MW-MOTIVIC COMPLEXES

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Abstract. The aim of this work is to develop a theory parallel to that of motivic complexes based on cycles and correspondences with coefficients in quadratic forms. This framework is closer to the point of view of $\mathbb{A}^1$-homotopy than the original one envisioned by Beilinson and set up by Voevodsky.

Contents

Introduction

The aim of this paper is to define the various categories of MW-motives built out of the category of finite Chow-Witt correspondences constructed in [CF14], and to study the motivic cohomology groups intrinsic to these categories. In Section 1, we start with a quick remainder of the basic properties of the category $\widetilde{\mathrm{Cor}}_k$. We then proceed with our first important result, namely that the sheaf (in either the Nisnevich or the étale topologies) associated to a MW-presheaf, i.e. an additive functor $\widetilde{\mathrm{Cor}}_k \to \mathrm{Ab}$, is a MW-sheaf. The method follows closely

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Voevodsky's method and relies on Lemma 1.2.6. We also discuss the monoidal structure on the category of MW-sheaves. In the second part of the paper, we prove the analogue for MW-sheaves of a famous theorem of Voevodsky saying that the sheaf (with transfers) associated to a homotopy invariant presheaf with transfers is strictly $\mathbb{A}^1$-invariant. Our method here is quite lazy. We heavily rely on the fact that an analogue theorem holds for quasi-stable sheaves with framed transfers by [GP15, Theorem 1.1]. Having this theorem at hand, it suffices to construct a functor from the category of linear framed presheaves to $\widetilde{\mathrm{Cor}}_k$ to prove the theorem. This functor is of independent interest and this is the reason why we take this shortcut. However, there is now a direct proof of this theorem due to H. A. Kolderup (still relying on ideas of Panin-Garkusha). In Section 3, we finally pass to the construction of the categories of MW-motives starting with a study of different model structures on the category of possibly unbounded complexes of MW-sheaves. The ideas here are closely related to [CD09b]. The category of effective motives $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$ (with coefficients in a ring R) is then defined as the category of $\mathbb{A}^1$-local objects in this category of complexes. Using the analogue of Voevodsky's theorem proved in Section 2, these objects are easily characterized by the fact that their homology sheaves are strictly $\mathbb{A}^1$-invariant. This allows as usual to give an explicit $\mathbb{A}^1$-localization functor, defined in terms of the Suslin (total) complex. The category of geometric objects is as in the classical case the subcategory of compact objects of $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$. Our next step is the formal inversion of the Tate motive in $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$ to obtain the stable category of MW-motives $\widetilde{\mathrm{DM}}(k,R)$ (with coefficients in R). We can then consider motivic cohomology as groups of extensions in this category, a point of view which allows to prove in Section 4 many basic property of this version of motivic cohomology, including a commutativity statement and a comparison theorem between motivic cohomology and Chow-Witt groups.

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Conventions

In all this work, we will fix a base field k assumed to be infinite perfect. All schemes considered will be assumed to be separated of finite type over k, unless explicitly stated.

We will fix a ring of coefficients R. We will also consider a Grothendieck topology t on the site of smooth k-schemes, which in practice will be either the Nisnevich of the étale topology. In section 3 and 4 we will restrict to these two latter cases.

1. MW-transfers on sheaves

1.1. **Reminder on MW-correspondences.**

1.1.1. We will use the definitions and constructions of [CF14].

In particular, for any smooth schemes X and Y (with Y connected of dimension d), we consider the following group of *finite* MW-*correspondences* from X to Y:

$$\tilde{\mathrm{c}}(X,Y) := \varinjlim_T \widetilde{\mathrm{CH}}^d_T\big(X \times Y, \omega_Y\big) \tag{1.1.1.a}$$

where T runs over the ordered set of reduced (but not necessarily irreducible) closed subschemes in $X \times Y$ whose projection to X is finite equidimensional and ω_Y is the pull back of the canonical sheaf of Y along the projection to the second factor. This definition is extended to the case where Y is non connected by additivity. When considering the coefficients ring R, we put:

$$\tilde{\mathrm{c}}(X,Y)_R := \tilde{\mathrm{c}}(X,Y) \otimes_{\mathbb{Z}} R.$$

In the sequel, we drop the index R from the notation when there is no possible confusion.

Because there is a natural morphism from Chow-Witt groups (twisted by any line bundle) to Chow groups, we get a canonical map:

$$\pi_{XY} : \tilde{\mathrm{c}}(X,Y) \to \mathrm{c}(X,Y) \tag{1.1.1.b}$$

for any smooth schemes X and Y, where the right hand side is the group of Voevodsky's finite correspondences which is compatible to the composition — see *loc. cit.* Remark 4.12. Let us recall the following result.

Lemma 1.1.2. *If $2 \in R^\times$, the induced map*

$$\pi_{XY} : \tilde{\mathrm{c}}(X,Y) \to \mathrm{c}(X,Y)$$

is a split epimorphism.

The lemma comes from the basic fact that the following composite map

$$\mathrm{K}^{\mathrm{M}}_n(F) \xrightarrow{(1)} \mathrm{K}^{\mathrm{MW}}_n(F,\mathcal{L}) \xrightarrow{(2)} \mathrm{K}^{\mathrm{M}}_n(F)$$

is multiplication by 2, where (1) is the map from Milnor K-theory of a field F to Milnor-Witt K-theory of F twisted by the 1-dimensional F-vector space $\mathcal{L}$ described in [CF14, §1] and (2) is the map killing η (see the discussion in *loc. cit.* after Definition 3.1).

Remark 1.1.3.
(1) In fact, a finite MW-correspondence amounts to a finite correspondence α together with a quadratic form over the function field of each irreducible component of the support of α satisfying some condition related with residues; see [CF14, Def. 4.6].
(2) Every finite MW-correspondence between smooth schemes X and Y has a well defined support ([CF14, Definition 4.6]). Roughly speaking, it is the minimal closed subset of $X \times Y$ on which the correspondence is defined.
(3) Recall that the Chow-Witt group in degree n of a smooth k-scheme X can be defined as the n-th Nisnevich cohomology group of the n-th unramified Milnor-Witt sheaf $\mathbf{K}^{\mathrm{MW}}_n$ (this cohomology being computed using an explicit flabby resolution of $\mathbf{K}^{\mathrm{MW}}_n$). This implies that the definition can be uniquely extended to the case where X is an essentially smooth k-scheme. Accordingly, one can extend the definition of finite MW-correspondences to the case of essentially smooth k-schemes using formula (1.1.1.a). The definition of composition obviously extends to that generalized setting. We will use that fact in the proof of Lemma 1.2.6.

(4) Consider the notations of the previous point. Assume that the essentially smooth k-scheme X is the projective limit of a projective system of essentially smooth k-schemes $(X_i)_{i\in I}$. Then the canonical map:

$$\Big(\varinjlim_{i\in I^{op}} \tilde{\mathrm{c}}(X_i,Y)\Big) \longrightarrow \tilde{\mathrm{c}}(X,Y)$$

is an isomorphism. This readily follows from formula (1.1.1.a) and the fact that Chow-Witt groups, as Nisnevich cohomology, commute with projective limits of schemes. See also [CF14, §5.1] for an extended discussion of these facts.

(5) For any smooth schemes X and Y, the group $\tilde{\mathrm{c}}(X,Y)$ is endowed with a structure of a left $\mathbf{K}_0^{\mathrm{MW}}(X)$-module and a right $\mathbf{K}_0^{\mathrm{MW}}(Y)$-module ([CF14, Example 4.10]). Pulling back along $X \to \operatorname{Spec} k$, it follows that $\tilde{\mathrm{c}}(X,Y)$ is a left $\mathbf{K}_0^{\mathrm{MW}}(k)$-module and it is readily verified that the category $\widetilde{\mathrm{Cor}}_k$ is in fact $\mathbf{K}_0^{\mathrm{MW}}(k)$-linear. Consequently, we can also consider $\mathbf{K}_0^{\mathrm{MW}}(k)$-algebras as coefficient rings.

1.1.4. Recall from *loc. cit.* that there is a composition product of MW-correspondences which is compatible with the projection map π_{XY}.

Definition 1.1.5. We denote by $\widetilde{\mathrm{Cor}}_k$ (resp. Cor_k) the additive category whose objects are smooth schemes and morphisms are finite MW-correspondences (resp. correspondences). If R is a ring, we let $\widetilde{\mathrm{Cor}}_{k,R}$ (resp. $\mathrm{Cor}_{k,R}$) be the category $\widetilde{\mathrm{Cor}}_k \otimes_{\mathbb{Z}} R$ (resp. $\mathrm{Cor}_k \otimes_{\mathbb{Z}} R$).

We denote by

$$\pi : \widetilde{\mathrm{Cor}}_k \to \mathrm{Cor}_k \tag{1.1.5.a}$$

the additive functor which is the identity on objects and the map π_{XY} on morphisms.

As a corollary of the above lemma, the induced functor

$$\pi : \widetilde{\mathrm{Cor}}_{k,R} \to \mathrm{Cor}_{k,R},$$

is full when $2 \in R^\times$. Note that the corresponding result without inverting 2 is wrong by [CF14, Remark 4.15].

1.1.6. The external product of finite MW-correspondences induces a symmetric monoidal structure on $\widetilde{\mathrm{Cor}}_k$ which on objects is given by the cartesian product of k-schemes. One can check that the functor π is symmetric monoidal, for the usual symmetric monoidal structure on the category Cor_k.

Finally, the graph of any morphism $f : X \to Y$ can be seen not only as a finite correspondence $\gamma(f)$ from X to Y but also as a finite MW-correspondence $\tilde{\gamma}(f)$ such that $\pi\tilde{\gamma}(f) = \gamma(f)$. One obtains in this way a canonical functor:

$$\tilde{\gamma} : \mathrm{Sm}_k \to \widetilde{\mathrm{Cor}}_k \tag{1.1.6.a}$$

which is faithful, symmetric monoidal, and such that $\pi \circ \tilde{\gamma} = \gamma$.

1.2. MW-**transfers.**

Definition 1.2.1. We let $\widetilde{\mathrm{PSh}}(k,R)$ (resp. $\mathrm{PSh}^{\mathrm{tr}}(k,R)$, resp. $\mathrm{PSh}(k,R)$) be the category of additive presheaves of R-modules on $\widetilde{\mathrm{Cor}}_k$ (resp. Cor_k, resp. Sm_k). Objects of $\widetilde{\mathrm{PSh}}(k,R)$ will be simply called MW-*presheaves*.

Definition 1.2.2. We denote by $\tilde{\mathrm{c}}_R(X)$ the representable presheaf $Y \mapsto \tilde{\mathrm{c}}(Y,X) \otimes_{\mathbb{Z}} R$. As usual, we also write $\tilde{\mathrm{c}}(X)$ in place of $\tilde{\mathrm{c}}_R(X)$ in case the context is clear.

The category of MW-presheaves is an abelian Grothendieck category.[1] It admits a unique symmetric monoidal structure such that the Yoneda embedding

$$\widetilde{\mathrm{Cor}}_k \to \widetilde{\mathrm{PSh}}(k,R),\ X \mapsto \tilde{\mathrm{c}}(X)$$

is symmetric monoidal (see e.g. [MVW06, Lecture 8]). From the functors (1.1.5.a) and (1.1.6.a), we derive as usual adjunctions of categories:

$$\mathrm{PSh}(k,R) \underset{\tilde{\gamma}_*}{\overset{\tilde{\gamma}^*}{\rightleftarrows}} \widetilde{\mathrm{PSh}}(k,R) \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathrm{PSh}^{\mathrm{tr}}(k,R)$$

such that $\tilde{\gamma}_*(F) = F \circ \tilde{\gamma}$, $\pi_*(F) = F \circ \pi$. The left adjoints $\tilde{\gamma}^*$ and π^* are easily described as follows. For a smooth scheme X, let $R(X)$ be the presheaf (of abelian groups) such that $R(X)(Y)$ is the free R-module generated by $\mathrm{Hom}(Y,X)$ for any smooth scheme Y. The Yoneda embedding yields $\mathrm{Hom}_{\mathrm{PSh}(k,R)}(R(X),F) = F(X)$ for any presheaf F, and in particular

$$\mathrm{Hom}_{\mathrm{PSh}(k,R)}(R(X),\tilde{\gamma}_*(F)) = F(X) = \mathrm{Hom}_{\widetilde{\mathrm{PSh}}(k,R)}(\tilde{\mathrm{c}}_R(X),F)$$

for any $F \in \widetilde{\mathrm{PSh}}(k,R)$. We can thus set $\tilde{\gamma}^*(R(X)) = \tilde{\mathrm{c}}_R(X)$. On the other hand, suppose that

$$F_1 \to F_2 \to F_3 \to 0$$

is an exact sequence in $\mathrm{PSh}(k,R)$. The functor $\mathrm{Hom}_{\mathrm{PSh}(k,R)}(\_,F)$ being left exact for any $F \in \mathrm{PSh}(k,R)$, we find an exact sequence of presheaves

$$0 \to \mathrm{Hom}_{\mathrm{PSh}(k,R)}(F_3,\tilde{\gamma}_*(G)) \to \mathrm{Hom}_{\mathrm{PSh}(k,R)}(F_2,\tilde{\gamma}_*(G)) \to \mathrm{Hom}_{\mathrm{PSh}(k,R)}(F_1,\tilde{\gamma}_*(G))$$

for any $G \in \widetilde{\mathrm{PSh}}(k,R)$ and by adjunction an exact sequence

$$0 \to \mathrm{Hom}_{\mathrm{PSh}(k,R)}(\tilde{\gamma}^*(F_3),G) \to \mathrm{Hom}_{\mathrm{PSh}(k,R)}(\tilde{\gamma}^*(F_2),G) \to \mathrm{Hom}_{\mathrm{PSh}(k,R)}(\tilde{\gamma}^*(F_1),G)$$

showing that $\tilde{\gamma}^*(F_3)$ is determined by $\tilde{\gamma}^*(F_2)$ and $\tilde{\gamma}^*(F_1)$, i.e. that the sequence

$$\tilde{\gamma}^*(F_1) \to \tilde{\gamma}^*(F_2) \to \tilde{\gamma}^*(F_3) \to 0$$

is exact. This gives the following formula. If F is a presheaf, we can choose a resolution by (infinite) direct sums of representable presheaves (e.g. [MVW06, proof of Lemma 8.1])

$$F_1 \to F_2 \to F \to 0$$

and compute $\tilde{\gamma}^*F$ as the cokernel of $\tilde{\gamma}^*F_1 \to \tilde{\gamma}^*F_2$. We let the reader define $\tilde{\gamma}^*$ on morphisms and check that it is independent (up to unique isomorphisms) of choices. A similar construction works for π^*. Note that the left adjoints $\tilde{\gamma}^*$ and π^* are symmetric monoidal and right-exact.

Lemma 1.2.3. *The functors $\tilde{\gamma}_*$ and π_* are faithful. If 2 is invertible in the ring R then the functor π_* is also full.*

[1]Recall that an abelian category is called Grothendieck abelian if it admits a family of generators, admits small sums and filtered colimits are exact. The category of presheaves over any essentially small category $\mathscr{S}$ with values in a the category of R-modules is a basic example of Grothendieck abelian category. In fact, it is generated by representable presheaves of R-modules. The existence of small sums is obvious and the fact filtered colimits are exact can be reduced to the similar fact for the category of R-modules by taking global sections over objects of $\mathscr{S}$.

Proof. The faithfulness of both $\tilde{\gamma}_*$ and π_* are obvious. To prove the second assertion, we use the fact that the map (1.1.1.b) from finite MW-correspondences to correspondences is surjective after inverting 2 (Lemma 1.1.2). In particular, given a MW-presheaf F, the property $F = \pi_*(F_0)$ is equivalent to the property on F that for any $\alpha, \alpha' \in \tilde{\mathrm{c}}(X, Y)$ with $\pi(\alpha) = \pi(\alpha')$ then $F(\alpha) = F(\alpha')$. Then it is clear that a natural transformation between two presheaves with transfers F_0 and G_0 is the same thing as a natural transformation between $F_0 \circ \pi$ and $G_0 \circ \pi$. □

Definition 1.2.4. We define a MW-*t-sheaf* (resp. *t*-sheaf with transfers) to be a presheaf with MW-transfers (resp. with transfers) F such that $\tilde{\gamma}_*(F) = F \circ \tilde{\gamma}$ (resp. $F \circ \gamma$) is a sheaf for the given topology t. When t is the Nisnevich topology, we will simply say MW-*sheaf* and when t is the étale topology we will say *étale* MW-*sheaf.*

We denote by $\widetilde{\mathrm{Sh}}_t(k, R)$ the category of MW-t-sheaves, seen as a full subcategory of the R-linear category $\widetilde{\mathrm{PSh}}(k, R)$. When t is the Nisnevich topology, we drop the index in this notation.

Note that there is an obvious forgetful functor

$$\tilde{\mathcal{O}}_t : \widetilde{\mathrm{Sh}}_t(k, R) \to \widetilde{\mathrm{PSh}}(k, R) \text{ (resp. } \mathcal{O}_t^{\mathrm{tr}} : \mathrm{Sh}_t^{\mathrm{tr}}(k, R) \to \mathrm{PSh}^{\mathrm{tr}}(k, R))$$

which is fully faithful. In what follows, we will drop the indication of the topology t in the above functors, as well as their adjoints.

Example 1.2.5. Given a smooth scheme X, the presheaf $\tilde{\mathrm{c}}(X)$ is in general not a MW-sheaf (see [CF14, 5.12]). Note however that $\tilde{\mathrm{c}}(\operatorname{Spec} k)$ is the unramified 0-th Milnor-Witt sheaf $\mathbf{K}_0^{\mathrm{MW}}$ (defined in [Mor12, §3]) by *loc. cit.* Ex. 4.4.

As in the case of the theory developed by Voevodsky, the theory of MW-sheaves rely on the following fundamental lemma, whose proof is adapted from Voevodsky's original argument.

Lemma 1.2.6. *Let X be a smooth scheme and $p : U \to X$ be a t-cover where t is the Nisnevich or the étale topology.*

Then the following complex

$$\ldots \xrightarrow{d_n} \tilde{\mathrm{c}}(U_X^n) \longrightarrow \ldots \longrightarrow \tilde{\mathrm{c}}(U \times_X U) \xrightarrow{d_1} \tilde{\mathrm{c}}(U) \xrightarrow{d_0} \tilde{\mathrm{c}}(X) \to 0$$

where d_n is the differential associated with the Čech simplicial scheme of U/X, is exact on the associated t-sheaves.

Proof. We have to prove that the fiber of the above complex at a t-point is an acyclic complex of R-modules. Taking into account Remark 1.1.3(4), we are reduced to prove, given an essentially smooth local henselian scheme S, that the following complex

$$C_* := \ldots \xrightarrow{d_n} \tilde{\mathrm{c}}(S, U_X^n) \longrightarrow \ldots \longrightarrow \tilde{\mathrm{c}}(S, U \times_X U) \xrightarrow{d_1} \tilde{\mathrm{c}}(S, U) \xrightarrow{d_0} \tilde{\mathrm{c}}(S, X) \to 0$$

is acyclic.

Let $\mathcal{A} = \mathcal{A}(S, X)$ be the set of admissible subsets in $S \times X$ ([CF14, Definition 4.1]). Given any $T \in \mathcal{A}$, and an integer $n \geq 0$, we let $C_n^{(T)}$ be the subgroup of $\tilde{\mathrm{c}}(S, U_X^n)$ consisting of MW-correspondences whose support is in the closed subset $U_T^n := T \times_X U_X^n$ of $S \times U_X^n$. The differentials are given by direct images along projections so they respect the support condition on MW-correspondence associated with $T \in \mathcal{F}$ and make $C_*^{(T)}$ into a subcomplex of C_*.

