(S^{\wedge 2} \wedge X_+) \\ {\scriptstyle \Omega^2_S(\tau)}\downarrow & & \downarrow{\scriptstyle \tau} \\ \widetilde{\mathrm{CH}}^{2n}(X) & \xrightarrow[u_1u_2]{} & \widetilde{\mathrm{CH}}^{2n+4}(S^{\wedge 2} \wedge X_+) \end{array}$$

where $u = (U,\varphi)$ is the universal bundle on HP^1, U_i are the respective factors with first Borel classes u_i and $U_i - H$ is the difference between U_i and the hyperbolic form H.

Since $\tilde{\chi}_{2n+4}$ is a generator of operations of weight $2n+4$, we are mainly interested in computing $\Omega^2_S(\tilde{\chi}_{2n+4})$. As observed above, it suffices to do it for $X = \mathrm{HP}^m$ with m large enough. In that case, it is enough to consider transformations either in the cohomology of the Witt sheaf or in the Chow ring (in view of Corollary 2.0.5 and Lemma 2.0.6).

4.1. **The Witt-valued operation.** In this section, we compute $\Omega^2_S(\tilde{\chi}^{\mathrm{W}}_{2n+4})$, where $\tilde{\chi}^{\mathrm{W}}_{2n+4}$ is the additive operation obtained via the Witt-valued Borel classes. More precisely, we compute the additive transformation determined by the diagram

$$\begin{array}{ccc} \mathrm{GW}^2(\mathrm{H}\mathbb{P}^m) & \xrightarrow{(U_1-H)(U_2-H)} & \mathrm{GW}^2(S^{\wedge 2}\wedge \mathrm{H}\mathbb{P}^m_+) \\ \downarrow{\scriptstyle \Omega^4_{\mathbb{P}^1}(\tilde{\chi}^{\mathrm{W}}_{2n+4})} & & \downarrow{\scriptstyle \tilde{\chi}^{\mathrm{W}}_{2n+4}} \\ \mathrm{H}^{2n}(\mathrm{H}\mathbb{P}^m, \mathbf{W}) & \xrightarrow[u_1u_2]{} & \mathrm{H}^{2n+4}(S^{\wedge 2}\wedge \mathrm{H}\mathbb{P}^m_+, \mathbf{W}) \end{array}$$

Let us start with a lemma recapitulating the values of the Witt-valued Borel classes of the product $U_1U_2U_3$ over $\mathrm{H}\mathbb{P}^1\times \mathrm{H}\mathbb{P}^1\times \mathrm{H}\mathbb{P}^m$ (for m large enough). The result is an immediate consequence of Theorem 3.3.1, using $h=u_1^2=u_2^2=0$.

Lemma 4.1.1. *The Witt-valued Borel classes of $U_1U_2U_3$ over $\mathrm{H}\mathbb{P}^1\times \mathrm{H}\mathbb{P}^1\times \mathrm{H}\mathbb{P}^m$*

$$\begin{aligned} b_1^{\mathrm{W}}(U_1U_2U_3) &= 0. \\ b_2^{\mathrm{W}}(U_1U_2U_3) &= -2u_3^2. \\ b_3^{\mathrm{W}}(U_1U_2U_3) &= -8u_1u_2u_3. \\ b_4^{\mathrm{W}}(U_1U_2U_3) &= u_3^4. \end{aligned}$$

and $b_i(U_1U_2U_3)=0$ for $i\geq 5$.

We now set $u_3=u$ for the rest of the section.

Lemma 4.1.2. *We have*

$$\tilde{\chi}^{\mathrm{W}}_{2n+4}(U_1U_2U_3)=\begin{cases} 4u^{n+2} & \text{if } n \text{ is even} \\ \alpha_n u_1u_2u^n & \text{if } n \text{ is odd}\end{cases}$$

for some $\alpha_n\in W(k)$. Moreover, the series $(\alpha_n)\in W(k)$ satisfies

$$\alpha_n=\begin{cases} -24 & \text{if } n=1. \\ -80 & \text{if } n=3 \\ 2\alpha_{n-2}-\alpha_{n-4}-32 & \text{if } n\geq 5 \text{ is odd.}\end{cases}$$