It is clear that C_* is the filtering union of the subcomplexes $C_*^{(T)}$ for $T \in \mathcal{F}$ so it suffices to prove that, for a given $T \in \mathcal{F}$, the complex $C_*^{(T)}$ is split. We prove the result when $R = \mathbb{Z}$, the general statement follows after tensoring with R. Because S is henselian and T is finite over S, the scheme T is a finite sum of local henselian schemes. Consequently, the t-cover $p_T : U_T \to T$, which is in particular étale and surjective, admits a splitting s. It follows from [Mil12, Proposition 2.15] that s is an isomorphism onto a connected component of U_T. We therefore obtain maps $s \times 1_{U_T^n} : U_T^n \to U_T^{n+1}$ such that $U_T^{n+1} = U_T^n \sqcup D_T^{n+1}$ for any $n \geq 0$ and a commutative diagram

$$\begin{array}{ccc} U_T^n & \longrightarrow & (S \times U_X^{n+1}) \setminus D_T^{n+1} \\ \downarrow & & \downarrow \\ U_T^{n+1} & \longrightarrow & S \times U_X^{n+1} \\ \downarrow & & \downarrow \\ U_T^n & \longrightarrow & S \times U_X^n \end{array}$$

in which the squares are Cartesian and the right-hand vertical maps are étale. By étale excision, we get isomorphisms

$$\widetilde{\mathrm{CH}}^*_{U_T^n}(S \times U_X^n, \omega_{U_X^n}) \to \widetilde{\mathrm{CH}}^*_{U_T^n}((S \times U_X^{n+1}) \setminus D_T^{n+1}, \omega_{U_X^{n+1}})$$

and

$$\widetilde{\mathrm{CH}}^*_{U_T^n}(S \times U_X^{n+1}, \omega_{U_X^{n+1}}) \to \widetilde{\mathrm{CH}}^*_{U_T^n}((S \times U_X^{n+1}) \setminus D_T^{n+1}, \omega_{U_X^{n+1}}).$$

Putting these isomorphisms together, we obtain an isomorphism

$$\widetilde{\mathrm{CH}}^*_{U_T^n}(S \times U_X^n, \omega_{U_X^n}) \to \widetilde{\mathrm{CH}}^*_{U_T^n}(S \times U_X^{n+1}, \omega_{U_X^{n+1}})$$

that we can compose with the extension of support to finally obtain a homomorphism

$$(s \times 1_{U_T^n})_* : \widetilde{\mathrm{CH}}^*_{U_T^n}(S \times U_X^n, \omega_{U_X^n}) \to \widetilde{\mathrm{CH}}^*_{U_T^{n+1}}(S \times U_X^{n+1}, \omega_{U_X^{n+1}})$$

yielding a contracting homotopy

$$(s \times 1_{U_T^n})_* : C_n^{(T)} \to C_{n+1}^{(T)}.$$

□

1.2.7. As in the classical case, one can derive from this lemma the existence of a left adjoint $\tilde{a}$ to the functor $\tilde{\mathcal{O}}$. The proof is exactly the same as in the case of sheaves with transfers (cf. [CD09b, 10.3.9] for example) but we include it here for the convenience of the reader.

Let us introduce a notation. If P is a presheaf on Sm_k, we define a presheaf with transfers

(1.2.7.a) $$\tilde{\gamma}^!(P) : Y \mapsto \mathrm{Hom}_{\mathrm{PSh}(k,R)}(\tilde{\gamma}_*(\tilde{\mathrm{c}}(Y)), P).$$

and we observe that $\tilde{\gamma}^!$ is right adjoint to the functor $\tilde{\gamma}_*$. The latter, having both a left and a right adjoint, is then exact. Given a natural transformation

$$\phi : P \to \tilde{\gamma}_* \tilde{\gamma}^!(P)$$

and smooth schemes X and Y, we define a pairing

$$P(X) \times \tilde{\mathrm{c}}(Y, X) \to P(Y), (\rho, \alpha) \mapsto \langle \rho, \alpha \rangle_\phi := [\phi_X(\rho)]_Y(\alpha)$$

where $\phi_X(\rho)$ is seen as a natural transformation $\tilde{\mathrm{c}}(X) \to P$. The following lemma is tautological.

Lemma 1.2.8. *Let P be a presheaf on Sm_k. Then there is a bijection between the following data:*

- *Presheaves with transfers $\tilde{P}$ such that $\tilde{\gamma}_*(\tilde{P}) = P$;*
- *Natural transformations $\phi : P \to \tilde{\gamma}_*\tilde{\gamma}^!(P)$ such that:*

(W1) $\forall \rho \in P(X), \langle \rho, Id_X \rangle_\phi = \rho$.

(W2) $\forall (\rho, \beta, \alpha) \in P(X) \times \tilde{\mathrm{c}}(Y,X) \times \tilde{\mathrm{c}}(Z,Y), \langle \langle \rho, \beta \rangle_\phi, \alpha \rangle_\phi = \langle \rho, \beta \circ \alpha \rangle_\phi$;

according to the following rules:

$$\tilde{P} \mapsto \big(P = \tilde{\gamma}_*(\tilde{P}) \xrightarrow{ad'} \tilde{\gamma}_*\tilde{\gamma}^!\tilde{\gamma}_*(\tilde{P}) = \tilde{\gamma}_*\tilde{\gamma}^!(P)\big)$$
$$(P, \langle ., \alpha \rangle_\phi) \leftarrow\!\shortmid \phi,$$

where ad' is the unit map for the adjunction $(\tilde{\gamma}_*, \tilde{\gamma}^!)$.

Before going further, we note the following corollary of the previous result.

Corollary 1.2.9. (1) *For any t-sheaf F on Sm_k, $\tilde{\gamma}^!(F)$ is a* MW*-t-sheaf.*

(2) *Let $\alpha \in \tilde{\mathrm{c}}(X,Y)$ be a finite* MW*-correspondence and $p : W \to Y$ a t-cover. Then there exists a t-cover $q : W' \to X$ and a finite* MW*-correspondence $\hat{\alpha} : W' \to W$ such that the following diagram commutes:*

(1.2.9.a)
$$\begin{array}{ccc} W' & \xrightarrow{\hat{\alpha}} & W \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[\alpha]{} & Y \end{array}$$

The first property is a direct consequence of Lemma 1.2.6 given Formula (1.2.7.a). The second property is an application of the fact that $\tilde{\mathrm{c}}(W) \to \tilde{\mathrm{c}}(X)$ is an epimorphism of sheaves, obtained from the same proposition.

We are ready to state and prove the main lemma which proves the existence of the right adjoint $\tilde{a}$ to $\tilde{\mathcal{O}}$.

Lemma 1.2.10. *Let $\tilde{P}$ be a* MW*-presheaf and $P := \tilde{\gamma}_*(\tilde{P})$. Let F be the t-sheaf associated with P and let $\tau : P \to F$ be the canonical natural transformation.*

Then there exists a unique pair $(\tilde{F}, \tilde{\tau})$ such that:

1. *$\tilde{F}$ is a* MW*-t-sheaf such that $\tilde{\gamma}_*(\tilde{F}) = F$.*
2. *$\tilde{\tau} : \tilde{P} \to \tilde{F}$ is a natural transformation of* MW*-presheaves such that the induced transformation*
$$P = \tilde{\gamma}_*(\tilde{P}) \xrightarrow{\tilde{\gamma}_*(\tilde{\tau})} \tilde{\gamma}_*(\tilde{F}) = F$$
coincides with τ.

Proof. Let us construct $\tilde{F}$. Applying Lemma 1.2.8 to $\tilde{P}$ and P, we get a canonical natural transformation: $\psi : P \to \tilde{\gamma}_*\tilde{\gamma}^!(P)$. Applying point (1) of Corollary 1.2.9 and the fact that F is the t-sheaf associated with P, there exists a unique natural transformation ϕ which fits into the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\psi} & \tilde{\gamma}_*\tilde{\gamma}^!(P) \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \tilde{\gamma}_*\tilde{\gamma}^!(\tau)} \\ F & \xrightarrow[\phi]{} & \tilde{\gamma}_*\tilde{\gamma}^!(F). \end{array}$$

To obtain the MW-sheaf $\tilde{F}$ satisfying (1), it is sufficient according to Lemma 1.2.8 to prove that conditions (W1) and (W2) are satisfied for the pairing $\langle ., . \rangle_\phi$. Before proving this, we note that the existence of $\tilde{\tau}$ satisfying property (2) is equivalent to the commutativity of the above diagram. In particular, the unicity of $(\tilde{F}, \tilde{\tau})$ comes from the unicity of the map ϕ.

Therefore, we only need to prove (W1) and (W2) for ϕ. Consider a couple $(\rho, \alpha) \in F(X) \times \tilde{c}(Y, X)$. Because F is the t-sheaf associated with P, there exists a t-cover $p : W \to X$ and a section $\hat{\rho} \in P(W)$ such that $p^*(\rho) = \tau_W(\hat{\rho})$. According to point (2) of Corollary 1.2.9, we get a t-cover $q : W' \to Y$ and a correspondence $\hat{\alpha} \in \tilde{c}(W', W)$ making the diagram (1.2.9.a) commutative. As ϕ is a natural transformation, we get

$$q^* \langle \rho, \alpha \rangle_\phi = \langle \rho, \alpha \circ q \rangle_\phi = \langle \rho, p \circ \hat{\alpha} \rangle_\phi = \langle p^* \rho, \hat{\alpha} \rangle_\phi = \langle \tau_W(\hat{\rho}), \hat{\alpha} \rangle_\phi = \langle \hat{\rho}, \hat{\alpha} \rangle_\psi.$$

Because $q^* : F(X) \to F(W)$ is injective, we deduce easily from this principle the properties (W1) and (W2) for ϕ from their analog properties for ψ. □

Proposition 1.2.11. (1) *The obvious forgetful functor* $\tilde{\mathcal{O}} : \widetilde{\mathrm{Sh}}_t(k, R) \to \widetilde{\mathrm{PSh}}(k, R)$ *admits a left adjoint* $\tilde{a}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{PSh}(k,R) & \xleftarrow{\tilde{\gamma}_*} & \widetilde{\mathrm{PSh}}(k,R) \\ \downarrow a & & \downarrow \tilde{a} \\ \mathrm{Sh}_t(k,R) & \xleftarrow{\tilde{\gamma}_*} & \widetilde{\mathrm{Sh}}_t(k,R) \end{array}$$

where a is the usual t-sheafification functor with respect to the smooth site.

(2) *The category* $\widetilde{\mathrm{Sh}}_t(k, R)$ *is a Grothendieck abelian category and the functor* $\tilde{a}$ *is exact.*

(3) *The functor* $\tilde{\gamma}_*$*, appearing in the lower line of the preceding diagram, admits a left adjoint* $\tilde{\gamma}^*$*, and commutes with every limit and colimit.*

Proof. The first point follows directly from the previous lemma: indeed, with the notation of this lemma, we can put: $\tilde{a}(P) = \tilde{F}$.

For point (2), we first remark that the functor $\tilde{a}$, being a left adjoint, commutes with every colimit. Moreover, the functor a is exact and $\tilde{\gamma}_* : \widetilde{\mathrm{PSh}}(k, R) \to \mathrm{PSh}(k, R)$ is also exact (Paragraph 1.2.7). Therefore, $\tilde{a}$ is exact because of the previous commutative square and the fact that $\tilde{\gamma}_*$ is faithful. Then, we easily deduce that $\widetilde{\mathrm{Sh}}_t(k, R)$ is a Grothendieck abelian category from the fact that $\widetilde{\mathrm{PSh}}(k, R)$ is such a category.

The existence of the left adjoint $\tilde{\gamma}^*$ follows formally. Thus $\tilde{\gamma}_*$ commutes with every limit. Because $\tilde{\gamma}_*$ is exact and commutes with arbitrary coproducts, we deduce that it commutes with arbitrary colimits, therefore proving point (3). □

Remark 1.2.12. The left adjoint $\tilde{\gamma}^*$ of $\tilde{\gamma}_* : \widetilde{\mathrm{Sh}}_t(k, R) \to \mathrm{Sh}_t(k, R)$ can be computed as the composite

$$\mathrm{Sh}_t(k, R) \xrightarrow{\mathcal{O}} \mathrm{PSh}(k, R) \xrightarrow{\tilde{\gamma}^*} \widetilde{\mathrm{PSh}}(k, R) \xrightarrow{\tilde{a}} \widetilde{\mathrm{Sh}}_t(k, R).$$

One can also observe that, according to point (2), a family of generators of the Grothendieck abelian category $\widetilde{\mathrm{Sh}}_t(k, R)$ is obtained by applying the functor $\tilde{a}$ to a family of generators of $\widetilde{\mathrm{PSh}}(k, R)$.

Definition 1.2.13. Given any smooth scheme X, we put $\tilde{R}_t(X) = \tilde{a}\,(\tilde{c}(X))$.

In particular, for a smooth scheme X, $\tilde{R}_t(X)$ is the t-sheaf associated with the presheaf $\tilde{c}(X)$, equipped with its canonical action of MW-correspondences (Lemma 1.2.10). The corresponding family, for all smooth schemes X, generates the abelian category $\widetilde{\mathrm{Sh}}_t(k, R)$.

1.2.14. One deduces from the monoidal structure on $\widetilde{\mathrm{Cor}}_k$ a monoidal structure on $\widetilde{\mathrm{Sh}}_t(k,R)$ whose tensor product $\tilde{\otimes}$ is uniquely characterized by the property that for any smooth schemes X and Y:

$$\tilde{R}_t(X)\,\tilde{\otimes}\,\tilde{R}_t(Y) = \tilde{R}_t(X\times Y). \tag{1.2.14.a}$$

Explicitly, the tensor product of any two sheaves $F,G \in \widetilde{\mathrm{Sh}}_t(k,R)$ is obtained by applying $\tilde{a}$ to the presheaf tensor product $F\otimes G$ mentioned after Definition 1.2.2. In particular, the bifunctor $\tilde{\otimes}$ commutes with colimits and therefore, as the abelian category $\widetilde{\mathrm{Sh}}_t(k,R)$ is a Grothendieck abelian category, the monoidal category $\widetilde{\mathrm{Sh}}_t(k,R)$ is closed. The internal Hom functor is characterized by the property that for any MW-t-sheaf F and any smooth scheme X,

$$\underline{\mathrm{Hom}}\left(\tilde{R}_t(X),F\right) = F(X\times -).$$

As a corollary of Proposition 1.2.11, we obtain functors between the category of sheaves we have considered so far.

Corollary 1.2.15. (1) *There exists a commutative diagram of symmetric monoidal functors*

$$\begin{array}{ccccc}
\mathrm{PSh}(k,R) & \xrightarrow{a_{\mathrm{Nis}}} & \mathrm{Sh}_{\mathrm{Nis}}(k,R) & \xrightarrow{a_{\text{ét}}} & \mathrm{Sh}_{\text{ét}}(k,R) \\
\downarrow{\scriptstyle \tilde{\gamma}^*} & & \downarrow{\scriptstyle \tilde{\gamma}^*_{\mathrm{Nis}}} & & \downarrow{\scriptstyle \tilde{\gamma}^*_{\text{ét}}} \\
\widetilde{\mathrm{PSh}}(k,R) & \xrightarrow{\tilde{a}_{\mathrm{Nis}}} & \widetilde{\mathrm{Sh}}_{\mathrm{Nis}}(k,R) & \xrightarrow{\tilde{a}_{\text{ét}}} & \widetilde{\mathrm{Sh}}_{\text{ét}}(k,R) \\
\downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*_{\mathrm{Nis}}} & & \downarrow{\scriptstyle \pi^*_{\text{ét}}} \\
\mathrm{PSh}^{\mathrm{tr}}(k,R) & \xrightarrow{a^{tr}_{\mathrm{Nis}}} & \mathrm{Sh}^{\mathrm{tr}}_{\mathrm{Nis}}(k,R) & \xrightarrow{a^{tr}_{\text{ét}}} & \mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(k,R)
\end{array}$$

which are all left adjoints of an obvious forgetful functor. Each of these functors respects the canonical family of abelian generators.

(2) *Let* $t = \mathrm{Nis}, \text{ét}$. *Then the right adjoint functor* $\tilde{\gamma}^t_* : \widetilde{\mathrm{Sh}}_t(k,R) \to \mathrm{Sh}_t(k,R)$ *is faithful. If* 2 *is invertible in* R, *the right adjoint functor* $\pi^t_* : \mathrm{Sh}^{\mathrm{tr}}_t(k,R) \to \widetilde{\mathrm{Sh}}_t(k,R)$ *is fully faithful.*

Indeed, the first point is a formal consequence of Proposition 1.2.11 and its analog for sheaves with transfers. The second point follows from the commutativity of the diagram in point (1), which induces an obvious commutative diagram for the right adjoint functors, the fact that the forgetful functor from sheaves to presheaves is always fully faithful and Lemma 1.2.3.

2. Framed correspondences

2.1. **Definitions and basic properties.** The aim of this section is to make a link between the category of linear framed correspondences (after Garkusha-Panin-Voevodsky) and the category of MW-presheaves. We start with a quick reminder on framed correspondences following [GP14].

Definition 2.1.1. Let U be a smooth k-scheme and $Z\subset U$ be a closed subset of codimension n. A set of regular functions $\phi_1,\dots,\phi_n \in k[U]$ is called a framing of Z in U if Z coincides with the closed subset $\phi_1 = \ldots = \phi_n = 0$.

Definition 2.1.2. Let X and Y be smooth k-schemes, and let $n \in \mathbb{N}$ be an integer. An explicit framed correspondence $c = (U, \phi, f)$ of level n from X to Y consists of the following data:

(1) A closed subset $Z \subset \mathbb{A}^n_X$ which is finite over X (here, Z is endowed with its reduced structure).
(2) An étale neighborhood $\alpha : U \to \mathbb{A}^n_X$ of Z.
(3) A framing $\phi = (\phi_1, \dots, \phi_n)$ of Z in U.
(4) A morphism $f : U \to Y$.

The closed subset Z is called the *support* of the explicit framed correspondence $c = (U, \phi, f)$.

Remark 2.1.3. One could give an alternative approach to the above definition. A framed correspondence (U, ϕ, f) corresponds to a pair of morphisms $\phi : U \to \mathbb{A}^n_k$ and $f : U \to Y$ yielding a unique morphism $\varphi : U \to \mathbb{A}^n_Y$. The closed subset $Z \subset U$ corresponds to the preimage of $Y \times \{0\} \subset Y \times \mathbb{A}^n_k = \mathbb{A}^n_Y$. This correspondence is unique.

Remark 2.1.4. Note that Z is not supposed to map surjectively onto a component of X. For instance $Z = \emptyset$ is an explicit framed correspondence of level n, denoted by 0_n. If Z is non-empty, then an easy dimension count shows that $Z \subset \mathbb{A}^n_X \to X$ is indeed surjective onto a component of X.

Remark 2.1.5. Suppose that X is a smooth connected scheme. By definition, an explicit framed correspondence of level $n = 0$ is either a morphism of schemes $f : X \to Y$ or 0_0.

Definition 2.1.6. Let $c = (U, \phi, f)$ and $c' = (U', \phi', f')$ be two explicit framed correspondences of level $n \geq 0$. Then, c and c' are said to be *equivalent* if they have the same support and there exists an open neighborhood V of Z in $U \times_{\mathbb{A}^n_X} U'$ such that the diagrams

$$\begin{array}{ccc} U \times_{\mathbb{A}^n_X} U' & \longrightarrow & U' \\ \downarrow & & \downarrow{\scriptstyle f'} \\ U & \xrightarrow[f]{} & Y \end{array}$$

and

$$\begin{array}{ccc} U \times_{\mathbb{A}^n_X} U' & \longrightarrow & U' \\ \downarrow & & \downarrow{\scriptstyle \phi'} \\ U & \xrightarrow[\phi]{} & \mathbb{A}^n_k \end{array}$$

are both commutative when restricted to V. A *framed* correspondence of level n is an equivalence class of explicit framed correspondences of level n.

Definition 2.1.7. Let X and Y be smooth schemes and let $n \in \mathbb{N}$. We denote by $\mathrm{Fr}_n(X, Y)$ the set of framed correspondences of level n from X to Y and by $\mathrm{Fr}_*(X, Y)$ the set $\sqcup_n \mathrm{Fr}_n(X, Y)$. Together with the composition of framed correspondences described in [GP14, §2], this defines a category whose objects are smooth schemes and morphisms are $\mathrm{Fr}_*(\_, \_)$. We denote this category by $\mathrm{Fr}_*(k)$ and refer to it as the *category of framed correspondences*.

We now pass to the linear version of the above category following [GP14, §7], starting with the following observation. Let X and Y be smooth schemes, and let $c_Z = (U, \phi, f)$ be an explicit framed correspondence of level n from X to Y with support Z of the form $Z = Z_1 \sqcup Z_2$.

Let $U_1 = U \setminus Z_2$ and $U_2 = U \setminus Z_1$. For $i = 1, 2$, we get étale morphisms $\alpha_i : U_i \to X$ and morphisms $\phi_i : U_i \to \mathbb{A}^n_k$, $f_i : U_i \to Y$ by precomposing the morphisms α, ϕ and f with the open immersion $U_i \to U$. Note that U_i is an étale neighborhood of Z_i for $i = 1, 2$ and that $c_{Z_i} = (U_i, \phi_i, f_i)$ are explicit framed correspondences of level n from X to Y with support Z_i.

Definition 2.1.8. Let X and Y be smooth schemes and let $n \in \mathbb{N}$. Let

$$\mathbb{Z}\mathrm{F}_n(X, Y) = \mathbb{Z}\mathrm{Fr}_n(X, Y)/H$$

where H is the subgroup generated by elements of the form $c_Z - c_{Z_1} - c_{Z_2}$ where $Z = Z_1 \sqcup Z_2$ is as above and $\mathbb{Z}\mathrm{Fr}_n(X, Y)$ is the free abelian group on $\mathrm{Fr}_n(X, Y)$. The category $\mathbb{Z}\mathrm{F}_*(k)$ of *linear framed correspondences* is the category whose objects are smooth schemes and whose morphisms are

$$\mathrm{Hom}_{\mathbb{Z}\mathrm{F}_*(k)}(X, Y) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}\mathrm{Fr}_n(X, Y).$$

Remark 2.1.9. Note that there is an obvious functor $\iota : \mathrm{Fr}_*(k) \to \mathbb{Z}\mathrm{F}_*(k)$ with $\iota(0_n) = 0$ for any $n \in \mathbb{N}$.

The stage being set, we now compare the category of finite MW-correspondences with the above categories.

Let U be a smooth k-scheme and let $\phi : U \to \mathbb{A}^n_k$ be a morphism corresponding to (nonzero) global sections $\phi_i \in \mathcal{O}(U)$. Each section ϕ_i can be seen as an element of $k(U)^\times$ and defines then an element of $\mathrm{K}^{\mathrm{MW}}_1(k(U))$. Let $|\phi_i|$ be the support of f_i, i.e. its vanishing locus, and let $Z = |\phi_1| \cap \ldots \cap |\phi_n|$. Consider the residue map

$$d : \mathrm{K}^{\mathrm{MW}}_1(k(U)) \to \bigoplus_{x \in U^{(1)}} \mathrm{K}^{\mathrm{MW}}_0(k(x), \omega_x).$$

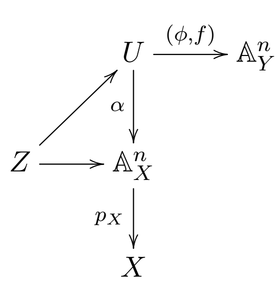
Then, $d(\phi_i)$ defines an element supported on $|\phi_i|$. As it is a boundary, it defines a cycle $Z(\phi_i) \in H^1_{|\phi_i|}(U, \mathbf{K}^{\mathrm{MW}}_1)$. Now, we can consider the intersection product

$$\mathrm{H}^1_{|\phi_1|}(U, \mathbf{K}^{\mathrm{MW}}_1) \times \ldots \times \mathrm{H}^1_{|\phi_n|}(U, \mathbf{K}^{\mathrm{MW}}_1) \to \mathrm{H}^n_Z(U, \mathbf{K}^{\mathrm{MW}}_n)$$

to get an element $Z(\phi_1) \cdot \ldots \cdot Z(\phi_n)$ that we denote by $Z(\phi)$.

Lemma 2.1.10. *Any explicit framed correspondence $c = (U, \phi, f)$ induces a finite* MW-*correspondence $\alpha(c)$ from X to Y. Moreover, two equivalent explicit framed correspondences c and c' induce the same finite* MW-*correspondence.*

Proof. Let us start with the first assertion. If Z is empty, its image is defined to be zero. If c is of level 0, then it corresponds to a morphism of schemes and we use the functor $\mathrm{Sm}_k \to \widetilde{\mathrm{Cor}}_k$ to define the image of c. We thus suppose that Z is non-empty (thus finite and surjective on some components of X) of level $n \geq 1$. Consider the following diagram



defining an explicit framed correspondence (U,ϕ,f) of level n. The framing ϕ defines an element $Z(\phi)\in \mathrm{H}^n_Z(U,\mathbf{K}^{\mathrm{MW}}_n)$ as explained above. Now, α is étale and therefore induces an isomorphism $\alpha^*\omega_{\mathbb{A}^n_X}\simeq\omega_U$. Choosing the usual orientation for $\mathbb{A}^n_k$, we get an isomorphism $\mathcal{O}_{\mathbb{A}^n_X}\simeq\omega_{\mathbb{A}^n_X}\otimes(p_X)^*\omega_X^\vee$ and therefore an isomorphism

$$\mathcal{O}_U\simeq\alpha^*(\mathcal{O}_{\mathbb{A}^n_X})\simeq\alpha^*(\omega_{\mathbb{A}^n_X}\otimes p_X^*\omega_X^\vee)\simeq\omega_U\otimes(p_X\alpha)^*\omega_X^\vee.$$

We can then see $Z(\phi)$ as an element of the group $\mathrm{H}^n_Z(U,\mathbf{K}^{\mathrm{MW}}_n,\omega_U\otimes(p_X\alpha)^*\omega_X^\vee)$. Consider next the map $(p_X\alpha,f):U\to X\times Y$ and the image T of Z under the map of underlying topological spaces. It follows from [MVW06, Lemma 1.4] that T is closed, finite and surjective over (some components of) X. Moreover, the morphism $Z\to T$ is finite and it follows that we have a push-forward homomorphism

$$(p_X\alpha,f)_*:\mathrm{H}^n_Z(U,\mathbf{K}^{\mathrm{MW}}_n,\omega_U\otimes(p_X\alpha)^*\omega_X^\vee)\to\mathrm{H}^n_T(X\times Y,\mathbf{K}^{\mathrm{MW}}_n,\omega_{X\times Y/X})$$

yielding, together with the canonical isomorphism $\omega_{X\times Y/X}\simeq\omega_Y$, a finite Chow-Witt correspondence $\alpha(c):=(p_X\alpha,f)_*(Z(\phi))$ between X and Y.

Suppose next that $c=(U,\phi,f)$ and $c'=(U',\phi',f')$ are two equivalent explicit framed correspondences of level n. Following the above construction, we obtain two cocycles $\tilde\alpha(c)\in\mathrm{H}^n_Z(U,\mathbf{K}^{\mathrm{MW}}_n,\omega_U\otimes(p_X\alpha)^*\omega_X^\vee)$ and $\tilde\alpha(c')\in\mathrm{H}^n_Z(U',\mathbf{K}^{\mathrm{MW}}_n,\omega_{U'}\otimes(p_X\alpha')^*\omega_X^\vee)$. Now, the pull-backs along the projections

$$\begin{array}{ccc} U\times_{\mathbb{A}^n_X}U' & \xrightarrow{p_2} & U' \\ \downarrow{\scriptstyle p_1} & & \\ U & & \end{array}$$

yield homomorphisms

$$p_1^*:\mathrm{H}^n_Z(U,\mathbf{K}^{\mathrm{MW}}_n,\omega_U\otimes(p_X\alpha)^*\omega_X^\vee)\simeq\mathrm{H}^n_{p_1^{-1}(Z)}(U\times_{\mathbb{A}^n_X}U',\mathbf{K}^{\mathrm{MW}}_n,\omega_{U\times_{\mathbb{A}^n_X}U'}\otimes(p_X\alpha p_1)^*\omega_X^\vee)$$

and

$$p_2^*:\mathrm{H}^n_Z(U',\mathbf{K}^{\mathrm{MW}}_n,\omega_{U'}\otimes(p_X\alpha')^*\omega_X^\vee)\simeq\mathrm{H}^n_{p_2^{-1}(Z)}(U\times_{\mathbb{A}^n_X}U',\mathbf{K}^{\mathrm{MW}}_n,\omega_{U\times_{\mathbb{A}^n_X}U'}\otimes(p_X\alpha p_2)^*\omega_X^\vee),$$

while the pull-back along the open immersion $i:V\to U\times_{\mathbb{A}^n_X}U'$ induces homomorphisms

$$i^*:\mathrm{H}^n_{p_1^{-1}(Z)}(U\times_{\mathbb{A}^n_X}U',\mathbf{K}^{\mathrm{MW}}_n,\omega_{U\times_{\mathbb{A}^n_X}U'}\otimes(p_X\alpha p_1)^*\omega_X^\vee)\simeq\mathrm{H}^n_Z(V,\mathbf{K}^{\mathrm{MW}}_n,\omega_V\otimes(p_X\alpha p_1 i)^*\omega_X^\vee)$$

and

$$i^*:\mathrm{H}^n_{p_2^{-1}(Z)}(U\times_{\mathbb{A}^n_X}U',\mathbf{K}^{\mathrm{MW}}_n,\omega_{U\times_{\mathbb{A}^n_X}U'}\otimes(p_X\alpha p_2)^*\omega_X^\vee)\simeq\mathrm{H}^n_Z(V,\mathbf{K}^{\mathrm{MW}}_n,\omega_V\otimes(p_X\alpha p_2 i)^*\omega_X^\vee).$$

Note that $p_X\alpha p_2=p_X\alpha p_1$ and that $i^*p_1^*(\tilde\alpha(c))=i^*p_2^*(\tilde\alpha(c'))$ by construction. Pushing forward along $V\to U\times_{\mathbb{A}^n_X}U'\to U\to X\times Y$, we get the result. □

Example 2.1.11. Let X be a smooth k-scheme. Consider the explicit framed correspondence σ_X of level 1 from X to X given by $(\mathbb{A}^1_X,q,p_X)$ where $q:\mathbb{A}^1_X=\mathbb{A}^1\times X\to\mathbb{A}^1$ is the projection to the first factor and $p_X:\mathbb{A}^1_X\to X$ is the projection to the second factor. We claim that $\alpha(\sigma_X)=Id\in\widetilde{\mathrm{Cor}}_k(X,X)$. To see this, observe that we have a commutative diagram

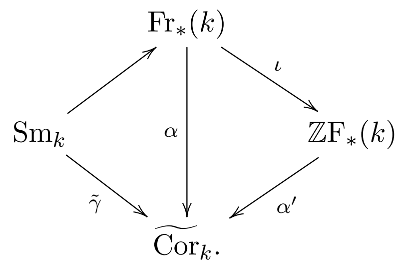
$$\begin{array}{ccc} \mathbb{A}^1_X & \xrightarrow{(p_X,p_X)} & X\times X \\ \downarrow{\scriptstyle p_X} & \nearrow{\scriptstyle \triangle} & \\ X & & \end{array}$$

where $\triangle$ is the diagonal map. Following the process of the above lemma, we start by observing that $Z(q) \in \mathrm{H}^1_X(\mathbb{A}^1_X, \mathbf{K}_1^{\mathrm{MW}})$ is the class of $\langle 1\rangle \otimes \bar{t} \in \mathbf{K}_0^{\mathrm{MW}}(k(X), (\mathfrak{m}/\mathfrak{m}^2)^*)$ where $\mathfrak{m}$ is the maximal ideal corresponding to X in the appropriate local ring and t is a coordinate of $\mathbb{A}^1$. Now, we choose the canonical orientation of $\mathbb{A}^1$ and the class of $Z(q)$ corresponds then to the class of $\langle 1\rangle \in \mathbf{K}_0^{\mathrm{MW}}(k(X))$ in $\mathrm{H}^1_X(\mathbb{A}^1_X, \mathbf{K}_1^{\mathrm{MW}}, \omega_{(\mathbb{A}^1_X/X)})$. Its push-forward under

$$(p_X)_* : \mathrm{H}^1_X(\mathbb{A}^1_X, \mathbf{K}_1^{\mathrm{MW}}, \omega_{(\mathbb{A}^1_X/X)}) \to \mathrm{H}^0(X, \mathbf{K}_0^{\mathrm{MW}})$$

is the class of $\langle 1\rangle$ and the claim follows from the fact that $(p_X, p_X)_* = \triangle_*(p_X)_*$ and the definition of the identity in $\widetilde{\mathrm{Cor}}_k(X, X)$.

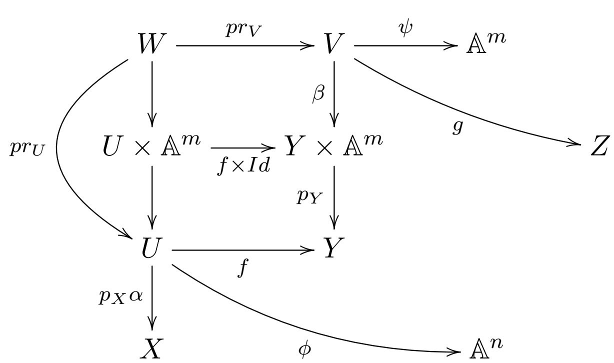
Proposition 2.1.12. *The assignment $c = (U, \phi, f) \mapsto \alpha(c)$ made explicit in Lemma 2.1.10 define functors $\alpha : \mathrm{Fr}_*(k) \to \widetilde{\mathrm{Cor}}_k$ and $\alpha' : \mathbb{Z}\mathrm{F}_*(k) \to \widetilde{\mathrm{Cor}}_k$ such that we have a commutative diagram of functors*



Proof. For any smooth schemes X, Y and any integer $n \geq 0$, we have a well-defined map $\alpha : \mathrm{Fr}_n(X, Y) \to \widetilde{\mathrm{Cor}}_k(X, Y)$ and therefore a well-defined map $\mathbb{Z}\mathrm{Fr}_n(X, Y) \to \widetilde{\mathrm{Cor}}_k(X, Y)$. Let $c = (U, \phi, f)$ be an explicit framed correspondence of level n with support Z of the form $Z = Z_1 \sqcup Z_2$. Let $c_i = (U_i, \phi_i, f_i)$ be the explicit framed correspondences with support Z_i obtained as in Definition 2.1.8. By construction, we get $\alpha(c) = \alpha(c_1) + \alpha(c_2)$ and it follows that $\alpha : \mathrm{Fr}_n(X, Y) \to \widetilde{\mathrm{Cor}}_k(X, Y)$ induces a homomorphism $\alpha' : \mathbb{Z}\mathrm{F}_n(X, Y) \to \widetilde{\mathrm{Cor}}_k(X, Y)$.

It remains then to show that the functors $\alpha : \mathrm{Fr}_k \to \widetilde{\mathrm{Cor}}_k$ and $\alpha' : \mathbb{Z}\mathrm{F}_*(k) \to \widetilde{\mathrm{Cor}}_k$ are well-defined, which amounts to prove that the respective compositions are preserved. Suppose then that (U, ϕ, f) is an explicit framed correspondence of level n between X and Y, and that (V, ψ, g) is an explicit framed correspondence of level m between Y and Z. We use the diagram

(2.1.12.a)



in which the squares are all cartesian. The composition of (U, ϕ, f) with (V, ψ, g) is given by $(W, (\phi \circ pr_U, \psi \circ pr_V), g \circ pr_V)$.

On the other hand, the morphisms $(p_X\alpha, f)\circ pr_U : W \to X \times Y$ and $(p_Y\beta, g)\circ pr_V : W \to Y \times Z$ yield a morphism $\rho : W \to X \times Y \times Z$ and then a diagram

(2.1.12.b)
$$\begin{array}{ccccc} & & W & \xrightarrow{pr_V} & V \\ & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle (p_Y\beta, g)} \\ W & \xrightarrow{\rho} & X\times Y\times Z & \xrightarrow{p_{Y\times Z}} & Y\times Z \\ \downarrow{\scriptstyle pr_U} & & \downarrow{\scriptstyle p_{X\times Y}} & & \downarrow \\ U & \xrightarrow[(p_X\alpha, f)]{} & X\times Y & \longrightarrow & Y \end{array}$$

in which all squares are cartesian. By base change ([CF14, Proposition 3.2, Remark 3.3]), we have $(p_{X\times Y})^*(p_X\alpha, f)_* = \rho_*(pr_U)^*$ and $(p_{Y\times Z})^*(p_Y\beta, g)_* = \rho_*(pr_V)^*$. By definition of the pull-back and the product, we have $(pr_U)^*(Z(\phi)) = Z(\phi\circ pr_U)$ and $(pr_V)^*(Z(\psi)) = Z(\psi\circ pr_V)$. It follows that
$$Z(\phi\circ pr_U, \psi\circ pr_V) = (pr_U)^*(Z(\phi))\cdot(pr_V)^*(Z(\psi)).$$
Finally, observe that there is a commutative diagram

$$\begin{array}{ccccc} W & & \xrightarrow{g\circ pr_V} & & Z \\ \| & & & & \uparrow \\ W & \xrightarrow{\rho} & X\times Y\times Z & \xrightarrow{p_{X\times Z}} & X\times Z \\ \| & & & & \downarrow \\ W & & \xrightarrow[p_X\circ\alpha\circ pr_U]{} & & X. \end{array}$$

Using these ingredients, we see that the composition is preserved. □

Remark 2.1.13. Note that the functor $\alpha' : \mathbb{Z}\mathrm{F}_*(k) \to \widetilde{\mathrm{Cor}}_k$ is additive. It follows from Example 2.1.11 that it is not faithful.

2.2. **Presheaves.** Let X be a smooth scheme. Recall from Example 2.1.11 that we have for any smooth scheme X an explicit framed correspondence σ_X of level 1 given by the triple $(\mathbb{A}^1_X, q, p_X)$ where q and p_X are respectively the projections onto $\mathbb{A}^1_k$ and X. The following definition can be found in [GP15, §1].

Definition 2.2.1. Let R be a ring. A presheaf of R-modules F on $\mathbb{Z}\mathrm{F}_*(k)$ is *quasi-stable* if for any smooth scheme X, the pull-back map $F(\sigma_X) : F(X) \to F(X)$ is an isomorphism. A quasi-stable presheaf is *stable* if $F(\sigma_X) : F(X) \to F(X)$ is the identity map for any X. We denote by $\mathrm{PSh}^{\mathrm{Fr}}(k, R)$ the category of presheaves on $\mathbb{Z}\mathrm{F}_*(k)$, by $\mathcal{Q}\mathrm{PSh}^{\mathrm{Fr}}(k, R)$ the category of quasi-stable presheaves on $\mathbb{Z}\mathrm{F}_*(k)$ and by $\mathcal{S}\mathrm{PSh}^{\mathrm{Fr}}(k, R)$ the category of stable presheaves.

Now, the functor $\alpha' : \mathbb{Z}\mathrm{F}_*(k) \to \widetilde{\mathrm{Cor}}_k$ induces a functor $\widetilde{\mathrm{PSh}}(k, R) \to \mathrm{PSh}^{\mathrm{Fr}}(k, R)$. By Example 2.1.11, this functor induces a functor
$$(\alpha')^* : \widetilde{\mathrm{PSh}}(k, R) \to \mathcal{S}\mathrm{PSh}^{\mathrm{Fr}}(k, R).$$
Recall next that a presheaf F on Sm_k is $\mathbb{A}^1$-invariant if the map $F(X) \to F(X\times\mathbb{A}^1)$ induced by the projection $X\times\mathbb{A}^1 \to X$ is an isomorphism for any smooth scheme X. A Nisnevich sheaf of abelian groups F is strictly $\mathbb{A}^1$-invariant if the homomorphisms $\mathrm{H}^i_{\mathrm{Nis}}(X, F) \to \mathrm{H}^i_{\mathrm{Nis}}(X\times\mathbb{A}^1, F)$ induced by the projection are isomorphisms for $i \geq 0$.