Proof. We proceed by induction on $n\in\mathbb{N}$ and omit to write $U_1U_2U_3$ in the argument. A straightforward computation using (4.0.7.a) and the above lemma shows that

$$\tilde{\chi}^{\mathrm{W}}_{2n+4}=\begin{cases} 4u^2 & \text{if } n=0. \\ -24u_1u_2u & \text{if } n=1. \\ 4u^4 & \text{if } n=2. \\ -80u_1u_2u^3 & \text{if } n=3\end{cases}$$

We now prove the result by induction on n, using the relation for $n\geq 5$

$$\tilde{\chi}^{\mathrm{W}}_{2n+4}=-b_2^{\mathrm{W}}\tilde{\chi}^{\mathrm{W}}_{2n}+b_3^{\mathrm{W}}\tilde{\chi}^{\mathrm{W}}_{2n-2}-b_4^{\mathrm{W}}\tilde{\chi}^{\mathrm{W}}_{2n-4}=2u^2\tilde{\chi}^{\mathrm{W}}_{2n}-8u_1u_2u\tilde{\chi}^{\mathrm{W}}_{2n-2}-u^4\tilde{\chi}^{\mathrm{W}}_{2n-4}.$$

Suppose first that n is even. In that case, we have $\tilde{\chi}^{\mathrm{W}}_{2n}=\tilde{\chi}^{\mathrm{W}}_{2(n-2)+4}=4u^n$ and similarly $\tilde{\chi}^{\mathrm{W}}_{2n-4}=4u^{n-2}$. On the other hand, $\tilde{\chi}^{\mathrm{W}}_{2n-2}$ is a multiple of $u_1u_2u^{n-3}$ and the result now follows using $u_1^2=0$.

Suppose next that n is odd. In that case $\tilde{\chi}^{\mathrm{W}}_{2n}=\tilde{\chi}^{\mathrm{W}}_{2(n-2)+4}=\alpha_{n-2}u_1u_2u^{n-2}$ and the same computation yields $\tilde{\chi}^{\mathrm{W}}_{2n-4}=\alpha_{n-4}u_1u_2u^{n-4}$. Moreover, we have $\tilde{\chi}^{\mathrm{W}}_{2n-2}=4u^{n-3}$ and a simple computation yields the result. ∎

Expressing the sequence in terms of n, we find the following corollary.

Corollary 4.1.3. *We have*

$$\tilde{\chi}^{\mathrm{W}}_{2n+4}(U_1U_2U_3) = \begin{cases} 4u^{n+2} & \text{if } n \text{ is even.} \\ -4(n+2)(n+1)u_1u_2u^n & \text{if } n \text{ is odd.} \end{cases}$$

This finally allows to compute the operations $\Omega^4_{\mathbb{P}^1}(\tilde{\chi}_n)$ for any $n \in \mathbb{N}$.

Theorem 4.1.4. *The operations $\tilde{\chi}^{\mathrm{W}}_{2n+4}$ satisfy*

$$\Omega^2_S(\tilde{\chi}^{\mathrm{W}}_{2n+4}) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -4(n+2)(n+1)\tilde{\chi}^{\mathrm{W}}_{2n} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Contemplating the diagram

$$\begin{array}{ccc} \mathrm{GW}^2(\mathrm{H}\mathbb{P}^m) & \xrightarrow{(U_1-H)(U_2-H)} & \mathrm{GW}^2(S^{\wedge 2} \wedge \mathrm{H}\mathbb{P}^m_+) \\ {\scriptstyle \Omega^4_{\mathbb{P}^1}(\tilde{\chi}^{\mathrm{W}}_{2n+4})}\downarrow & & \downarrow{\scriptstyle \tilde{\chi}^{\mathrm{W}}_{2n+4}} \\ \mathrm{H}^{2n}(\mathrm{H}\mathbb{P}^m, \mathbf{W}) & \xrightarrow[u_1u_2]{} & \mathrm{H}^{2n+4}(S^{\wedge 2} \wedge \mathrm{H}\mathbb{P}^m_+, \mathbf{W}) \end{array}$$

we see that we have to compute

$$\tilde{\chi}^{\mathrm{W}}_{2n+4}(U_1U_2U_3) - \tilde{\chi}^{\mathrm{W}}_{2n+4}(U_1HU_3) - \tilde{\chi}^{\mathrm{W}}_{2n+4}(HU_2U_3) + \tilde{\chi}^{\mathrm{W}}_{2n+4}(H^2U_3).$$