We can now state the main theorem of [GP15].

Theorem 2.2.2. *Let F be an $\mathbb{A}^1$-invariant quasi-stable $\mathbb{Z}\mathrm{F}_*(k)$-presheaf of R-modules F.*

1. *If the base field k is infinite, the associated Nisnevich sheaf F_{Nis} of R-modules is $\mathbb{A}^1$-invariant and quasi-stable.*
2. *Assume the base field k is infinite and perfect and the presheaf of R-modules F is in addition a Nisnevich sheaf. Then F is strictly $\mathbb{A}^1$-invariant, as a Nisnevich sheaf of R-modules.*

Proof. These results are proved in [GP15] in the case where $R = \mathbb{Z}$ but we can deduce from the latter the case of an arbitrary ring of coefficients, simply by forgetting the scalars. Let us give the details below.

Let $\varphi : \mathbb{Z} \to R$ be the unique morphism of rings attached with the ring R. We will consider the restriction of scalars functor

$$\varphi_* : R - \mathrm{mod} \to \mathbb{Z} - \mathrm{mod}.$$

As this functor admits both a left and a right adjoint (extension of scalars and coinduced module), it commutes with every limits and colimits. Besides, it is conservative.

For any category $\mathscr{S}$, one extends the functor φ_* to presheaves on $\mathscr{S}$ by applying it termwise:

$$\hat{\varphi}_* : \mathrm{PSh}(\mathscr{S}, R) \to \mathrm{PSh}(\mathscr{S}, \mathbb{Z}).$$

By construction, for any object X of $\mathscr{S}$ and any presheaf F of R-modules, we have the relation:

$$\Gamma(X, \hat{\varphi}_* F) = \varphi_*(\Gamma(X, F)). \tag{2.2.2.a}$$

Suppose now that $\mathscr{S}$ is endowed with a Grothendieck topology. Then, as φ_* is exact and commutes with products, the functor $\hat{\varphi}_*$ respects sheaves so that we get an induced functor:

$$\tilde{\varphi}_* : \mathrm{Sh}(\mathscr{S}, R) \to \mathrm{Sh}(\mathscr{S}, \mathbb{Z}).$$

Then, using again the fact φ_* commutes with colimits, and the classical formula defining the associated sheaf functor $a : \mathrm{PSh}(\mathscr{S}, ?) \to \mathrm{Sh}(\mathscr{S}, ?)$, we get, for any presheaf F of R-modules, a canonical isomorphism:

$$a\big(\hat{\varphi}_*(F)\big) \simeq \tilde{\varphi}_* a(F). \tag{2.2.2.b}$$

The fact φ_* is conservative together with relations (2.2.2.a) and (2.2.2.b) are sufficient to prove assertion (1). Indeed, these facts imply that it is sufficient to check the $\mathbb{A}^1$-invariance and quasi-stability of the presheaf $\hat{\varphi}_*(F)$ to conclude.

Let us come back to the abstract situation to prove the remaining relation. Recall that one can compute cohomology of an object X in $\mathscr{S}$ with coefficients in a sheaf F by considering the colimit of the Čech cohomology of the various hypercovers of X. Thus, relation (2.2.2.a) and the fact that φ_* commutes with colimits and products implies that, for any integer n, we get the following isomorphism of abelian groups, natural in X:

$$H^n\big(X, \tilde{\varphi}_*(F)\big) \simeq \varphi_* H^n\big(X, F\big). \tag{2.2.2.c}$$

Therefore to prove assertion (2), using the latter relation and once again the fact φ_* is conservative, we are reduced to consider the sheaf $\tilde{\varphi}_*(F)$ of abelian groups, which as a presheaf is just $\hat{\varphi}_*(F)$. Using relation (2.2.2.a), the latter is $\mathbb{A}^1$-invariant and quasi-stable so that we are indeed reduced to the case of abelian groups as expected. □

Remark 2.2.3. It is worth to mention that the considerations of the preceding proof are part of a standard machinery of changing coefficients for sheaves that can be applied in particular in our context.

We left the exact formulation to the reader, but describe it in the general case of a morphism of rings $\varphi : R \to R'$.

The restriction of scalars functor φ_* together with its left adjoint φ^* (extension of scalars) and its right adjoint $\varphi_!$ (associated coinduced module), can be extended (using the arguments of the preceding proof or similar arguments) to the category of MW-presheaves or MW-sheaves as two pairs of adjoint functors, written for simplicity here by (φ^*, φ_*) and $(\varphi_*, \varphi^!)$. Note that the extended functor φ_* will still be conservative and that the functor φ^* will be monoidal.

Moreover, using the definitions of the following section, the pair of adjoint functors (φ^*, φ_*) will induce adjoint functors on the associated effective and stable $\mathbb{A}^1$-derived categories, such that in particular the induced functor φ_* is still conservative on $\mathrm{DM}^{\mathrm{eff}}(k, R)$ and $\mathrm{DM}(k, R)$. Similarly the pair of adjoint functors $(\varphi_*, \varphi^!)$ can also be derived. Such considerations have been used for example in [CD16, §5.4].

3. MW-motivic complexes

3.1. **Derived category.** For any abelian category $\mathcal{A}$, we denote by $\mathrm{C}(\mathcal{A})$ the category of (possibly unbounded) complexes of objects of $\mathcal{A}$ and by $\mathrm{K}(\mathcal{A})$ the category of complexes with morphisms up to homotopy. Finally, we denote by $\mathrm{D}(\mathcal{A})$ the derived category of $\mathrm{C}(\mathcal{A})$. We refer to [Wei94, §10] for all these notions.

3.1.1. Recall from our notations that t is now either the Nisnevich or the étale topology.

As usual in motivic homotopy theory, our first task is to equip the category of complexes of MW-t-sheaves with a good model structure. This is done using the method of [CD09a], thanks to Lemma 1.2.6 and the fact that $\mathrm{Sh}_t(k, R)$ is a Grothendieck abelian category (Proposition 1.2.11(2)).

Except for one subtlety in the case of the étale topology, our construction is analogous to that of sheaves with transfers. In particular, the proof of the main point is essentially an adaptation of [CD09b, 5.1.26]. In order to make a short and streamlined proof, we first recall a few facts from model category theory.

3.1.2. We will be using the adjunction of Grothendieck abelian categories:

$$\tilde{\gamma}^* : \mathrm{Sh}_t(k, R) \leftrightarrows \widetilde{\mathrm{Sh}}_t(k, R) : \tilde{\gamma}_*$$

of Corollary 1.2.15. Recall from Lemma 1.2.3 that the functor $\tilde{\gamma}_*$ is conservative and exact.

First, there exists the so-called injective model structure on $\mathrm{C}(\mathrm{Sh}_t(k, R))$ and $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k, R))$ which is defined such that the cofibrations are monomorphisms (thus every object is cofibrant) and weak equivalences are quasi-isomorphisms (this is classical; see e.g. [CD09a, 2.1]). The fibrant objects for this model structure are called *injectively fibrant*.

Second, there exists the t-descent model structure on the category $\mathrm{C}(\mathrm{Sh}_t(k, R))$ (see [CD09a, Ex. 2.3]) characterized by the following properties:

- the class of *cofibrations* is given by the smallest class of morphisms of complexes closed under suspensions, pushouts, transfinite compositions and retracts generated by the inclusions

$$R_t(X) \to C\big(R_t(X) \xrightarrow{Id} R_t(X)\big)[-1] \tag{3.1.2.a}$$

for a smooth scheme X, where $R_t(X)$ is the free sheaf of R-modules on X.
- weak equivalences are quasi-isomorphisms.

Our aim is to obtain the same kind of model structure on the category $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k,R))$ of complexes of MW-t-sheaves. Let us recall from [CD09a] that one can describe nicely the fibrant objects for the t-descent model structure. This relies on the following definition for a complex K of t-sheaves:

- the complex K is *local* if for any smooth scheme X and any integer $n \in \mathbb{Z}$, the canonical map:

$$\mathrm{H}^n\big(K(X)\big) = \mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(k,R))}(R_t(X), K[n]) \to \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_t(k,R))}(R_t(X), K[n]) \tag{3.1.2.b}$$

  is an isomorphism;
- the complex K is *t-flasque* if for any smooth scheme X and any t-hypercover $p : \mathcal{X} \to X$, the induced map:

(3.1.2.c)

$$\mathrm{H}^n\big(K(X)\big) = \mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(k,R))}(R_t(X), K[n]) \xrightarrow{p^*} \mathrm{Hom}_{\mathrm{K}(\mathrm{Sh}_t(k,R))}(R_t(\mathcal{X}), K[n]) = \mathrm{H}^n\big(K(\mathcal{X})\big)$$

  is an isomorphism.

  Our reference for t-hypercovers is [DHI04]. Recall in particular that $\mathcal{X}$ is a simplicial scheme whose terms are arbitrary direct sums of smooth schemes. Then the notation $R_t(\mathcal{X})$ stands for the complex associated with the simplicial t-sheaves obtained by applying the obvious extension of the functor R_t to the category of direct sums of smooth schemes. Similarly, $K(\mathcal{X})$ is the total complex (with respect to products) of the obvious double complex.

Then, let us state for further reference the following theorem ([CD09a, Theorem 2.5]).

Theorem 3.1.3. *Let K be a complex of t-sheaves on the smooth site. Then the following three properties on K are equivalent:*

(i) *K is fibrant for the t-descent model structure,*
(ii) *K is local,*
(iii) *K is t-flasque.*

Under these equivalent conditions, we will say that K is t-fibrant.[2]

3.1.4. Consider now the case of MW-t-sheaves. We will define *cofibrations* in $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k,R))$ as in the previous paragraph by replacing R_t by $\tilde{R}_t$ in (3.1.2.a), i.e. the cofibrations are the morphisms in the smallest class of morphisms of complexes of MW-t-sheaves closed under suspensions, pushouts, transfinite compositions and retracts generated by the inclusions

$$\tilde{R}_t(X) \to C\big(\tilde{R}_t(X) \xrightarrow{Id} \tilde{R}_t(X)\big)[-1] \tag{3.1.4.a}$$

for a smooth scheme X. In particular, note that bounded above complexes of MW-t-sheaves whose components are direct sums of sheaves of the form $\tilde{R}_t(X)$ are cofibrant. This is easily seen by taking the push-out of (3.1.2.a) along the morphism $\tilde{R}_t(X) \to 0$.

Similarly, a complex K in $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k,R))$ will be called *local* (resp. *t-flasque*) if it satisfies the definition in the preceding paragraph after replacing respectively $\mathrm{Sh}_t(k,R)$ and R_t by $\widetilde{\mathrm{Sh}}_t(k,R)$ and $\tilde{R}_t$ in (3.1.2.b) (resp. (3.1.2.c)).

[2]Moreover, fibrations for the t-descent model structure are epimorphisms of complexes whose kernel is t-fibrant.

In order to show that cofibrations and quasi-isomorphisms define a model structure on $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k,R))$, we will have to prove the following result in analogy with the previous theorem.

Theorem 3.1.5. *Let K be a complex of* MW*-t-sheaves. Then the following conditions are equivalent:*

(i) *K is local;*
(ii) *K is t-flasque.*

The proof is essentially an adaptation of the proof of [CD09b, 5.1.26, 10.3.17], except that the case of the étale topology needs a new argument. It will be completed as a corollary of two lemmas, the first of which is a reinforcement of Lemma 1.2.6.

Lemma 3.1.6. *Let $p : \mathcal{X} \to X$ be a t-hypercover of a smooth scheme X. Then the induced map:*

$$p_* : \tilde{R}_t(\mathcal{X}) \to \tilde{R}_t(X)$$

is a quasi-isomorphism of complexes of MW*-t-sheaves.*

Proof. In fact, we have to prove that the complex $\tilde{R}_t(\mathcal{X})$ is acyclic in positive degree and that p_* induces an isomorphism $\mathrm{H}_0(\tilde{R}_t(\mathcal{X})) = \tilde{R}_t(X)$.[3] In particular, as these assertions only concerns the n-th homology sheaf of $\tilde{R}_t(\mathcal{X})$, we can always assume that $\mathcal{X} \simeq \mathrm{cosk}_n(\mathcal{X})$ for a large enough integer n (because these two simplicial objects have the same $(n-1)$-skeleton). In other words, we can assume that $\mathcal{X}$ is a bounded t-hypercover in the terminology of [DHI04, Def. 4.10].

As a consequence of the existence of the injective model structure, the category $\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))$ is naturally enriched over the derived category of R-modules. Let us denote by $\mathbf{R}\,\mathrm{Hom}^\bullet$ the corresponding Hom-object. We have only to prove that for any complex K of MW-t-sheaves, the natural map:

$$p^* : \mathbf{R}\,\mathrm{Hom}^\bullet(\tilde{R}_t(X), K) \to \mathbf{R}\,\mathrm{Hom}^\bullet(\tilde{R}_t(\mathcal{X}), K)$$

is an isomorphism in the derived category of R-modules. Because there exists an injectively fibrant resolution of any complex K, and $\mathbf{R}\,\mathrm{Hom}^\bullet$ preserves quasi-isomorphisms, it is enough to consider the case of an injectively fibrant complex K of MW-t-sheaves.

In this case, $\mathbf{R}\,\mathrm{Hom}^\bullet(-, K) = \mathrm{Hom}^\bullet(-, K)$ (as any complex is cofibrant for the injective model structure) and we are reduced to prove that the following complex of presheaves on the smooth site:

$$X \mapsto \mathrm{Hom}^\bullet(\tilde{R}_t(X), K)$$

satisfies t-descent with respect to bounded t-hypercovers *i.e.* sends bounded t-hypercovers $\mathcal{X}/X$ to quasi-isomorphisms of complexes of R-modules. But Lemma 1.2.6 (and the fact that K is injectively fibrant) tells us that this is the case when $\mathcal{X}$ is the t-hypercover associated with a t-cover. So we conclude using [DHI04, A.6]. □

The second lemma for the proof of Theorem 3.1.5 is based on the previous one.

Lemma 3.1.7. *Let us denote by* C, K, D *(respectively by* $\tilde{\mathrm{C}}$, $\tilde{\mathrm{K}}$, $\tilde{\mathrm{D}}$*) the category of complexes, complexes up to homotopy and derived category of the category* $\mathrm{Sh}_t(k,R)$ *(respectively* $\widetilde{\mathrm{Sh}}_t(k,R)$*).*

[3] Note that the second fact follows from Lemma 1.2.6 and the definition of t-hypercovers, but our proof works more directly.

Given a simplicial scheme $\mathcal{X}$ whose components are (possibly infinite) coproducts of smooth k-schemes and a complex K of MW-*t-sheaves, we consider the isomorphism of R-modules obtained from the adjunction $(\tilde{\gamma}^*, \tilde{\gamma}_*)$:*

$$\epsilon_{\mathcal{X},K} : \mathrm{Hom}_{\tilde{\mathrm{C}}}(\tilde{R}_t(\mathcal{X}), K) \to \mathrm{Hom}_{\mathrm{C}}(R_t(\mathcal{X}), \tilde{\gamma}_*(K)).$$

Then there exist unique isomorphisms $\epsilon'_{\mathcal{X},K}$ and $\epsilon''_{\mathcal{X},K}$ of R-modules making the following diagram commutative:

$$\begin{array}{ccc}
\mathrm{Hom}_{\tilde{\mathrm{C}}}(\tilde{R}_t(\mathcal{X}), K) & \xrightarrow{\epsilon_{\mathcal{X},K}} & \mathrm{Hom}_{\mathrm{C}}(R_t(\mathcal{X}), \tilde{\gamma}_*(K)) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\tilde{\mathrm{K}}}(\tilde{R}_t(\mathcal{X}), K) & \xrightarrow{\epsilon'_{\mathcal{X},K}} & \mathrm{Hom}_{\mathrm{K}}(R_t(\mathcal{X}), \tilde{\gamma}_*(K)) \\
{\scriptstyle \pi_{\mathcal{X},K}}\downarrow & & \downarrow{\scriptstyle \pi'_{\mathcal{X},K}} \\
\mathrm{Hom}_{\tilde{\mathrm{D}}}(\tilde{R}_t(\mathcal{X}), K) & \xrightarrow{\epsilon''_{\mathcal{X},K}} & \mathrm{Hom}_{\mathrm{D}}(R_t(\mathcal{X}), \tilde{\gamma}_*(K))
\end{array}$$

where the vertical morphisms are the natural localization maps.

Proof. The existence and unicity of $\epsilon'_{\mathcal{X},K}$ simply follows from the fact $\tilde{\gamma}^*$ and $\tilde{\gamma}_*$ are additive functors, so in particular $\epsilon_{\mathcal{X},K}$ is compatible with chain homotopy equivalences.

For the case of $\epsilon''_{\mathcal{X},K}$, we assume that the complex K is injectively fibrant. In this case, the map $\pi_{\mathcal{X},K}$ is an isomorphism. This already implies the existence and unicity of the map $\epsilon''_{\mathcal{X},K}$. Besides, according to the previous lemma and the fact that the map $\pi_{\mathcal{X},K}$ is an isomorphism natural in $\mathcal{X}$, we obtain that K is t-flasque (in the sense of Paragraph 3.1.4). Because $\epsilon'_{\mathcal{X},K}$ is an isomorphism natural in $\mathcal{X}$, we deduce that $\tilde{\gamma}_*(K)$ is t-flasque. In view of Theorem 3.1.3, it is t-fibrant. As $R_t(\mathcal{X})$ is cofibrant for the t-descent model structure on C, we deduce that $\pi'_{\mathcal{X},K}$ is an isomorphism. Therefore, $\epsilon''_{\mathcal{X},K}$ is an isomorphism.

The case of a general complex K now follows from the existence of an injectively fibrant resolution $K \to K'$ of any complex of MW-t-sheaves K. □

proof of Theorem 3.1.5. The previous lemma shows that the following conditions on a complex K of MW-t-sheaves are equivalent:

- K is local (resp. t-flasque) in $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k, R))$;
- $\tilde{\gamma}_*(K)$ is local (resp. t-flasque) in $\mathrm{C}(\mathrm{Sh}_t(k, R))$.

Then Theorem 3.1.5 follows from Theorem 3.1.3. □

Here is an important corollary (analogous to [VSF00, chap. 5, 3.1.8]) which is simply a restatement of Lemma 3.1.7.

Corollary 3.1.8. *Let K be a complex of* MW-*t-sheaves and X be a smooth scheme. Then for any integer $n \in \mathbb{Z}$, there exists a canonical isomorphism, functorial in X and K:*

$$\mathrm{Hom}_{\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))}(\tilde{R}_t(X), K[n]) = \mathbb{H}^n_t(X, K)$$

where the right hand side stands for the t-hypercohomology of X with coefficients in the complex $\tilde{\gamma}_*(K)$ (obtained after forgetting MW-*transfers).*

Recall that the category $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k, R))$ is symmetric monoidal, with tensor product induced as usual from the tensor product on $\widetilde{\mathrm{Sh}}_t(k, R)$ (see Paragraph 1.2.14).

Corollary 3.1.9. *The category* $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k,R))$ *has a proper cellular model structure (see* [Hir03, 12.1.1 and 13.1.1]*) with quasi-isomorphisms as weak equivalences and cofibrations as defined in Paragraph 3.1.4. Moreover, the fibrations for this model structure are epimorphisms of complexes whose kernel are t-flasque (or equivalently local) complexes of* MW*-t-sheaves. Finally, this is a symmetric monoidal model structure; in other words, the tensor products (resp. internal Hom functor) admits a total left (resp. right) derived functor.*

Proof. Each claim is a consequence of [CD09a, 2.5, 5.5 and 3.2], applied to the Grothendieck abelian category $\widetilde{\mathrm{Sh}}_t(k,R)$ with respect to the descent structure $(\mathcal{G},\mathcal{H})$ (see [CD09a, Def. 2.2] for the notion of descent structure) defined as follows:

- $\mathcal{G}$ is the class of MW-t-sheaves of the form $\tilde{R}_t(X)$ for smooth scheme X;
- $\mathcal{H}$ is the (small) family of complexes which are cones of morphisms $p_* : \tilde{R}_t(\mathcal{X}) \to \tilde{R}_t(X)$ for a t-hypercover p.

Indeed, $\mathcal{G}$ generates the category $\widetilde{\mathrm{Sh}}_t(k,R)$ (see after Definition 1.2.13) and the condition to be a descent structure is given by Theorem 3.1.5.

In the end, we can apply [CD09a, 3.2] to derive the tensor product as the tensor structure is weakly flat (in the sense of [CD09a, §3.1]) due to the preceding definition and formula (1.2.14.a). □

Remark 3.1.10. We can follow the procedure of [MVW06, §8] to compute the tensor product of two bounded above complexes of MW-t-sheaves. This follows from [CD09a, Proposition 3.2] and the fact that bounded above complexes of MW-t-sheaves whose components are direct sums of representable sheaves are cofibrant.

Definition 3.1.11. The model structure on $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k,R))$ of the above corollary is called the *t-descent model structure.*

In particular, the category $\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))$ is a triangulated symmetric closed monoidal category.

3.1.12. We also deduce from the t-descent model structure that the vertical adjunctions of Corollary 1.2.15 induce Quillen adjunctions with respect to the t-descent model structure on each category involved and so admit derived functors as follows:

(3.1.12.a)
$$\begin{array}{ccccc}
\mathrm{D}(\mathrm{Sh}(k,R)) & \underset{\tilde{\gamma}_*}{\overset{\mathbf{L}\tilde{\gamma}^*}{\rightleftarrows}} & \mathrm{D}(\widetilde{\mathrm{Sh}}(k,R)) & \underset{\pi_*}{\overset{\mathbf{L}\pi^*}{\rightleftarrows}} & \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}(k,R)) \\
\mathbf{R}\mathcal{O}\uparrow\downarrow a & & \mathbf{R}\mathcal{O}\uparrow\downarrow \tilde{a} & & \mathbf{R}\mathcal{O}\uparrow\downarrow a^{\mathrm{tr}} \\
\mathrm{D}(\mathrm{Sh}_{\acute{e}t}(k,R)) & \underset{\tilde{\gamma}_{\acute{e}t*}}{\overset{\mathbf{L}\tilde{\gamma}^*_{\acute{e}t}}{\rightleftarrows}} & \mathrm{D}(\widetilde{\mathrm{Sh}}_{\acute{e}t}(k,R)) & \underset{\pi_{\acute{e}t*}}{\overset{\mathbf{L}\pi^*_{\acute{e}t}}{\rightleftarrows}} & \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}_{\acute{e}t}(k,R))
\end{array}$$

where we have not indicated the topology in the notations when it is the Nisnevich topology, denoted by $(a,\mathcal{O})$ for the adjoint pair associated étale sheaf and forgetful functor and similarly for MW-transfers and transfers. When the functors are exact, they are trivially derived and so we have used the same notation than for their counterpart for sheaves.

Note that by definition, the left adjoints in this diagram are all monoidal functors and sends the object represented by a smooth scheme X (say in degree 0) to the analogous object

(3.1.12.b)
$$\mathbf{L}\tilde{\gamma}^*\big(R_t(X)\big) = \tilde{R}_t(X), \mathbf{L}\pi^*\big(\tilde{R}_t(X)\big) = R^{\mathrm{tr}}(X).$$

3.2. **The $\mathbb{A}^1$-derived category.** We will now adapt the usual $\mathbb{A}^1$-homotopy machinery to our context.

Definition 3.2.1. We define the category $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}}(k,R)$ of MW-motivic complexes for the topology t as the localization of the triangulated category $\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))$ with respect to the localizing triangulated subcategory[4] $\mathcal{T}_{\mathbb{A}^1}$ generated by complexes of the form:

$$\cdots 0 \to \tilde{R}_t(\mathbb{A}^1_X) \xrightarrow{p_*} \tilde{R}_t(X) \to 0 \cdots$$

where p is the projection of the affine line relative to an arbitrary smooth k-scheme X. As usual, we define the MW-*motive* $\tilde{\mathrm{M}}(X)$ associated to a smooth scheme X as the complex concentrated in degree 0 and equal to the representable MW-t-sheaf $\tilde{R}_t(X)$. Respecting our previous conventions, we mean the Nisnevich topology when the topology is not indicated.

According to this definition, it is formal that the localizing triangulated subcategory $\mathcal{T}_{\mathbb{A}^1}$ is stable under the derived tensor product of $\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))$ (cf. Corollary 3.1.9). In particular, it induces a triangulated monoidal structure on $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}}(k,R)$.

3.2.2. As usual, we can apply the classical techniques of localization to our triangulated categories and also to our underlying model structure. So a complex of MW-t-sheaf E is called $\mathbb{A}^1$-*local* if for any smooth scheme X and any integer $i \in \mathbb{Z}$, the induced map

$$\mathrm{Hom}_{\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))}(\tilde{R}_t(X), E[i]) \to \mathrm{Hom}_{\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))}(\tilde{R}_t(\mathbb{A}^1_X), E[i])$$

is an isomorphism. In view of Corollary 3.1.8, it amounts to ask that the t-cohomology of $\tilde{\gamma}_*(E)$ is $\mathbb{A}^1$-invariant, or in equivalent words, that E is strictly $\mathbb{A}^1$-local.

Applying Neeman's localization theorem (see [Nee01b, 9.1.19]),[5] the category $\widetilde{\mathrm{DM}}_t^{\mathrm{eff}}(k,R)$ can be viewed as the full subcategory of $\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))$ whose objects are the $\mathbb{A}^1$-local complexes. Equivalently, the canonical functor $\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R)) \to \widetilde{\mathrm{DM}}_t^{\mathrm{eff}}(k,R)$ admits a fully faithful right adjoint whose essential image consists in $\mathbb{A}^1$-local complexes. In particular, one deduces formally the existence of an $\mathbb{A}^1$-localization functor $L_{\mathbb{A}^1} : \mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R)) \to \mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))$.

Besides, we get the following proposition by applying the general left Bousfield localization procedure for proper cellular model categories (see [Hir03, 4.1.1]). We say that a morphism ϕ of $\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))$ is an *weak* $\mathbb{A}^1$-*equivalence* if for any $\mathbb{A}^1$-local object E, the induced map $\mathrm{Hom}(\phi, E)$ is an isomorphism.

Proposition 3.2.3. *The category* $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k,R))$ *has a symmetric monoidal model structure with weak* $\mathbb{A}^1$*-equivalences as weak equivalences and cofibrations as defined in Paragraph 3.1.4. This model structure is proper and cellular. Moreover, the fibrations for this model structure are epimorphisms of complexes whose kernel are* t*-flasque and* $\mathbb{A}^1$*-local complexes.*

The resulting model structure on $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k,R))$ will be called the $\mathbb{A}^1$-*local model structure*. The proof of the proposition follows formally from Corollary 3.1.9 by the usual localization procedure of model categories, see [CD09a, §3] for details. Note that the tensor product of two bounded above complexes can be computed as in the derived category.

[4]Recall that according to Neeman [Nee01b, 3.2.6], localizing means stable by coproducts.

[5]Indeed recall the derived category of $\widetilde{\mathrm{Sh}}_t(k,R)$ is a well generated triangulated category according to [Nee01a, Th. 0.2].

3.2.4. As a consequence of the above discussion, the category $\widetilde{\mathrm{DM}}^{\mathrm{eff}}_t(k,R)$ is a triangulated symmetric monoidal closed category. Besides, it is clear that the functors of Corollary (1.2.15) induce Quillen adjunctions for the $\mathbb{A}^1$-local model structures. Equivalently, Diagram (3.1.12.a) is compatible with $\mathbb{A}^1$-localization and induces adjunctions of triangulated categories:

$$\begin{array}{ccccc} \mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(k,R) & \underset{\tilde\gamma_*}{\overset{\mathbf{L}\tilde\gamma^*}{\rightleftarrows}} & \widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R) & \underset{\pi_*}{\overset{\mathbf{L}\pi^*}{\rightleftarrows}} & \mathrm{DM}^{\mathrm{eff}}(k,R) \\ {\scriptstyle \mathbf{R}\mathcal{O}}\uparrow\downarrow{\scriptstyle a} & & {\scriptstyle \mathbf{R}\mathcal{O}}\uparrow\downarrow{\scriptstyle \tilde a} & & {\scriptstyle \mathbf{R}\mathcal{O}}\uparrow\downarrow{\scriptstyle a^{tr}} \\ \mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1,\mathrm{\acute{e}t}}(k,R) & \underset{\tilde\gamma_{\mathrm{\acute{e}t}*}}{\overset{\mathbf{L}\tilde\gamma^*_{\mathrm{\acute{e}t}}}{\rightleftarrows}} & \widetilde{\mathrm{DM}}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(k,R) & \underset{\pi_{\mathrm{\acute{e}t}*}}{\overset{\mathbf{L}\pi^*_{\mathrm{\acute{e}t}}}{\rightleftarrows}} & \mathrm{DM}^{\mathrm{eff}}_{\mathrm{\acute{e}t}}(k,R). \end{array} \tag{3.2.4.a}$$

In this diagram, the left adjoints are all monoidal and send the different variant of motives represented by a smooth scheme X to the analogous motive. In particular,

$$\mathbf{L}\pi^*\tilde{\mathrm{M}}(X) = \mathrm{M}(X).$$

Also, the functors $\tilde\gamma_{t*}$ and π_{t*} for $t = \mathrm{Nis}, \mathrm{\acute{e}t}$ (or following our conventions, $t = \varnothing, \mathrm{\acute{e}t}$) are conservative. Note moreover that their analogues in diagram (3.1.12.a) preserve $\mathbb{A}^1$-local objects and so commute with the $\mathbb{A}^1$-localization functor. Therefore one deduces from Morel's $\mathbb{A}^1$-connectivity theorem [Mor05] the following result.

Theorem 3.2.5. *Assume k is a perfect field. Let E be a complex of* MW*-sheaves concentrated in positive degrees. Then the complex $L_{\mathbb{A}^1}E$ is concentrated in positive degrees.*

Indeed, to check this, one needs only to apply the functor $\tilde\gamma_*$ as it is conservative and so we are reduced to Morel's theorem [Mor05, 6.1.8].

Corollary 3.2.6. *Under the assumption of the previous theorem, the triangulated category $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$ admits a unique t-structure such that the functor $\tilde\gamma_* : \widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R) \to \mathrm{D}\left(\widetilde{\mathrm{Sh}}(k,R)\right)$ is t-exact.*

Note that the truncation functor on $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$ is obtained by applying to an $\mathbb{A}^1$-local complex the usual truncation functor of $\mathrm{D}(\widetilde{\mathrm{Sh}}(k,R))$ and then the $\mathbb{A}^1$-localization functor.

Definition 3.2.7. If k is a perfect field, the t-structure on $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$ obtained in the previous corollary will be called the *homotopy t-structure*.

Remark 3.2.8. Of course, the triangulated categories $\mathrm{DM}^{\mathrm{eff}}(k,R)$ and $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(k,R)$ are also equipped with t-structures, called in each cases homotopy t-structures — in the first case, it is due to Voevodsky and in the second to Morel.

It is clear from the definitions that the functors $\tilde\gamma_*$ and π_* in Diagram (3.2.4.a) are t-exact.

As in the case of Nisnevich sheaves with transfers, we can describe nicely $\mathbb{A}^1$-local objects and the $\mathbb{A}^1$-localization functor due to the following theorem.

Theorem 3.2.9. *Assume k is an infinite perfect field. Let F be an $\mathbb{A}^1$-invariant* MW*-presheaf. Then, the associated* MW*-sheaf $\tilde a(F)$ is strictly $\mathbb{A}^1$-invariant. Moreover, the Zariski sheaf associated with F coincides with $\tilde a(F)$ and the natural map*

$$\mathrm{H}^i_{\mathrm{Zar}}(X,\tilde a(F)) \to \mathrm{H}^i_{\mathrm{Nis}}(X,\tilde a(F))$$

is an isomorphism for any smooth scheme X.

Proof. Recall that we have a functor $(\alpha')^* : \widetilde{\mathrm{PSh}}(k, R) \to \mathcal{S}\mathrm{PSh}^{\mathrm{Fr}}(k, R)$. In view of Theorem 2.2.2, the Nisnevich sheaf associated to the presheaf $(\alpha')^*(F)$ is strictly $\mathbb{A}^1$-invariant and quasi-stable. It follows that $\tilde{a}(F)$ is strictly $\mathbb{A}^1$-invariant by Corollary 1.2.15. Now, a strictly $\mathbb{A}^1$-invariant sheaf admits a Rost-Schmid complex in the sense of [Mor12, §5] and the result follows from [Mor12, Corollary 5.43]. □

Remark 3.2.10. It would be good to have a proof intrinsic to MW-motives of the above theorem. This will be worked out in Håkon Andreas Kolderup's thesis ([Kol17]).

As in the case of Voeovdsky's theory of motivic complexes, this theorem has several important corollaries.

Corollary 3.2.11. *Assume k is an infinite perfect field. Then, a complex E of* MW*-sheaves is $\mathbb{A}^1$-local if and only if its homology (Nisnevich) sheaves are $\mathbb{A}^1$-invariant. The heart of the homotopy t-structure (Def. 3.2.7) on $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k, R)$ consists of $\mathbb{A}^1$-invariant* MW*-sheaves.*

The proof is classical. We recall it for the comfort of the reader.

Proof. Let K be an $\mathbb{A}^1$-local complex of MW-sheaves over k. Let us show that its homology sheaves are $\mathbb{A}^1$-invariant. According to the existence of the $\mathbb{A}^1$-local model structure on $\mathrm{C}(\mathrm{Sh}_t(k, R))$ (Proposition 3.2.3), there exist a Nisnevich fibrant and $\mathbb{A}^1$-local complex K' and a weak $\mathbb{A}^1$-equivalence:

$$K \xrightarrow{\phi} K'.$$

As K and K' are $\mathbb{A}^1$-local, the map ϕ is a quasi-isomorphism. In particular, we can replace K by K'. In other words, we can assume K is Nisnevich fibrant thus local (Theorem 3.1.3). Then we get:

$$\mathrm{H}^n\big(K(X)\big) \simeq \mathrm{Hom}_{\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))}\big(\tilde{R}_t(X), K[n]\big)$$

according to the definition of local in Paragraph 3.1.4. This implies in particular that the cohomology presheaves of K are $\mathbb{A}^1$-invariant. We conclude using Theorem 3.2.9.

Assume conversely that K is a complex of MW-sheaves whose homology sheaves are $\mathbb{A}^1$-invariant. Let us show that K is $\mathbb{A}^1$-local. According to Corollary 3.1.8, we need only to show its Nisnevich hypercohomology is $\mathbb{A}^1$-invariant. Then we apply the Nisnevich hypercohomology spectral sequence for any smooth scheme X:

$$E_2^{p,q} = \mathrm{H}^p_{\mathrm{Nis}}(X, \mathrm{H}^q_{\mathrm{Nis}}(K)) \Rightarrow \mathbb{H}^n_{\mathrm{Nis}}(X, K).$$

As the cohomological dimension of the Nisnevich topology is bounded by the dimension of X, the E_2-term is concentrated in degree $p \in [0, \dim(X)]$ and the spectral sequence converges ([SV00, Theorem 0.3]). It is moreover functorial in X. Therefore it is enough to show the map induced by the projection

$$\mathrm{H}^p_{\mathrm{Nis}}(X, \mathrm{H}^q_{\mathrm{Nis}}(K)) \to \mathrm{H}^p_{\mathrm{Nis}}(\mathbb{A}^1_X, \mathrm{H}^q_{\mathrm{Nis}}(K))$$

is an isomorphism to conclude. By assumption the sheaf $\mathrm{H}^q_{\mathrm{Nis}}(K)$ is $\mathbb{A}^1$-invariant so Theorem 3.2.9 applies again.

As the functor $\tilde{\gamma}_*$ is t-exact by Corollary 3.2.6, the conclusion about the heart of $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k, R)$ follows. □

Corollary 3.2.12. *Let K be an $\mathbb{A}^1$-local complex of* MW*-sheaves. Then we have*

$$\mathbb{H}^i_{\mathrm{Zar}}(X, K) = \mathbb{H}^i_{\mathrm{Nis}}(X, K)$$

for any smooth scheme X and any $i \in \mathbb{Z}$.

Proof. The proof uses the same principle as in the previous result. Let us first consider the change of topology adjunction:

$$\alpha^* : \mathrm{Sh}_{\mathrm{Zar}}(k,R) \leftrightarrows \mathrm{Sh}_{\mathrm{Nis}}(k,R) : \alpha_*$$

where α^* is obtained using the functor "associated Nisnevich sheaf functor" and α_* is just the forgetful functor. This adjunction can be derived (using for example the injective model structures) and induces:

$$\alpha^* : \mathrm{D}(\mathrm{Sh}_{\mathrm{Zar}}(k,R)) \leftrightarrows \mathrm{D}(\mathrm{Sh}_{\mathrm{Nis}}(k,R)) : \mathbf{R}\alpha_*$$

— note indeed that α^* is exact. Coming back to the statement of the corollary, we have to show that the adjunction map:

$$\tilde{\gamma}_*(K) \to \mathbf{R}\alpha_*\alpha^*(\tilde{\gamma}_*(K)) \tag{3.2.12.a}$$

is a quasi-isomorphism. Let us denote abusively by K the sheaf $\tilde{\gamma}_*(K)$. Note that this will be harmless as $\tilde{\gamma}_*(\mathrm{H}^q_{\mathrm{Nis}}(K)) = \mathrm{H}^q_{\mathrm{Nis}}(\tilde{\gamma}_*K)$. With this abuse of notation, one has canonical identifications:

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\mathrm{Nis}}(k,R))}(\tilde{\mathbb{Z}}_t(X), \mathbf{R}\alpha_*\alpha^*(K)) = \mathrm{H}^p_{\mathrm{Zar}}(X,K)$$
$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}_{\mathrm{Nis}}(k,R))}(\tilde{\mathbb{Z}}_t(X), \mathbf{R}\alpha_*\alpha^*(\mathrm{H}^q_{\mathrm{Nis}}(K))) = \mathrm{H}^p_{\mathrm{Zar}}(X, \mathrm{H}^q_{\mathrm{Nis}}(K))$$

Using now the tower of truncation of K for the standard t-structure on $\mathrm{D}(\widetilde{\mathrm{Sh}}_t(k,R))$ — or equivalently $\mathrm{D}(\mathrm{Sh}_t(k,R))$ — and the preceding identifications, one gets a spectral sequence:

$${}^{\mathrm{Zar}}E_2^{p,q} = \mathrm{H}^p_{\mathrm{Zar}}(X, \mathrm{H}^q_{\mathrm{Nis}}(K)) \Rightarrow \mathbb{H}^{p+q}_{\mathrm{Zar}}(X,K)$$

and the morphism (3.2.12.a) induces a morphism of spectral sequence:

$$\begin{aligned} E_2^{p,q} = \mathrm{H}^p_{\mathrm{Nis}}(X, \mathrm{H}^q_{\mathrm{Nis}}(K)) \longrightarrow {}^{\mathrm{Zar}}E_2^{p,q} &= \mathrm{H}^p_{\mathrm{Zar}}(X, \mathrm{H}^q_{\mathrm{Nis}}(K)) \\ \Rightarrow \mathbb{H}^{p+q}_{\mathrm{Nis}}(X,K) &\longrightarrow \mathbb{H}^{p+q}_{\mathrm{Zar}}(X,K). \end{aligned}$$

The two spectral sequences converge (as the Zariski and cohomological dimension of X are bounded). According to Theorem 3.2.9, the map on the E_2-term is an isomorphism so the map on the abutment must be an isomorphism and this concludes. □

3.2.13. Following Voevodsky, given a complex E of MW-sheaves, we define its associated Suslin complex as the following complex of sheaves:[6]

$$\mathrm{C}^{sing}_*(E) := \underline{\mathrm{Hom}}(\tilde{R}_t(\Delta^*), E)$$

where Δ^* is the standard cosimplicial scheme.