We know the value of the first term by the above corollary. By naturality of the operations, the values of the other operations are obtained by pulling-back to the respective factors, which amounts to respectively setting $u_2 = 0$, $u_1 = 0$ and $u_1 = u_2 = 0$. The claim follows. □

4.2. **The Chow-valued operation.** In this section, we prove the analogue of Theorem 4.1.4 for Chow-valued operations. To this end, recall from [Rio10, Proposition 6.2.2.1] that there are operations

$$\chi_i : \mathrm{Gr} \to \mathrm{K}(\mathbb{Z}(i), 2i).$$

We still denote by χ_i the composite operation

$$\mathrm{HGr} \to \mathrm{Gr} \to \mathrm{K}(\mathbb{Z}(i), 2i).$$

On the other hand, we defined operations

$$\tilde{\chi}^{\mathrm{CH}}_{2n} : \mathrm{HGr} \to \mathrm{K}(\mathbb{Z}(2n), 4n)$$

using the Chow-valued Borel classes.

Lemma 4.2.1. *For any ≥ 1, we have*

$$2\tilde{\chi}^{\mathrm{CH}}_{2n} = \chi_{2n}.$$

Proof. We first observe that by definition of the Borel classes, we have $b_1^{\mathrm{CH}}(U) = -c_2(U)$ if U is a symplectic bundle of rank 2. The splitting principle then shows that $b_i^{\mathrm{CH}}(E) = (-1)^i c_{2i}(E)$ for any symplectic bundle E and that $c_{2i+1}(E) = 0$ for any $i \in \mathbb{N}$. Now, we prove the result by induction on n. For $n = 1$, we get $\tilde{\chi}^{\mathrm{CH}}_2(E) = b_1^{\mathrm{CH}}(E) = -c_2(E)$ while on the other hand

$$\chi_2(U) = c_1^2(U) - 2c_2(U) = -2c_2(U).$$

Suppose then that the result is proved for $n \geq 1$. We have

$$\begin{aligned} \chi_{2n+2} &= c_1\chi_{2n+1} - c_2\chi_{2n} + \ldots + c_{2n+1}\chi_1 - (2n+2)c_{2n+2} \\ &= -(\sum_{i=1}^{n} c_{2i}\chi_{2n-2i+2}) - (2n+2)c_{2n+2} \\ &= -2(\sum_{i=1}^{n}(-1)^i b_i^{\mathrm{CH}}\tilde{\chi}^{\mathrm{CH}}_{2n-2i+2}) - (2n+2)(-1)^{n+1}b^{\mathrm{CH}}_{n+1} \\ &= -2(\sum_{i=1}^{n}(-1)^i b_i^{\mathrm{CH}}\tilde{\chi}^{\mathrm{CH}}_{2n-2i+2} + (-1)^{n+1}(n+1)b^{\mathrm{CH}}_{n+1}) \\ &= 2\tilde{\chi}^{\mathrm{CH}}_{2n+2}. \end{aligned}$$

□

Corollary 4.2.2. *We have*

$$\Omega^2_S(\tilde{\chi}^{\mathrm{CH}}_{2n+4}) = 24\binom{2n+4}{4}\tilde{\chi}^{\mathrm{CH}}_{2n} = (2n+4)(2n+3)(2n+2)(2n+1)\tilde{\chi}^{\mathrm{CH}}_{2n}.$$

for any $n \in \mathbb{N}$.

Proof. We know from [Rio10, 6.2.3.2] that $\Omega_{\mathbb{P}^1}(\chi_{2n+4}) = (2n+4)\chi_{2n+3}$ as an operation on Gr, and a fortiori as an operation on HGr. The result now follows from Lemma 4.2.1, using the fact that $\tilde{\chi}^{\mathrm{CH}}_{2n}$ is a generator of the group of operation, which is isomorphic to $\mathbb{Z}$. □

Putting together the results of Theorem 4.1.4 and Corollary 4.2.2, we find the following result.

Theorem 4.2.3. *For any $n \geq 0$, the (Chow-Witt-valued) operation $\tilde{\chi}_{2n+4}$ satisfy*

$$\Omega^2_S(\tilde{\chi}_{2n+4}) = \begin{cases} 12\binom{2n+4}{4}\langle 1,-1\rangle\tilde{\chi}_{2n} & \text{if } n \text{ is even} \\ 4(n+2)(n+1)(\langle -1\rangle + (2n^2+4n+1)\langle 1,-1\rangle)\tilde{\chi}_{2n} & \text{if } n \text{ is odd.} \end{cases}$$

Definition 4.2.4. For n odd, we set

$$\psi_{2n+4} := 4(n+2)(n+1)(\langle -1\rangle + (2n^2+4n+1)\langle 1,-1\rangle),$$

so that $\Omega^2_S(\tilde{\chi}_{2n+4}) = \psi_{2n+4}\tilde{\chi}_{2n}$.

We observe that if the base field is of characteristic $p > 0$, then ψ_{2n+4} is hyperbolic and the formulas for $\Omega^2_S(\tilde{\chi}_{2n+4})$ are the same for any $n \in \mathbb{N}$. This follows easily from the fact that the Witt group of a finite field is 4-torsion.

In case k is of characteristic 0, then ψ_{2n+4} is defined over $\mathbb{Q}$ and has a non trivial signature and a non trivial rank. To define our operations, we'll have to formally invert ψ_{2n+4} (for any n odd) in $\mathrm{GW}(k)$. We explain this process in the next section.

4.3. **Inverting the relevant forms.** As explained above, we consider the localization of $\mathrm{GW}(k)$ with respect to the multiplicative system S_ψ generated by $\{\psi_{2n+4} | n \text{ odd}\}$.

Lemma 4.3.1. *Let P be an ordering of k and let $s_P : \mathrm{W}(k) \to \mathbb{Z}$ be the corresponding signature homomorphism, i.e. the homomorphism characterized by*

$$s_P(\langle a\rangle) = \begin{cases} 1 & \text{if } a \text{ is positive w.r.t. } P. \\ -1 & \text{if } a \text{ is negative w.r.t. } P. \end{cases}$$

Then $s_P(\psi_{2n+4}) = -8m(m+1) \neq 0$ for any $m \in \mathbb{N}$.

Proof. It suffices to observe that 1 is a square, and then is positive w.r.t. P. It follows that -1 is negative and we conclude. □

Proposition 4.3.2. *Let $\mathcal{P}$ be the set of orderings of k. Then,*

$$S_\psi^{-1}\mathrm{W}(k) \simeq \bigoplus_{P\in\mathcal{P}} \mathbb{Q}.$$

Proof. The above lemma shows that the map

$$\mathrm{W}(k) \xrightarrow{\sum s_P} \bigoplus_{P\in\mathcal{P}} \mathbb{Z}$$

induces a well-defined map as in the statement, i.e. we have a commutative diagram

$$\begin{array}{ccc} \mathrm{W}(k) & \xrightarrow{\sum s_P} & \bigoplus_{P\in\mathcal{P}} \mathbb{Z} \\ \downarrow & & \downarrow \\ S_\psi^{-1}\mathrm{W}(k) & \longrightarrow & \bigoplus_{P\in\mathcal{P}} \mathbb{Q}. \end{array}$$

Besides, the kernel and cokernel of the top homomorphism are 2-primary torsion. It follows immediately that the bottom map is surjective. Let now y be in the kernel of

$$S_\psi^{-1}\mathrm{W}(k) \xrightarrow{\sum s_P} \bigoplus_{P\in\mathcal{P}} \mathbb{Q},$$

We may write $y = \frac{x}{s}$ with $s \in S_\psi$ and $x \in W(k)$ and then $\sum s_P(x)(\sum s_P(s))^{-1} = 0$ showing that $\sum s_P(x) = 0$. It follows that $2^r x = 0$ for some $r \in \mathbb{N}$. Setting $m = 2^r - 1$, we have $\psi_{2m+1} = -8m(m+1) = -8m2^r$ and $\psi_{2m+1} \cdot x = 0$, showing that the map is injective. □