Corollary 3.2.14. *Assume k is an infinite perfect field.*

Then for any complex E of MW*-sheaves, there exists a canonical quasi-isomorphism:*

$$L_{\mathbb{A}^1}(E) \simeq \mathrm{C}^{sing}_*(E).$$

Proof. Indeed, according to Corollary 3.2.11, it is clear that $\mathrm{C}^{sing}_*(E)$ is $\mathbb{A}^1$-local. Thus the canonical map

$$c : E \to \mathrm{C}^{sing}_*(E)$$

induces a morphism of complexes:

$$L_{\mathbb{A}^1}(c) : L_{\mathbb{A}^1}(E) \to L_{\mathbb{A}^1}(\mathrm{C}^{sing}_*(E)) = \mathrm{C}^{sing}_*(E).$$

[6] Explicitly, this complex associates to a smooth scheme X the total complex (for coproducts) of the bicomplex $E(\Delta^* \times X)$.

As $\Delta^n \simeq \mathbb{A}^n_k$, one checks easily that the map c is an $\mathbb{A}^1$-weak equivalence. Therefore the map $L_{\mathbb{A}^1}(c)$ is an $\mathbb{A}^1$-weak equivalence of $\mathbb{A}^1$-local complexes, thus a quasi-isomorphism. □

3.2.15. As usual, one defines the Tate object in $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$ by the formula

$$\tilde{R}(1) := \tilde{\mathrm{M}}(\mathbb{P}^1_k)/\tilde{\mathrm{M}}(\{\infty\})[-2] \simeq \tilde{\mathrm{M}}(\mathbb{A}^1_k)/\tilde{\mathrm{M}}(\mathbb{A}^1_k - \{0\})[-2] \simeq \tilde{\mathrm{M}}(\mathbb{G}_m)/\tilde{\mathrm{M}}(\{1\})[-1].$$

Then one defines the effective MW-motivic cohomology of a smooth scheme X in degree $(n,i) \in \mathbb{Z} \times \mathbb{N}$:[7] as

$$\mathrm{H}^{n,i}_{\mathrm{MW}}(X,R) = \mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)}(\tilde{R}(X), \tilde{R}(i)[n])$$

where $\tilde{R}(i) = \tilde{R}(1)^{\otimes i}$.

Corollary 3.2.16. *Assume k is an infinite perfect field. The effective* MW*-motivic cohomology defined above coincides with the generalized motivic cohomology groups defined (for $R = \mathbb{Z}$) in* [CF14, Definition 6.6].

Proof. By Corollary 3.2.14, the Suslin complex of $R(i)$ is $\mathbb{A}^1$-local. It follows then from Corollary 3.2.12 that its Nisnevich hypercohomology and its Zariski hypercohomology coincide. We conclude using [FØ16, Corollary 4.0.4]. □

We now spend a few lines in order to compare ordinary motivic cohomology with MW-motivic cohomology, following [CF14, Definition 6.8]. In this part, we suppose that R is flat over $\mathbb{Z}$. If X is a smooth scheme, recall from [CF14, Definition 5.15] that the presheaf with MW-transfers $\mathrm{I}\tilde{\mathrm{c}}(X)$ defined by

$$\mathrm{I}\tilde{\mathrm{c}}(X)(Y) = \varinjlim_T \mathrm{H}^d_T\big(X \times Y, \mathrm{I}^{d+1}, \omega_Y\big)$$

fits in an exact sequence

$$0 \to \mathrm{I}\tilde{\mathrm{c}}(X) \to \tilde{\mathrm{c}}(X) \to \mathbb{Z}^{\mathrm{tr}}(X) \to 0$$

As $\tilde{\mathrm{c}}(X)$ is a sheaf in the Zariski topology, it follows that $\mathrm{I}\tilde{\mathrm{c}}(X)$ is also such a sheaf. We can also consider the Zariski sheaf $\mathrm{I}\tilde{\mathrm{c}}_R(X)$ defined by

$$\mathrm{I}\tilde{\mathrm{c}}_R(X)(Y) = \mathrm{I}\tilde{\mathrm{c}}(X)(Y) \otimes R.$$

Definition 3.2.17. We denote by $\mathrm{I}\tilde{R}_t(X)$ the t-sheaf associated to the presheaf $\mathrm{I}\tilde{\mathrm{c}}_R(X)$.

In view of Proposition 1.2.11, $\mathrm{I}\tilde{\mathrm{c}}_R(X)$ has MW-transfers. Moreover, sheafification being exact and R being flat, we have an exact sequence

$$0 \to \mathrm{I}\tilde{R}_t(X) \to \tilde{R}_t(X) \to R^{\mathrm{tr}}(X) \to 0$$

of MW-t-sheaves (note the slight abuse of notation when we write $R^{\mathrm{tr}}(X)$ in place of $\pi^t_* R^{\mathrm{tr}}(X)$). We deduce from [MVW06, Lemma 2.13] an exact sequence

$$0 \to \mathrm{I}\tilde{R}_t\{q\} \to \tilde{R}_t\{q\} \to R^{\mathrm{tr}}_t\{q\} \to 0$$

for any $q \in \mathbb{N}$, where $\mathrm{I}\tilde{R}_t\{m\} = \mathrm{I}\tilde{R}_t(\mathbb{G}_m^{\wedge m})$.

Definition 3.2.18. For any $q \in \mathbb{N}$, we set $\mathrm{I}\tilde{R}_t(q) = \mathrm{I}\tilde{R}_t\{q\}[-q]$ in $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$ and

$$\mathrm{H}^{p,q}_{\mathrm{I}}(X,R) = \mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)}(\tilde{M}(X), \mathrm{I}\tilde{R}_t(q)[p])$$

for any smooth scheme X.

[7] Negative twists will be introduced in the next section.

As Zariski cohomology and Nisnevich cohomology coincide by Corollary 3.2.12, these groups coincide with the ones defined in [CF14, Definition 6.8] (when $R = \mathbb{Z}$). We now construct a long exact sequence

$$\ldots \to \mathrm{H}_{\mathrm{I}}^{p,q}(X,R) \to \mathrm{H}_{\mathrm{MW}}^{p,q}(X,R) \to \mathrm{H}^{p,q}(X,R) \to \mathrm{H}_{\mathrm{I}}^{p+1,q}(X,R) \to \ldots$$

for any smooth scheme X and any $q \in \mathbb{N}$. The method we use was explained to us by Grigory Garkusha, whom we warmly thank.

For a smooth scheme X, let $B(X)$ be the quotient $\tilde{R}_t(X)/\mathrm{I}\tilde{R}_t(X)$. This association defines a presheaf B with MW-transfers, whose associated sheaf is $R^{\mathrm{tr}}\{q\}$. Next, observe that Suslin's construction is exact on presheaves, and therefore we obtain an exact sequence of complexes of MW-presheaves

$$0 \to \mathrm{C}_*^{sing}(\mathrm{I}\tilde{R}\{q\}) \to \mathrm{C}_*^{sing}(\tilde{R}\{q\}) \to \mathrm{C}_*^{sing}(B) \to 0.$$

In particular, we obtain an exact sequence of complexes of R-modules

$$0 \to \mathrm{C}_*^{sing}(\mathrm{I}\tilde{R}\{q\})(Z) \to \mathrm{C}_*^{sing}(\tilde{R}\{q\})(Z) \to \mathrm{C}_*^{sing}(B)(Z) \to 0.$$

for any local scheme Z. Now, [FØ16, Corollary 4.0.4] shows that the morphism $\mathrm{C}_*^{sing}(B)(Z) \to \mathrm{C}_*^{sing}(R^{\mathrm{tr}}\{q\})(Z)$ is a quasi-isomorphism. Consequently, we obtain an exact triangle in the derived category of MW-sheaves

$$\mathrm{C}_*^{sing}(\mathrm{I}\tilde{R}\{q\}) \to \mathrm{C}_*^{sing}(\tilde{R}\{q\}) \to \mathrm{C}_*^{sing}(R^{\mathrm{tr}}\{q\}) \to \mathrm{C}_*^{sing}(\mathrm{I}\tilde{R}\{q\})[1]$$

and the existence of the long exact sequence

$$\ldots \to \mathrm{H}_{\mathrm{I}}^{p,q}(X,R) \to \mathrm{H}_{\mathrm{MW}}^{p,q}(X,R) \to \mathrm{H}^{p,q}(X,R) \to \mathrm{H}_{\mathrm{I}}^{p+1,q}(X,R) \to \ldots$$

follows.

3.2.19. To end this section, we now discuss the effective geometric MW-motives, which are built as in the classical case.

Definition 3.2.20. One defines the category $\widetilde{\mathrm{DM}}_{gm}^{\mathrm{eff}}(k,\mathbb{Z})$ of *geometric effective motives* over the field k as the pseudo-abelianization of the Verdier localization of the homotopy category $\mathrm{K}^b(\widetilde{\mathrm{Cor}}_k)$ associated to the additive category $\widetilde{\mathrm{Cor}}_k$ with respect to the thick triangulated subcategory containing complexes of the form:

(1) $\ldots \to [W] \xrightarrow{k_*-g_*} [V] \oplus [U] \xrightarrow{f_*+j_*} [X] \to \ldots$ for an elementary Nisnevich distinguished square of smooth schemes:

$$\begin{array}{ccc} W & \xrightarrow{k} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & X; \end{array}$$

(2) $\ldots \to [\mathbb{A}^1_X] \xrightarrow{p_*} [X] \to \ldots$ where p is the canonical projection and X is a smooth scheme.

It is clear that the natural map $\mathrm{K}^b(\widetilde{\mathrm{Cor}}_k) \to \mathrm{D}(\widetilde{\mathrm{Sh}}(k,\mathbb{Z}))$ induces a canonical functor:

$$\iota : \widetilde{\mathrm{DM}}_{gm}^{\mathrm{eff}}(k,\mathbb{Z}) \to \widetilde{\mathrm{DM}}^{\mathrm{eff}}(k).$$

As a consequence of [CD09a, Theorem 6.2] (see also Example 6.3 of *op. cit.*) and Theorem 3.2.9, we get the following result.

Proposition 3.2.21. *The functor ι is fully faithful and its essential image coincides with each one of the following subcategories:*

- *the subcategory of compact objects of $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)$;*
- *the smallest thick triangulated subcategory of $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)$ which contains the motives $\tilde{\mathrm{M}}(X)$ for a smooth scheme X.*

Moreover, when k is an infinite perfect field, one can reduce in point (1) in the definition of $\widetilde{\mathrm{DM}}^{\mathrm{eff}}_{gm}(k, \mathbb{Z})$ to consider those complexes associated to a Zariski open cover $U \cup V$ of a smooth scheme X.

Remark 3.2.22. Note that the previous proposition states in particular that the objects of the form $\tilde{\mathrm{M}}(X)$ for a smooth scheme X are compact in $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k, R)$. Therefore, they form a family of compact generators of $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k, R)$ in the sense that $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k, R)$ is equal to its smallest triangulated category containing $\tilde{\mathrm{M}}(X)$ for a smooth scheme X and stable under coproducts.

3.3. **The stable $\mathbb{A}^1$-derived category.**

3.3.1. As usual in motivic homotopy theory, we now describe the $\mathbb{P}^1$-stabilization of the category of MW-motivic complexes for the topology t (again, $t = \mathrm{Nis}, \mathrm{ét}$).

Recall that the Tate twist in $\widetilde{\mathrm{DM}}^{\mathrm{eff}}_t(k, R)$ is defined by one of the following equivalent formulas:

$$\tilde{R}(1) := \tilde{\mathrm{M}}(\mathbb{P}^1_k)/\tilde{\mathrm{M}}(\{\infty\})[-2] \simeq \tilde{\mathrm{M}}(\mathbb{A}^1_k)/\tilde{\mathrm{M}}(\mathbb{A}^1_k - \{0\})[-2] \simeq \tilde{\mathrm{M}}(\mathbb{G}_m)/\tilde{\mathrm{M}}(\{1\})[-1].$$

In the construction of the $\mathbb{P}^1$-stable category as well as in the study of the homotopy t-structure, it is useful to introduce a redundant notation of $\mathbb{G}_m$-twist:

$$\tilde{R}\{1\} := \tilde{\mathrm{M}}(\mathbb{G}_m)/\tilde{\mathrm{M}}(\{1\})$$

so that $\tilde{R}\{1\} = \tilde{R}(1)[1]$. The advantage of this definition is that we can consider $\tilde{R}\{1\}$ as a MW-t-sheaf instead of a complex. For $m \geq 1$, we set $\tilde{R}\{m\} = \tilde{R}\{1\}^{\otimes m}$ and we observe that $\tilde{R}(m) = \tilde{R}\{m\}[-m]$.

Let us recall the general process of $\otimes$-inversion of the Tate twist in the context of model categories, as described in our particular case in [CD09a, §7]. We define the category $\widetilde{\mathrm{Sp}}_t(k, R)$ of (abelian) Tate MW-t-spectra as the additive category whose object are couples $(\mathcal{F}_*, \epsilon_*)$ where $\mathcal{F}_*$ is a sequence of MW-t-sheaves such that $\mathcal{F}_n$ is equipped with an action of the symmetric group $\mathfrak{S}_n$ and, for any integer $n \geq 0$,

$$\epsilon_n : (\mathcal{F}_n\{1\} := \mathcal{F}_n \otimes \tilde{R}\{1\}) \to \mathcal{F}_{n+1}$$

is a morphism of MW-t-sheaves, called the *suspension map*, such that the iterated morphism

$$\mathcal{F}_n\{m\} \to \mathcal{F}_{n+m}$$

is $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant for any $n \geq 0$ and $m \geq 1$. (see *loc. cit.* for more details). The morphisms in $\widetilde{\mathrm{Sp}}_t(k, R)$ between couples $(\mathcal{F}_*, \epsilon_*)$ and $(\mathcal{G}_*, \tau_*)$ are sequences of $\mathfrak{S}_n$-equivariant morphisms $f_n : \mathcal{F}_n \to \mathcal{G}_n$ such that the following diagram of $\mathfrak{S}_n \times \mathfrak{S}_m$-equivariant maps

$$\begin{array}{ccc} \mathcal{F}_n\{m\} & \longrightarrow & \mathcal{F}_{n+m} \\ {\scriptstyle f_n\{m\}}\downarrow & & \downarrow{\scriptstyle f_{n+m}} \\ \mathcal{G}_n\{m\} & \longrightarrow & \mathcal{G}_{n+m} \end{array}$$

is commutative for any $n \geq 0$ and $m \geq 1$.

This is a Grothendieck abelian, closed symmetric monoidal category with tensor product described in [CD09a, §7.3, §7.4] (together with [ML98, Chapter VII, §4, Exercise 6]). Further, we have a canonical adjunction of abelian categories:

$$\Sigma^\infty : \widetilde{\mathrm{Sh}}_t(k,R) \leftrightarrows \widetilde{\mathrm{Sp}}_t(k,R) : \Omega^\infty \tag{3.3.1.a}$$

such that $\Sigma^\infty(\mathcal{F}) = (\mathcal{F}\{n\})_{n\geq 0}$ with the obvious suspension maps and $\Omega^\infty(\mathcal{F}_*, \epsilon_*) = \mathcal{F}_0$. Recall the Tate MW-$t$-spectrum $\Sigma^\infty(\mathcal{F})$ is called the *infinite spectrum* associated with $\mathcal{F}$. The functor Σ^∞ is monoidal (cf. [CD09a, §7.8]).

One can define the $\mathbb{A}^1$-stable cohomology of a complex $\mathbb{E} = (\mathbb{E}_*, \sigma_*)$ of Tate MW-t-spectra, for any smooth scheme X and any couple $(n,m) \in \mathbb{Z}^2$:

$$\mathrm{H}^{n,m}_{st-\mathbb{A}^1}(X,\mathbb{E}) := \varinjlim_{r\geq \max(0,-m)} \Big(\mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)}(\tilde{\mathrm{M}}(X)\{r\}, \mathbb{E}_{m+r}[n]) \Big) \tag{3.3.1.b}$$

where the transition maps are induced by the suspension maps σ_* and $\tilde{\mathrm{M}}(X)\{r\} = \tilde{\mathrm{M}}(X) \otimes \tilde{R}\{r\}$.

Definition 3.3.2. We say that a morphism $\varphi : \mathbb{E} \to \mathbb{F}$ of complexes of Tate MW-t-spectra is a *stable* $\mathbb{A}^1$*-equivalence* if for any smooth scheme X and any couple $(n,m) \in \mathbb{Z}^2$, the induced map

$$\varphi_* : \mathrm{H}^{n,m}_{st-\mathbb{A}^1}(X,\mathbb{E}) \to \mathrm{H}^{n,m}_{st-\mathbb{A}^1}(X,\mathbb{F})$$

is an isomorphism.

One defines the category $\widetilde{\mathrm{DM}}_t(k,R)$ of MW-*motivic spectra* for the topology t as the localization of the triangulated category $\mathrm{D}(\widetilde{\mathrm{Sp}}_t(k,R))$ with respect to stable $\mathbb{A}^1$-derived equivalences.

3.3.3. As usual, we can describe the above category as the homotopy category of a suitable model category.

First, recall that we can define the negative twist of an abelian Tate MW-t-spectrum $\mathcal{F}_*$ by the formula, for $n > 0$:

$$[(\mathcal{F}_*)\{-n\}]_m = \begin{cases} \mathbb{Z}[\mathfrak{S}_m] \otimes_{\mathbb{Z}[\mathfrak{S}_{m-n}]} \mathcal{F}_{m-n} & \text{if } m \geq n. \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.3.a}$$

with the obvious suspension maps. Note for future references that one has according to this definition and that of the tensor product:

$$\mathcal{F}_*\{-n\} = \mathcal{F}_* \otimes (\Sigma^\infty \tilde{R})\{-n\}. \tag{3.3.3.b}$$

We then define the class of cofibrations in $\mathrm{C}(\widetilde{\mathrm{Sp}}_t(k,R))$ as the smallest class of morphisms of complexes closed under suspensions, negative twists, pushouts, transfinite compositions and retracts generated by the infinite suspensions of cofibrations in $\mathrm{C}(\widetilde{\mathrm{Sh}}_t(k,R))$.

Applying [CD09a, Prop. 7.13], we get:

Proposition 3.3.4. *The category* $\mathrm{C}(\widetilde{\mathrm{Sp}}_t(k,R))$ *of complexes of Tate* MW-t*-spectra has a symmetric monoidal model structure with stable* $\mathbb{A}^1$*-equivalences as weak equivalences and cofibrations as defined above. This model structure is proper and cellular.*

Moreover, the fibrations for this model structure are epimorphisms of complexes whose kernel is a complex $\mathbb{E}$ *such that:*

- *for any* $n \geq 0$, $\mathbb{E}_n$ *is a* t*-flasque and* $\mathbb{A}^1$*-local complex;*

- *for any $n \geq 0$, the map obtained by adjunction from the obvious suspension map:*
$$\mathbb{E}_{n+1} \to \mathbf{R}\,\underline{\mathrm{Hom}}(\tilde{R}\{1\}, \mathbb{E}_n)$$
is an isomorphism.

Therefore, the homotopy category $\widetilde{\mathrm{DM}}_t(k,R)$ is a triangulated symmetric monoidal category with internal Hom. The adjoint pair (3.3.1.a) can be derived and induces an adjunction of triangulated categories:
$$\Sigma^\infty : \widetilde{\mathrm{DM}}^{\mathrm{eff}}_t(k,R) \leftrightarrows \widetilde{\mathrm{DM}}_t(k,R) : \Omega^\infty.$$
As a left derived functor of a monoidal functor, the functor Σ^∞ is monoidal. Slightly abusing notation, we still denote by $\tilde{\mathrm{M}}(X)$ the MW-motivic spectrum $\Sigma^\infty(\tilde{\mathrm{M}}(X))$.