Corollary 4.3.3. *The signature and rank homomorphisms induce an isomorphism*

$$S_\psi^{-1}\mathrm{GW}(k) \simeq \mathbb{Q} \oplus \bigoplus_{P\in\mathcal{P}} \mathbb{Q} \simeq \mathrm{GW}(k) \otimes \mathbb{Q}$$

Proof. We have an exact sequence of $\mathrm{GW}(k)$-modules

$$0 \to \mathrm{GW}(k) \to \mathbb{Z} \oplus \mathrm{W}(k) \to \mathrm{W}(k)/2 \to 0$$

Localization being exact, we deduce an exact sequence of $S_\psi^{-1}\mathrm{GW}(k)$-modules. The above proposition shows that

$$S_\psi^{-1}\mathrm{GW}(k) \simeq S_\psi^{-1}\mathbb{Z} \oplus \bigoplus_{P\in\mathcal{P}} \mathbb{Q}$$

and the result follows from the fact that m divides the rank of ψ_{2m+1} for any $m \in \mathbb{N}$. □

4.4. **The stable operation.** We are now in position to define our stable operation. For each odd integer n, we have an operation

$$\frac{1}{\psi_{2n+4}\cdot\psi_{2n}\cdot\ldots\cdot\psi_6}\tilde{\chi}_{2n+4}:\mathrm{HGr}\to \mathrm{K}(\tilde{\mathbb{Q}}(2n+4),4n+8).$$

We define

$$\mathrm{B}_2:\mathrm{HGr}\to\mathrm{K}(\tilde{\mathbb{Q}}(2),4)$$

by $\mathrm{B}_2=b_1$ and for odd $n\geq 3$

$$\mathrm{B}_{2n}:\mathrm{HGr}\to\mathrm{K}(\tilde{\mathbb{Q}}(2n),4n)$$

by $\mathrm{B}_{2n}:=\frac{1}{\psi_{2n}\cdot\psi_{2n-4}\cdot\ldots\cdot\psi_6}\tilde{\chi}_{2n}$.

In view of Theorem 4.2.3, it follows that these operations induce a morphism of $S^{\wedge 2}$-spectra

$$\mathrm{B}_2:\mathbf{HGr}\to\mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 2\rangle.$$

Lemma 4.4.1. *We have a commutative diagram of spectra*

$$\begin{array}{ccc}\mathbf{HGr} & \xrightarrow{B_2} & \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 2\rangle\\ \downarrow & & \downarrow\\ \mathbf{KGL} & \xrightarrow[\mathrm{ch}_2]{} & \mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle 2\rangle\end{array}$$

in which the vertical arrows are the forgetful functors and the bottom arrow is the (shifted) Chern character (e.g. [Rio10, Definition 6.2.3.9]*).*

Proof. By Lemma 4.2.1, we know that $2\tilde{\chi}_{2n}^{\mathrm{CH}}=\chi_{2n}$ and in particular, we have $\tilde{\chi}_2^{\mathrm{CH}}=\frac{1}{2}\chi_2$. Now the composite

$$\mathbf{HGr}\xrightarrow{\mathrm{B}_1}\mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 2\rangle\to\mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle 2\rangle$$

is the one induced by the Chow-valued operations $\tilde{\chi}^{\mathrm{CH}}$ and the corresponding operations $\mathrm{B}_{2n}^{\mathrm{CH}}$. By Corollary 4.2.2, we have

$$\mathrm{B}_{2n}^{\mathrm{CH}}=\frac{2}{(2n)!}\tilde{\chi}_{2n}^{\mathrm{CH}}$$

and we can conclude using Lemma 4.2.1. □

Let now $\mathbf{S}$ be the $S^{\wedge^2}$-spectrum whose term in degree n is S^{2+4n} and whose transition maps are the obvious one. In other terms, $\mathbf{S}=\Sigma^\infty_{S^{\wedge 2}}S$. Using the fact that $\Omega^2_S\,\mathrm{HGr}\simeq\mathrm{HGr}$, we obtain a morphism of spectra

$$\eta\langle 2\rangle:\mathbf{S}\to\mathbf{HGr}$$

On the other hand, we also have a morphism of spectra

$$\eta^{\mathrm{MW}}\langle 2\rangle:\mathbf{S}\to\mathbf{H}_{\mathrm{MW}}\langle 2\rangle$$

inducing a morphism $\mathbf{S}\to\mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 2\rangle$ that we still denote by $\eta^{\mathrm{MW}}\langle 2\rangle$. The following result is essentially a reinterpretation of the fact that the diagrams

$$\begin{array}{ccc}\mathrm{GW}^2(\mathrm{H}\mathbb{P}^m) & \xrightarrow{(U_1-H)(U_2-H)} & \mathrm{GW}^2(S^{\wedge 2}\wedge\mathrm{H}\mathbb{P}^m_+)\\ \downarrow{\scriptstyle \mathrm{B}_{2n}} & & \downarrow{\scriptstyle \mathrm{B}_{2n+4}}\\ \widetilde{\mathrm{CH}}^{2n}(\mathrm{H}\mathbb{P}^m) & \xrightarrow[u_1u_2]{} & \widetilde{\mathrm{CH}}^{2n+4}(S^{\wedge 2}\wedge\mathrm{H}\mathbb{P}^m_+)\end{array}$$

commute for n odd, but we state it for further reference.

Lemma 4.4.2. *The diagram of $S^{\wedge 2}$-spectra*

$$\begin{array}{ccc} \mathbf{S} & \xrightarrow{\eta\langle 2\rangle} & \mathbf{HGr} \\ & \searrow^{\eta^{\mathrm{MW}}\langle 2\rangle} & \downarrow{\scriptstyle \mathrm{B}_2} \\ & & \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 2\rangle \end{array}$$

is commutative.

5. The Borel character

First of all, recall that the obvious forgetful functor induces an equivalence of categories between the homotopy category of S-spectra and $S^{\wedge 2}$-spectra. The morphisms of spectra we defined in the previous section therefore give morphism of S-spectra (up to homotopy) and we can consider the shifted version (here, we use $S \simeq (\mathbb{P}^1)^{\wedge 2}$ implicitly in the notation)

$$\mathrm{B}_2\langle -2\rangle : \mathbf{HGr}\langle -2\rangle \to \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})$$

Now, we have a canonical weak equivalence $\mathbf{KQ} \to \mathbf{HGr}\langle -2\rangle$ and we can consider the morphism of S-spectra

$$\mathrm{B} : \mathbf{KQ} \to \mathbf{H}_{\mathrm{MW}}(\mathbb{Q}).$$

In view of Lemmas 4.4.1 and 4.4.2, we have commutative diagrams

$$\begin{array}{ccc} \mathbf{KQ} & \xrightarrow{\mathrm{B}} & \mathbf{H}_{\mathrm{MW}}(\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathbf{KGL} & \xrightarrow[\mathrm{ch}]{} & \mathbf{H}_{\mathrm{M}}(\mathbb{Q}) \end{array}$$

and

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\eta} & \mathbf{KQ} \\ & \searrow^{\eta^{\mathrm{MW}}} & \downarrow{\scriptstyle \mathrm{B}} \\ & & \mathbf{H}_{\mathrm{MW}}(\mathbb{Q}) \end{array}$$

We can consider all these morphisms in the category $\mathrm{SH}(k)_{\mathbb{Q},-}$, in which the following result holds.

Proposition 5.0.1. *The morphism $\eta^{\mathrm{MW}}_{\mathbb{Q},-}$ in the diagram*

$$\begin{array}{ccc} \mathbb{1}_{\mathbb{Q},-} & \xrightarrow{\eta_{\mathbb{Q},-}} & \mathbf{KQ}_{\mathbb{Q},-} \\ & \searrow^{\eta^{\mathrm{MW}}_{\mathbb{Q},-}} & \downarrow{\scriptstyle \mathrm{B}_-} \\ & & \mathrm{KW}_{\mathbb{Q}} \end{array}$$

is an isomorphism.