3.3.5. By construction, the MW-motivic spectrum $\tilde{R}\{1\}$, and thus $\tilde{R}(1)$ is $\otimes$-invertible in $\widetilde{\mathrm{DM}}_t(k,R)$ (see [CD09a, Prop. 7.14]). Moreover, using formulas (3.3.3.a) and (3.3.3.b), one obtains a canonical map in $\widetilde{\mathrm{Sp}}_t(k,R)$:
$$\phi : \Sigma^\infty\tilde{R}\{1\} \otimes \big((\Sigma^\infty\tilde{R})\{-1\}\big) \to \big(\Sigma^\infty\tilde{R}\{1\}\big)\{-1\} \to \tilde{R}.$$
The following proposition justifies the definition of Paragraph 3.3.3 of negative twists.

Proposition 3.3.6. *The map ϕ is a stable $\mathbb{A}^1$-equivalence. The* MW*-motive $(\Sigma^\infty\tilde{R})\{-1\}$ is the tensor inverse of $\tilde{R}\{1\}$. For any* MW*-t-spectra $\mathbb{E}$, the map obtained by adjunction from $\mathbb{E}\otimes\phi$:*
$$\mathbb{E}\{-1\} \to \mathbf{R}\,\underline{\mathrm{Hom}}(\tilde{R}\{1\}, \mathbb{E})$$
is a stable $\mathbb{A}^1$-equivalence.

Proof. The first assertion follows from a direct computation using the definition of stable $\mathbb{A}^1$-equivalences via the $\mathbb{A}^1$-stable cohomology (3.3.1.b). The other assertions are formal consequences of the first one. □

As in the effective case, we derive from the functors of Corollary 1.2.15 Quillen adjunctions for the stable $\mathbb{A}^1$-local model structures and consequently adjunctions of triangulated categories:

(3.3.6.a)
$$\begin{array}{ccccc}
\mathrm{D}_{\mathbb{A}^1}(k,R) & \underset{\tilde{\gamma}_*}{\overset{\mathbf{L}\tilde{\gamma}^*}{\rightleftarrows}} & \widetilde{\mathrm{DM}}(k,R) & \underset{\pi_*}{\overset{\mathbf{L}\pi^*}{\rightleftarrows}} & \mathrm{DM}(k,R) \\
\mathbf{R}\mathcal{O}\uparrow\downarrow a & & \mathbf{R}\mathcal{O}\uparrow\downarrow \tilde{a} & & \mathbf{R}\mathcal{O}\uparrow\downarrow a^{tr} \\
\mathrm{D}_{\mathbb{A}^1,\text{ét}}(k,R) & \underset{\tilde{\gamma}_{\text{ét}*}}{\overset{\mathbf{L}\tilde{\gamma}^*_{\text{ét}}}{\rightleftarrows}} & \widetilde{\mathrm{DM}}_{\text{ét}}(k,R) & \underset{\pi_{\text{ét}*}}{\overset{\mathbf{L}\pi^*_{\text{ét}}}{\rightleftarrows}} & \mathrm{DM}_{\text{ét}}(k,R)
\end{array}$$

where each left adjoints is monoidal and sends a motive of a smooth scheme to the corresponding motive.

Formally, one can compute morphisms of MW-motivic spectra as follows.

Proposition 3.3.7. *For any smooth scheme X, any pair of integers $(n,m)\in\mathbb{Z}^2$ and any* MW*-motivic spectrum $\mathbb{E}$, one has a canonical functorial isomorphism:*
$$\begin{aligned}\mathrm{Hom}_{\widetilde{\mathrm{DM}}(k,R)}(\tilde{\mathrm{M}}(X), \mathbb{E}(n)[m]) &\simeq \mathrm{H}^{n,m}_{st-\mathbb{A}^1}(X,\mathbb{E}) \\ &= \varinjlim_{r\geq \max(0,-m)}\Big(\mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)}(\tilde{\mathrm{M}}(X)\{r\}, \mathbb{E}_{m+r}[n])\Big)\end{aligned}$$

Proof. This follows from [Ayo07, 4.3.61 and 4.3.79] which can be applied because of Remark 3.2.22 and the fact the cyclic permutation of order 3 acts on $\tilde{R}\{3\}$ as the identity in $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$ (the proof of this fact is postponed until Corollary 4.1.5). □

In fact, as in the case of motivic complexes, one gets a better result if we assume the base field k is infinite perfect. This is due to the following analogue of Voevodsky's cancellation theorem [Voe10], which is proved in [FØ16].

Theorem 3.3.8. *Let k be a perfect infinite field. Then for any complexes K and L of* MW*-sheaves, the morphism*

$$\mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)}(K,L) \to \mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)}(K(1),L(1)),$$

obtained by tensoring with the Tate twist, is an isomorphism.

We then formally deduce the following corollary from this result.

Corollary 3.3.9. *If k is an infinite perfect field, the functor*

$$\Sigma^\infty : \widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R) \to \widetilde{\mathrm{DM}}(k,R)$$

is fully faithful.

4. MW-motivic cohomology

4.1. MW-motivic cohomology as Ext-groups.

Given our construction of the triangulated category $\widetilde{\mathrm{DM}}(k,R)$, we can now define, in the style of Beilinson, a generalization of motivic cohomology as follows.

Definition 4.1.1. We define the MW-*motivic cohomology* of a smooth scheme X in degree $(n,i) \in \mathbb{Z}^2$ and coefficients in R as:

$$\mathrm{H}^{n,i}_{\mathrm{MW}}(X,R) = \mathrm{Hom}_{\widetilde{\mathrm{DM}}(k,R)}(\tilde{\mathrm{M}}(X), \tilde{R}(i)[n]).$$

As usual, we deduce a cup-product on MW-motivic cohomology. We define its étale variant by taking morphisms in $\widetilde{\mathrm{DM}}_{\text{ét}}(k,R)$. Then we derive from the preceding (essentially) commutative diagram the following morphisms of cohomology theories, all compatible with products and pullbacks:

(4.1.1.a)
$$\begin{array}{ccccc} \mathrm{H}^{n,i}_{\mathbb{A}^1}(X,R) & \longrightarrow & \mathrm{H}^{n,i}_{\mathrm{MW}}(X,R) & \longrightarrow & \mathrm{H}^{n,i}_{M}(X,R) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^{n,i}_{\mathbb{A}^1,\text{ét}}(X,R) & \longrightarrow & \mathrm{H}^{n,i}_{\mathrm{MW},\text{ét}}(X,R) & \longrightarrow & \mathrm{H}^{n,i}_{L}(X,R). \end{array}$$

where $\mathrm{H}_{\mathbb{A}^1}(X,R)$ and $\mathrm{H}_{\mathbb{A}^1,\text{ét}}(X,R)$ are respectively Morel's stable $\mathbb{A}^1$-derived cohomology and its étale version while $\mathrm{H}^{n,i}_{M}(X,R)$ and $\mathrm{H}^{n,i}_{L}(X,R)$ are respectively the motivic cohomology and the Lichtenbaum motivic cohomology (also called *étale motivic cohomology*).

Gathering all the informations we have obtained in the previous section on MW-motivic complexes, we get the following computation.

Proposition 4.1.2. *Assume that k is an infinite perfect field. For any smooth scheme X and any couple of integers $(n,m) \in \mathbb{Z}^2$, the* MW*-motivic cohomology $\mathrm{H}^{n,m}_{\mathrm{MW}}(X,\mathbb{Z})$ defined previously coincides with the generalized motivic cohomology defined in* [CF14].

More explicitly,

$$\mathrm{H}^{n,m}_{\mathrm{MW}}(X,\mathbb{Z}) = \begin{cases} \mathrm{H}^n_{\mathrm{Zar}}(X,\tilde{\mathbb{Z}}(m)) & \text{if } m>0,\\ \mathrm{H}^n_{\mathrm{Zar}}(X,\mathbf{K}^{\mathrm{MW}}_0) & \text{if } m=0,\\ \mathrm{H}^{n-m}_{\mathrm{Zar}}(X,\mathbf{W}) & \text{if } m<0 \end{cases}$$

where $\mathbf{K}^{\mathrm{MW}}_0$ *(resp.* $\mathbf{W}$*) is the unramified sheaf associated with Milnor-Witt K-theory in degree* 0 *(resp. unramified Witt sheaf) – see* [Mor12, §3].

Proof. The cases $m>0$ and $m=0$ are clear from the previous corollary and Corollary 3.2.16.

Consider the case $m<0$. Then we can use the following computation:

$$\begin{aligned}\mathrm{H}^{n,m}_{\mathrm{MW}}(X,\mathbb{Z}) &= \mathrm{Hom}_{\widetilde{\mathrm{DM}}(k,R)}\big(\tilde{\mathrm{M}}(X),\tilde{\mathbb{Z}}\{m\}[n-m]\big)\\ &= \mathrm{Hom}_{\widetilde{\mathrm{DM}}(k,R)}\big(\tilde{\mathrm{M}}(X)\otimes\tilde{\mathbb{Z}}\{-m\},\tilde{\mathbb{Z}}[n-m]\big)\\ &= \mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)}\big(\tilde{\mathrm{M}}(X),\mathbf{R}\,\underline{\mathrm{Hom}}(\tilde{\mathbb{Z}}\{-m\},\tilde{\mathbb{Z}})[n-m]\big)\end{aligned}$$

where the last identification follows from the preceding corollary and the usual adjunction.

As the MW-motivic complex $\tilde{\mathbb{Z}}\{-m\}$ is cofibrant and the motivic complex $\tilde{\mathbb{Z}} = \mathbf{K}^{\mathrm{MW}}_0$ is Nisnevich-local and $\mathbb{A}^1$-invariant (cf. [CF14, Ex. 4.4] and [Fas08, Cor. 11.3.3]), we get:

$$\mathbf{R}\,\underline{\mathrm{Hom}}(\tilde{\mathbb{Z}}\{-m\},\tilde{\mathbb{Z}}) = \underline{\mathrm{Hom}}(\tilde{\mathbb{Z}}\{-m\},\tilde{\mathbb{Z}})$$

and this last sheaf is isomorphic to $\mathbf{W}$ according to [CF14, Lemma 5.23]. So the assertion now follows from Corollaries 3.2.11 and 3.1.8. □

4.1.3. We next prove a commutativity result for MW-motivic cohomology. First, note that the sheaf $\tilde{R}\{1\} = \tilde{R}(\mathbb{G}_{m,k})/\tilde{R}(\{1\})$ is a direct factor of $\tilde{R}(\mathbb{G}_{m,k})$ and that the permutation map

$$\sigma : \tilde{R}(\mathbb{G}_{m,k})\otimes\tilde{R}(\mathbb{G}_{m,k}) \to \tilde{R}(\mathbb{G}_{m,k})\otimes\tilde{R}(\mathbb{G}_{m,k})$$

given by the morphism of schemes $\mathbb{G}_{m,k}\times\mathbb{G}_{m,k}\to\mathbb{G}_{m,k}\times\mathbb{G}_{m,k}$ defined by $(x,y)\mapsto(y,x)$ induces a map

$$\sigma : \tilde{R}\{1\}\otimes\tilde{R}\{1\}\to\tilde{R}\{1\}\otimes\tilde{R}\{1\}.$$

On the other hand, recall from Remark 1.1.3 (5), that $\widetilde{\mathrm{Cor}}_k$ is $\mathbf{K}^{\mathrm{MW}}_0(k)$-linear. In particular, the class of $\epsilon = -\langle -1\rangle \in \mathbf{K}^{\mathrm{MW}}_0(k)$ (and its corresponding element in $\mathbf{K}^{\mathrm{MW}}_0(k)\otimes_{\mathbb{Z}} R$ that we still denote by ϵ) yields a MW-correspondence

$$\epsilon = \epsilon\cdot Id : \tilde{R}(\mathbb{G}_{m,k})\otimes\tilde{R}(\mathbb{G}_{m,k}) \to \tilde{R}(\mathbb{G}_{m,k})\otimes\tilde{R}(\mathbb{G}_{m,k})$$

that also induces a MW-correspondence

$$\epsilon : \tilde{R}\{1\}\otimes\tilde{R}\{1\}\to\tilde{R}\{1\}\otimes\tilde{R}\{1\}.$$

We can now state the following lemma ([FØ16, Lemma 3.0.6]).

Lemma 4.1.4. *The* MW*-correspondences* σ *and* ϵ *are* $\mathbb{A}^1$*-homotopic.*

As an obvious corollary, we obtain the following result.

Corollary 4.1.5. *For any* $i,j\in\mathbb{Z}$*, the switch* $\tilde{R}(i)\otimes\tilde{R}(j)\to\tilde{R}(j)\otimes\tilde{R}(i)$ *is* $\mathbb{A}^1$*-homotopic to* $\langle(-1)^{ij}\rangle$.

Proof. By definition, we have $\tilde{R}(i) := \tilde{R}\{i\}[-i]$ and $\tilde{R}(j) := \tilde{R}\{j\}[-j]$. We know from the previous lemma that the switch $\tilde{R}\{i\}\otimes\tilde{R}\{j\} \to \tilde{R}\{j\}\otimes\tilde{R}\{i\}$ is homotopic to $(\epsilon)^{ij}$. The result now follows from the compatibility isomorphisms for tensor triangulated categories (see e.g. [MVW06, Exercise 8A.2]) and the fact that $(-1)^{ij}(\epsilon)^{ij} = \langle(-1)^{ij}\rangle$. ☐

Theorem 4.1.6. *Let $i, j \in \mathbb{Z}$ be integers. For any smooth scheme X, the pairing*

$$\mathrm{H}^{p,i}_{\mathrm{MW}}(X,R)\otimes \mathrm{H}^{q,j}_{\mathrm{MW}}(X,R)\to \mathrm{H}^{p+q,i+j}_{\mathrm{MW}}(X,R)$$

is $(-1)^{pq}\langle(-1)^{ij}\rangle$-commutative.

Proof. The proof of [MVW06, Theorem 15.9] applies mutatis mutandis. ☐

4.2. **Comparison with Chow-Witt groups.**

4.2.1. *The naive Milnor-Witt presheaf.* . Let S be a ring and let $S^\times$ be the group of units in S. We define the naive Milnor-Witt presheaf of S as in the case of fields by considering the free $\mathbb{Z}$-graded algebra $A(S)$ generated by the symbols $[a]$ with $a \in S^\times$ in degree 1 and a symbol η in degree -1 subject to the usual relations:

(1) $[ab] = [a] + [b] + \eta[a][b]$ for any $a, b \in S^\times$.
(2) $[a][1-a] = 0$ for any $a \in S^\times \setminus \{1\}$.
(3) $\eta[a] = [a]\eta$ for any $a \in S^\times$
(4) $\eta(\eta[-1] + 2) = 0$.

This definition is clearly functorial in S and it follows that we obtain a presheaf of $\mathbb{Z}$-graded algebras on the category of smooth schemes via

$$X \mapsto \mathrm{K}^{\mathrm{MW}}_*(\mathcal{O}(X)).$$

We denote by $\mathbf{K}^{\mathrm{MW}}_{*,\mathrm{naive}}$ the associated Nisnevich sheaf of graded $\mathbb{Z}$-algebras and observe that this definition naturally extends to essentially smooth k-schemes. Our next aim is to show that this naive definition in fact coincides with the definition of the unramified Milnor-Witt K-theory sheaf given in [Mor12, §3] (see also [CF14, §1]). Indeed, let X be a smooth integral scheme. The ring homomorphism $\mathcal{O}(X) \to k(X)$ induces a ring homomorphism $\mathrm{K}^{\mathrm{MW}}_*(\mathcal{O}(X)) \to \mathrm{K}^{\mathrm{MW}}_*(k(X))$ and it is straightforward to check that elements in the image are unramified, i.e. that the previous homomorphism induces a ring homomorphism $\mathrm{K}^{\mathrm{MW}}_*(\mathcal{O}(X)) \to \mathbf{K}^{\mathrm{MW}}_*(X)$. By the universal property of the associated sheaf, we obtain a morphism of sheaves

$$\mathbf{K}^{\mathrm{MW}}_{*,\mathrm{naive}} \to \mathbf{K}^{\mathrm{MW}}_*.$$

If X is an essentially smooth local k-scheme, it follows from [GSZ16, Theorem 6.3] that the map $\mathbf{K}^{\mathrm{MW}}_{*,\mathrm{naive}}(X) \to \mathbf{K}^{\mathrm{MW}}_*(X)$ is an isomorphism, showing that the above morphism is indeed an isomorphism.

4.2.1. A comparison map. Let now X be a smooth connected scheme and let $a \in \mathcal{O}(X)^\times$ be an invertible global section. It corresponds to a morphism $X \to \mathbb{G}_{m,k}$ and in turn to an element in $\widetilde{\mathrm{Cor}}_k(X, \mathbb{G}_{m,k})$ yielding a map

$$s : \mathcal{O}(X)^\times \to \mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)}(\tilde{M}(X), \tilde{\mathbb{Z}}\{1\}) = \mathrm{H}^{1,1}_{\mathrm{MW}}(X,\mathbb{Z}).$$

Consider next the correspondence $\eta[t] \in \widetilde{\mathrm{CH}}^0(X \times \mathbb{G}_{m,k}) = \widetilde{\mathrm{Cor}}_k(X \times \mathbb{G}_{m,k}, \operatorname{Spec} k)$ and observe that it restricts trivially when composed with the map $X \to X \times \mathbb{G}_{m,k}$ given by

$x \mapsto (x,1)$. It follows that we obtain an element

$$s(\eta) \in \mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)}(\tilde{M}(X) \otimes \tilde{\mathbb{Z}}\{1\}, \tilde{\mathbb{Z}}) = \mathrm{Hom}_{\widetilde{\mathrm{DM}}(k,R)}(\tilde{M}(X), \tilde{\mathbb{Z}}\{-1\}) = \mathrm{H}_{\mathrm{MW}}^{-1,-1}(X,\mathbb{Z}).$$

Using the product structure of the cohomology ring, we finally obtain a (graded, functorial in X) ring homomorphism

$$s : A(\mathcal{O}(X)) \to \bigoplus_{n\in\mathbb{Z}} \mathrm{H}_{\mathrm{MW}}^{n,n}(X,\mathbb{Z}),$$

where $A(\mathcal{O}(X))$ is the free $\mathbb{Z}$-graded (unital, associative) algebra generated in degree 1 by the elements $s(a)$ and in degree -1 by $s(\eta)$.

Theorem 4.2.2. *Let X be a smooth scheme. Then, the graded ring homomorphism*

$$s : A(\mathcal{O}(X)) \to \bigoplus_{n\in\mathbb{Z}} \mathrm{H}_{\mathrm{MW}}^{n,n}(X,\mathbb{Z})$$

induces a graded ring homomorphism

$$s : \mathrm{K}_*^{\mathrm{MW}}(\mathcal{O}(X)) \to \bigoplus_{n\in\mathbb{Z}} \mathrm{H}_{\mathrm{MW}}^{n,n}(X,\mathbb{Z})$$

which is functorial in X.

Proof. We have to check that the four relations defining Milnor-Witt K-theory hold in the graded ring on the right-hand side. First, note that Theorem 4.1.6 yields $\epsilon s(\eta)s(a) = s(a)s(\eta)$ and the third relation follows from the fact that $\epsilon s(\eta) = s(\eta)$ by construction. Observe next that $s(\eta)s(-1)+1 = \langle -1\rangle$ by [FØ16, Lemma 6.0.1] and it follows easily that $s(\eta)(s(\eta)s(-1)+2) = 0$. Next, consider the multiplication map

$$m : \mathbb{G}_{m,k} \times \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$$

and the respective projections on the j-th factor

$$p_j : \mathbb{G}_{m,k} \times \mathbb{G}_{m,k} \to \mathbb{G}_{m,k}$$

for $j = 1,2$. They all define correspondences that we still denote by the same symbols and it is straightforward to check that $m - p_1 - p_2$ defines a morphism $\tilde{\mathbb{Z}}\{1\} \otimes \mathbb{Z}\{1\} \to \tilde{\mathbb{Z}}\{1\}$ in $\widetilde{\mathrm{Cor}}_k$. It follows from [FØ16, Lemma 6.0.2] that this correspondence corresponds to $s(\eta)$ through the isomorphism

$$\mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)}(\tilde{\mathbb{Z}}\{1\}, \tilde{\mathbb{Z}}) \to \mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)}(\tilde{\mathbb{Z}}\{1\} \otimes \mathbb{Z}\{1\}, \tilde{\mathbb{Z}}\{1\})$$

given by the cancellation theorem. As a corollary, we see that the defining relation (1) of Milnor-Witt K-theory is satisfied in $\bigoplus_{n\in\mathbb{Z}} \mathrm{H}_{\mathrm{MW}}^{n,n}(X,\mathbb{Z})$. Indeed, if $a, b \in \mathcal{O}(X)^\times$, then $s(a)s(b)$ is represented by the morphism $X \to \mathbb{G}_{m,k} \times \mathbb{G}_{m,k}$ corresponding to (a,b). Applying $m - p_1 - p_2$ to this correspondence, we get $s(ab) - s(a) - s(b)$ which is $s(\eta)s(a)s(b)$ by the above discussion.

To check that the Steinberg relation holds in the right-hand side, we first consider the morphism

$$\mathbb{A}^1 \setminus \{0,1\} \to \mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\}$$

defined by $a \mapsto (a, 1-a)$. Composing with the correspondence $\tilde{M}(\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1 \setminus \{0\}) \to \tilde{M}((\mathbb{G}_{m,k})^{\wedge 2})$, we obtain a morphism

$$\tilde{M}(\mathbb{A}^1 \setminus \{0,1\}) \to \tilde{M}((\mathbb{G}_{m,k})^{\wedge 2}).$$

We can perform the same computation in $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(k)$ where this morphism is trivial by [HK01] and we conclude that it is also trivial in $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k)$ by applying the functor $\mathbf{L}\tilde{\gamma}^*$. □

For any $p,q\in\mathbb{Z}$, we denote by $\mathbf{H}^{p,q}_{\mathrm{MW}}$ the (Nisnevich) sheaf associated to the presheaf $X\mapsto \mathrm{H}^{p,q}_{\mathrm{MW}}(X,\mathbb{Z})$. The homomorphism of the previous theorem induces a morphism on induced sheaves and we have the following result.

Theorem 4.2.3. *The homomorphism of sheaves of graded rings*

$$s:\mathbf{K}^{\mathrm{MW}}_*\to\bigoplus_{n\in\mathbb{Z}}\mathbf{H}^{n,n}_{\mathrm{MW}}$$

is an isomorphism.

Proof. Let L/k be a finitely generated field extension. Then, it follows from [CF14, Theorem 6.19] that the homomorphism s_L is an isomorphism. Now, the presheaf on $\widetilde{\mathrm{Cor}}_k$ given by $X\mapsto\bigoplus_{n\in\mathbb{Z}}\mathrm{H}^{n,n}_W(X,\mathbb{Z})$ is homotopy invariant by definition. It follows from Theorem 2.2.2 that the associated sheaf is strictly $\mathbb{A}^1$-invariant. Now, $\mathbf{K}^{\mathrm{MW}}_*$ is also strictly $\mathbb{A}^1$-invariant and it follows from [Mor12, Definition 2.1, Remark 2.3, Theorem 2.11] that s is an isomorphism if and only if s_L is an isomorphism for any finitely generated field extension L/k. □

Theorem 4.2.4. *For any smooth scheme X and any integers $p,n\in\mathbb{Z}$, the hypercohomology spectral sequence induces isomorphisms*

$$\mathrm{H}^{p,n}_{\mathrm{MW}}(X,\mathbb{Z})\to\mathrm{H}^{p-n}(X,\mathbf{K}^{\mathrm{MW}}_n)$$

provided $p\geq 2n-1$.

Proof. In view of Proposition 4.1.2, we may suppose that $n>0$. For any $q\in\mathbb{Z}$, we denote by $\mathbf{H}^{q,n}_{\mathrm{MW}}$ the Nisnevich sheaf associated to the presheaf $X\mapsto\mathrm{H}^{q,n}_{\mathrm{MW}}(X,\mathbb{Z})$ and observe that they coincide with the cohomology sheaves of the complexes $\tilde{\mathbb{Z}}(n)$. Now, the latter are concentrated in cohomological levels $\leq n$ and it follows that $\mathbf{H}^{q,n}_{\mathrm{MW}}=0$ if $q>n$. On the other hand, the sheaves $\mathbf{H}^{q,n}_{\mathrm{MW}}$ are strictly $\mathbb{A}^1$-invariant, and as such admit a Gersten complex whose components in degree m are of the form

$$\bigoplus_{x\in X^{(p)}}(\mathbf{H}^{q,n}_{\mathrm{MW}})_{-p}(k(x),\wedge^p(\mathfrak{m}_x/\mathfrak{m}_x^2)^*)$$

by [Mor12, §5]. By the cancellation theorem 3.3.8, we have a canonical isomorphism of sheaves $\mathbf{H}^{q-p,n-p}_{\mathrm{MW}}\simeq(\mathbf{H}^{q,n}_{\mathrm{MW}})_{-p}$ and it follows that the terms in the Gersten resolution are of the form

$$\bigoplus_{x\in X^{(p)}}(\mathbf{H}^{q-p,n-p}_{\mathrm{MW}})(k(x),\wedge^p(\mathfrak{m}_x/\mathfrak{m}_x^2)^*).$$

If $p\geq n$, then $\tilde{\mathbb{Z}}(n-p)\simeq\mathbf{K}^{\mathrm{MW}}_{n-p}[p-n]$ and it follows that $\mathbf{H}^{q-p,n-p}_{\mathrm{MW}}$ is the sheaf associated to the presheaf $X\mapsto\mathrm{H}^{q-n}(X,\mathbf{K}^{\mathrm{MW}}_{n-p})$, which is trivial if $q\neq n$. Altogether, we see that

$$\mathrm{H}^p(X,\mathbf{H}^{q,n}_{\mathrm{MW}})=\begin{cases}0 & \text{if } q>n.\\ 0 & \text{if } p\geq n \text{ and } q\neq n.\end{cases}$$

We now consider the hypercohomology spectral sequence for the complex $\tilde{\mathbb{Z}}(n)$ ([SV00, Theorem 0.3]) $E_2^{p,q}:=\mathrm{H}^p(X,\mathbf{H}^{q,n}_{\mathrm{MW}})\implies\mathrm{H}^{p+q,n}(X,\mathbb{Z})$ which is strongly convergent. Our computations of the sheaves $\mathbf{H}^{q,n}_{\mathrm{MW}}$ immediately imply that $\mathrm{H}^{p-n}(X,\mathbf{H}^{n,n}_{\mathrm{MW}})=\mathrm{H}^{p,n}_{\mathrm{MW}}(X,\mathbb{Z})$ if $p\geq 2n-1$. We conclude using Theorem 4.2.3. □

Remark 4.2.5. The isomorphisms $\mathrm{H}^{p,n}_{\mathrm{MW}}(X,\mathbb{Z}) \to \mathrm{H}^{p-n}(X,\mathbf{K}^{\mathrm{MW}}_n)$ are functorial in X. Indeed, the result comes from the analysis of the hypercohomology spectral sequence for the complexes $\tilde{\mathbb{Z}}(n)$, which is functorial in X.

Setting $p = 2n$ in the previous theorem, and using the fact that $\mathrm{H}^n(X,\mathbf{K}^{\mathrm{MW}}_n) = \widetilde{\mathrm{CH}}^n(X)$ by definition (for $n \in \mathbb{N}$), we get the following corollary.

Corollary 4.2.6. *For any smooth scheme X and any $n \in \mathbb{N}$, the hypercohomology spectral sequence induces isomorphisms*

$$\mathrm{H}^{2n,n}_{\mathrm{MW}}(X,\mathbb{Z}) \to \widetilde{\mathrm{CH}}^n(X).$$

Remark 4.2.7. Both Theorems 4.2.4 and Corollary 4.2.6 are still valid if one considers cohomology with support on a closed subset $Y \subset X$, i.e. the hypercohomology spectral sequence (taken with support) yields an isomorphism

$$\mathrm{H}^{p,n}_{\mathrm{MW},Y}(X,\mathbb{Z}) \to \mathrm{H}^{p-n}_Y(X,\mathbf{K}^{\mathrm{MW}}_n)$$

provided $p \geq 2n-1$.

Let now E be a rank r vector bundle over X, $s : X \to E$ be the zero section and $E^0 = E \setminus s(X)$. The Thom space of E is the object of $\widetilde{\mathrm{DM}}(k,\mathbb{Z})$ defined by

$$Th(E) = \Sigma^\infty \tilde{\mathrm{M}}(\tilde{\mathbb{Z}}(E)/\tilde{\mathbb{Z}}(E^0)).$$

It follows from Corollary 3.1.8 and [AF16, Proposition 3.13] that (for $n \in \mathbb{N}$)

$$\mathrm{Hom}_{\widetilde{\mathrm{DM}}(k,\mathbb{Z})}(\mathrm{Th}(E),\tilde{\mathbb{Z}}(n)[2n]) \simeq \mathrm{Hom}_{\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)}(\tilde{\mathrm{M}}(\tilde{\mathbb{Z}}(E)/\tilde{\mathbb{Z}}(E^0)),\tilde{\mathbb{Z}}(n)[2n]) \simeq \mathrm{H}^{2n,n}_{\mathrm{MW},X}(E,\mathbb{Z}).$$

Using the above result, we get $\mathrm{Hom}_{\widetilde{\mathrm{DM}}(k,\mathbb{Z})}(\mathrm{Th}(E),\tilde{\mathbb{Z}}(n)[2n]) \simeq \widetilde{\mathrm{CH}}^n_X(E)$. Using finally the Thom isomorphism ([Mor12, Corollary 5.30] or [Fas08, Remarque 10.4.8])

$$\widetilde{\mathrm{CH}}^{n-r}(X,\det(E)) \simeq \widetilde{\mathrm{CH}}^n_X(L),$$

we obtain an isomorphism

$$\mathrm{Hom}_{\widetilde{\mathrm{DM}}(k,\mathbb{Z})}(\mathrm{Th}(E),\tilde{\mathbb{Z}}(n)[2n]) \simeq \widetilde{\mathrm{CH}}^{n-r}(X,\det(E))$$

which is functorial (for schemes over X).

Remark 4.2.8. The isomorphisms of Corollary 4.2.6 induce a ring homomorphism

$$\bigoplus_{n\in\mathbb{N}} \mathrm{H}^{2n,n}_{\mathrm{MW}}(X,\mathbb{Z}) \to \bigoplus_{n\in\mathbb{N}} \widetilde{\mathrm{CH}}^n(X).$$

This follows readily from the fact that the isomorphism of Theorem 4.2.3 is an isomorphism of graded rings.

5. Relations with ordinary motives

Our aim in this section is to show that both the categories $\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$ and $\widetilde{\mathrm{DM}}(k,R)$ split into two factors when $2 \in R^\times$, one of the factors being the corresponding category of ordinary motives. We assume that $R = \mathbb{Z}[1/2]$, the general case being obtained from this one. To start with, let X be a smooth k-scheme and let $\mathcal{L}$ be a line bundle over X. On the small Nisnevich site of X, we have a Cartesian square of sheaves of graded abelian groups ([CF14, §1])

$$\begin{array}{ccc} \mathbf{K}_*^{\mathrm{MW}}(\mathcal{L}) & \longrightarrow & \mathbf{I}^*(\mathcal{L}) \\ \downarrow & & \downarrow \\ \mathbf{K}_*^{\mathrm{M}} & \longrightarrow & \overline{\mathbf{I}}^*(\mathcal{L}) \end{array}$$

where $\overline{\mathbf{I}}^*(\mathcal{L})$ is the sheaf associated to the presheaf $\mathbf{I}^*(\mathcal{L})/\mathbf{I}^{*+1}(\mathcal{L})$. Observe that $\overline{\mathbf{I}}^*(\mathcal{L})$ is in fact independent of $\mathcal{L}$ ([Fas08, Lemme E.1.2]), and we will routinely denote it by $\overline{\mathbf{I}}^*$ below. Next, observe that $\langle 1,1\rangle \mathbf{I}^*(Y) \subset \mathbf{I}^{*+1}(Y)$ for any smooth scheme Y, and it follows that $\overline{\mathbf{I}}^*$ is a 2-torsion sheaf. Inverting 2, we then obtain an isomorphism

$$\mathbf{K}_*^{\mathrm{MW}}(\mathcal{L})[1/2] \simeq \mathbf{I}^*(\mathcal{L})[1/2] \times \mathbf{K}_*^{\mathrm{M}}[1/2]. \tag{5.0.0.a}$$

This decomposition can be more concretely seen as follows. In $\mathbf{K}_0^{\mathrm{MW}}(X)[1/2]$ (no line bundle here), we may write

$$1 = (1+\langle -1\rangle)/2 + (1-\langle -1\rangle)/2.$$

We observe that both $e := (1+\langle -1\rangle)/2$ and $1-e = (1-\langle -1\rangle)/2$ are idempotent, and thus decompose $\mathbf{K}_0^{\mathrm{MW}}(X)[1/2]$ as

$$\mathbf{K}_0^{\mathrm{MW}}(X)[1/2] \simeq \mathbf{K}_0^{\mathrm{MW}}(X)[1/2]/e \oplus \mathbf{K}_0^{\mathrm{MW}}(X)[1/2]/(1-e).$$

Now, $\mathbf{K}_0^{\mathrm{MW}}(X)[1/2]/e = \mathbf{W}(X)[1/2]$ (as $2e = 2+\eta[-1] := h$), while the relation $\eta h = 0$ (together with $h = 2$ modulo $(1-e)$) imply $\mathbf{K}_0^{\mathrm{MW}}(X)[1/2]/(1-e) = \mathbf{K}_0^{\mathrm{M}}(X) = \mathbb{Z}$. As $\mathbf{K}_*^{\mathrm{MW}}(\mathcal{L})[1/2]$ is a sheaf of $\mathbf{K}_*^{\mathrm{MW}}$-algebra, we find

$$\mathbf{K}_*^{\mathrm{MW}}(\mathcal{L})[1/2] \simeq \mathbf{K}_*^{\mathrm{MW}}(\mathcal{L})[1/2]/e \oplus \mathbf{K}_*^{\mathrm{MW}}(\mathcal{L})[1/2]/(1-e)$$

and

$$\mathbf{K}_*^{\mathrm{MW}}(\mathcal{L})[1/2]/e = \mathbf{K}_*^{\mathrm{MW}}(\mathcal{L})[1/2]/h = \mathbf{K}_*^{\mathrm{W}}(\mathcal{L})[1/2]$$

where $\mathbf{K}^{\mathrm{W}}(\mathcal{L})$ is the Witt K-theory sheaf discussed in [Mor04] and

$$\mathbf{K}_*^{\mathrm{MW}}(\mathcal{L})[1/2]/(1-e) = \mathbf{K}_*^{\mathrm{M}}[1/2].$$

This splitting has very concrete consequences on the relevant categories of motives. To explain them in their proper context, recall first from [CF14, Remark 5.16] that one can introduce the category of finite W-correspondences as follows.

For smooth (connected) schemes X and Y, let $\mathrm{WCor}_k(X,Y)$ be the abelian group

$$\mathrm{WCor}_k(X,Y) := \varinjlim_{T\in\mathcal{A}(X,Y)} \mathrm{H}_T^{d_Y}\big(X\times Y, \mathbf{W}, \omega_Y\big)$$

where $\mathcal{A}(X,Y)$ is the poset of admissible subsets of $X\times Y$ ([CF14, Definition 4.1]), d_Y is the dimension of Y and ω_Y is the pull-back along the second projection of the canonical module on Y. We let WCor_k be the category whose objects are smooth schemes and whose morphisms are given by the above formula. The results of [CF14] apply mutatis mutandis, showing in particular that WCor_k is an additive category endowed with a tensor product. Moreover the results of the present paper also apply, allowing to use the main theorems in this new framework. In particular, we can build the category of effective W-motives along the same lines as those used above, and its stable version as well.

Definition 5.0.1. For any ring R, we denote by $\mathrm{WDM}^{\mathrm{eff}}(k,R)$ the category of effective W-motives, i.e. the full subcategory of $\mathbb{A}^1$-local objects of the derived category of W-sheaves. We denote by $\mathrm{WDM}(k,R)$ the category obtained from the previous one by inverting the Tate object.

The relations with the categories previously built can be described as follows. Observe that by definition we have

$$\mathrm{WCor}_k(X,Y) := \varinjlim_{T\in\mathcal{A}(X,Y)} \mathrm{H}^{d_Y}_T\big(X\times Y,\mathbf{W},\omega_Y\big) = \varinjlim_{T\in\mathcal{A}(X,Y)} \mathrm{H}^{d_Y}_T\big(X\times Y,\mathbf{I}^{d_Y},\omega_Y\big)$$

for any smooth schemes X and Y. The morphism of sheaves $\mathbf{K}^{\mathrm{MW}}_{d_Y}(\omega_Y)\to\mathbf{I}^{d_Y}(\omega_Y)$ thus yields a well defined functor

$$\beta:\widetilde{\mathrm{Cor}}_k\to\mathrm{WCor}_k$$

This functor in turn induces a functor β_* between the categories of presheaves and sheaves (in either the Nisnevich or the étal topologies), yielding finally (exact) functors fitting in the commutative diagram

$$\begin{array}{ccc}
\mathrm{WDM}^{\mathrm{eff}}(k,R) & \xrightarrow{\beta_*} & \widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R) \\
{\scriptstyle\Sigma^\infty}\downarrow & & \downarrow{\scriptstyle\Sigma^\infty} \\
\mathrm{WDM}(k,R) & \xrightarrow[\beta_*]{} & \widetilde{\mathrm{DM}}(k,R)
\end{array}$$

where the vertical functors are stabilization functors (which are fully faithful). Note that both functors β_* admit monoidal left adjoints $\mathbf{L}\beta^*$ preserving representable objects.

Theorem 5.0.2. *Suppose that $2\in R^\times$. We then have equivalences of categories*

$$(\beta_*,\pi_*):\mathrm{WDM}^{\mathrm{eff}}(k,R)\times\mathrm{DM}^{\mathrm{eff}}(k,R)\to\widetilde{\mathrm{DM}}^{\mathrm{eff}}(k,R)$$

and

$$(\beta_*,\pi_*):\mathrm{WDM}(k,R)\times\mathrm{DM}(k,R)\to\widetilde{\mathrm{DM}}(k,R).$$

Proof. The functors β and π are constructed using the isomorphism of sheaves (5.0.0.a)

$$\mathbf{K}^{\mathrm{MW}}_*(\mathcal{L})[1/2]\simeq\mathbf{I}^*(\mathcal{L})[1/2]\times\mathbf{K}^{\mathrm{M}}_*[1/2]$$

together with the first and second projection. We can construct functors in the other direction by using the inclusion into the relevant factor and then the inverse of the above isomorphism. The result is then clear. □

Remark 5.0.3. Still under the assumption that $2\in R^\times$, the above splitting induces isomorphisms

$$\mathrm{H}^{p,q}_{\mathrm{MW}}(X,R)\simeq\mathrm{H}^{p,q}_{\mathrm{W}}(X,R)\times\mathrm{H}^{p,q}(X,R)$$

functorial in X. Here, $\mathrm{H}^{p,q}_{\mathrm{W}}(X,R)$ is the W-motivic cohomology defined in the category $\mathrm{WDM}(k,R)$. Moreover, this isomorphism is compatible with pull-backs, push-forwards and products.

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