Proof. We know from [ALP17, Theorem 3.7] that the homotopy sheaves of the left-hand side are concentrated on the diagonal, for which they coincide with the graded ring whose component is $\mathbf{W} \otimes \mathbb{Q}$ in each degree. Second, we prove that the bottom term has the same homotopy sheaves. To start with, recall from [DF16, Remark 5.0.3] that rationally we have a splitting $\mathbf{H}_{\mathrm{MW}}(\mathbb{Q}) = \mathbf{H}_{\mathrm{M}}(\mathbb{Q}) \times \mathrm{KW}_{\mathbb{Q}}$. On the other hand, we know that the homotopy sheaves of $\mathbf{H}_{\mathrm{MW}}(\mathbb{Q})$ coincide with those of $\mathbf{H}_{\mathrm{M}}(\mathbb{Q})$ outside the diagonal, and it follows that the rational homotopy sheaves of $\mathrm{KW}_{\mathbb{Q}}$ are equal to $\mathbf{W} \otimes \mathbb{Q}$ in the diagonal and are trivial else.

We can now conclude the proof by observing that the unit map induces an isomorphism on homotopy sheaves, which is straightforward. □

Using 4-periodicity of $\mathbf{KQ}$, we can twist the above morphisms to get morphisms

$$\mathrm{B}_{4i} : \mathbf{KQ} \to \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 4i \rangle$$

which fit in commutative diagrams similar as those outlined above. In particular, we obtain a morphism of spectra

$$\prod_{i\in\mathbb{Z}} \mathrm{B}_{4i} : \mathbf{KQ} \to \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 4i \rangle$$

On the other hand, we can consider the composite

$$\mathrm{ch}_2 : \mathbf{KQ} \to \mathbf{KGL} \to \mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle 2 \rangle$$

and we finally obtain a morphism of S-spectra

$$\prod_{i\in\mathbb{Z}} \mathrm{B}_{4i} \times \prod_{i\in\mathbb{Z}} \mathrm{ch}_{2+4\mathrm{i}} : \mathbf{KQ} \to \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 4i \rangle \times \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle 2+4i \rangle$$

Definition 5.0.2. We denote by bo the above morphism and call it the *Borel character.*

Recall that we have a commutative diagram

$$\begin{array}{ccc} \mathbf{KQ} & \xrightarrow{\mathrm{B}} & \mathbf{H}_{\mathrm{MW}}(\mathbb{Q}) \\ \downarrow & & \downarrow \\ \mathbf{KGL} & \xrightarrow[\mathrm{ch}]{} & \mathbf{H}_{\mathrm{M}}(\mathbb{Q}) \end{array}$$

and it follows that we get a commutative diagram

(5.0.2.a)
$$\begin{array}{ccc} \mathbf{KQ} & \xrightarrow{\mathrm{bo}} & \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 4i \rangle \times \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle 2+4i \rangle \\ \downarrow & & \downarrow \\ \mathbf{KGL} & \xrightarrow[\mathrm{ch}_t]{} & \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle i \rangle \end{array}$$

in which the vertical maps are the forgetful maps. Thus, the Borel character is in some sense a refinement of the Chern character.

Theorem 5.0.3. *The Borel character*

$$\mathrm{bo} : \mathbf{KQ} \to \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 4i \rangle \times \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle 2+4i \rangle$$

constructed above agrees with that of [DFJK19, 2.12]*. In particular, it is a morphism of ring spectra, which is rationally an isomorphism.*

Proof. We can decompose the Borel character as

$$\mathbf{KQ} \to \mathbf{KQ}_{\mathbb{Q}} \to \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{MW}}(\mathbb{Q})\langle 4i\rangle \times \prod_{i\in\mathbb{Z}} \mathbf{H}_{\mathrm{M}}(\mathbb{Q})\langle 2+4i\rangle$$

where the first arrow is the rationalization. We are reduced to show that the second morphism is an isomorphism of (rational) ring-spectra. Now, we can decompose the stable rational homotopy category in its plus and minus parts, and it suffices to prove the result in each factor. We start with the plus part, in which case the result follows from (5.0.2.a) and [DFJK19, Proposition 2.10]. For the minus part, the result follows from Proposition 5.0.1 and [DFJK19, Theorem 2.3]. □

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