ORIENTATION THEORY IN ARITHMETIC GEOMETRY

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ABSTRACT. This work is devoted to study orientation theory in arithmetic geometry within the motivic homotopy theory of Morel and Voevodsky. The main tool is a formulation of the absolute purity property for an *arithmetic cohomology theory*, either represented by a cartesian section of the stable homotopy category or satisfying suitable axioms. We give many examples, formulate conjectures and prove a useful property of analytical invariance. Within this axiomatic, we thoroughly develop the theory of characteristic and fundamental classes, Gysin and residue morphisms. This is used to prove Riemann-Roch formulas, in Grothendieck style for arbitrary natural transformations of cohomologies, and a new one for residue morphisms. They are applied to rational motivic cohomology and étale rational ℓ-adic cohomology, as expected by Grothendieck in [SGA6, XIV, 6.1].

CONTENTS

INTRODUCTION

History. One of the most striking intuition of Riemann is that the natural domain of definition of abelian integrals are (branched) surfaces rather than the complex plane. Retrospectively, one is amazed that this single idea contained in seeds the modern development of both analytical and algebraic geometry, whose varieties are now studied through their sheaf of functions. In this long and deep evolution, the Riemann-Roch formula played a catalytic role, or rather that of a lighthouse.

In his 1857 masterpiece, [Rie57], Riemann studied his new complex functions, defined in modern terms on a compact Riemann surface Σ. Notably, he described the general form of these functions once we prescribed m given (simple) poles in Σ. The striking new idea is the appearance of a geometrical invariant of S that Riemann had discovered before, that we now know as the *genus p* of Σ. Riemann established that complex functions on S with m given poles depend upon (at least) $m - p + 1$ constants (see *loc. cit.*, §5). A formula that we now read as:

$$(R) \qquad\qquad l(D) \geq \deg(D) - p + 1,$$

where D is the divisor on S made by the formal sum of the m given points, with degree $\deg(D)$ equal to m, and $l(D)$ is the dimension of the space of functions f on Σ whose associated divisor is D – meaning it admits simple poles exactly at the m points of the support of D. A few years later, Roch in [Roc65] interpreted analytically the difference of the two members of (R) as the "number of linearly disjoint integrands which can vanish at the m given poles" (see [Gra98, second par. p. 802]). In modern terminology, this becomes the *Riemann-Roch formula*, which we write today as:

$$(RR) \qquad\qquad l(D) - l(K - D) = \deg(D) - p + 1$$

where K is the canonical divisor on Σ.

Looking through the glass of a century of research, one is amazed by the exceptional role that took up this simple formula. This is particularly visible in the algebraic reformulation of Riemann's ideas by Clebsch, and then Brill-Noether (Max), where one of the driving motivation was to define the genus of an algebraic curve (complex plane projective) in order to prove the Riemann-Roch formula. The same problem is addressed slightly later in the development of algebraic surfaces by M. Noether, and then by the Italian school (Castelnuovo, Enriques, Severi, ...),

whose guide was to formulate the correct extension of the Riemann inequality in dimension 2 and in particular to find the good notion of genus.

But the most historically surprising application of the formula came almost eighty years after its introduction by Riemann when F.K. Schmidt extended it to the case of function fields K over a finite field and use it to prove the rationality of the Zeta function associated with K (1931). The formula followed the path opened up by the influential 1882 work of Dedekind-Weber, who developed the birational point of view initiated by Riemann by transporting his work to the purely arithmetical world of function fields over the complex numbers. The impact of Schmidt's proof on the modern formulation of algebraic geometry is of primary importance, as it lead Weil to his work on abelian varieties and most of all to the formulation of his conjectures on Zeta functions.

As said by Dieudonné in his history of algebraic geometry, the followers of Riemann split into several branches without much interactions (see [Die74], beginning of chap. VI). So while the notion of cohomology in arithmetic geometry was slowly revealing itself, the algebraic geometers working on the Riemann-Roch problem for surfaces were discovering the theory of *canonical classes*: M. Noether for surfaces (1886), Severi (1932), Segre and Todd in higher dimensions. Meanwhile, Poincaré introduced singular homology and topologists started to study *characteristic classes* of vector bundles (Stiefel and Whitney 1935, Chern 1946) without any connections with the theory of canonical classes. The unifying tool was to be the theory of sheaves invented by Leray during World War II. Only a few years after its introduction, this theory was fully developed first by Cartan and Serre for analytical varieties and secondly by Kodaira and Spencer for Kahlerian varieties. The problem of extending the Riemann-Roch formula in higher dimensions was crystallized in those years of boiling development around the notion of sheaves, in particular through the attempts of Kodaira and Serre. The first one had already solved the extension problem for Kahlerian varieties of dimension 2 (1951) and dimension 3 (1952) and linked the problem with computations of the canonical classes of Todd, while Serre had remarked its link with duality, used Thom cobordism theory to treat a special case[1] and conjectured a general form for the extended Riemann-Roch formula. Shortly after these advances, it belonged to Hirzebruch (1954, see also [Hir66]) to prove (and make precise[2]) the formula conjectured by Serre, formula that we now call the Hirzebruch-Riemann-Roch formula:

(HRR) $$\chi(X, E) = \deg \big(\operatorname{ch}(E).\operatorname{Td}(E) \big)$$

where E is a vector bundle on an analytical variety X, $\chi(X, E)$ (resp. $\operatorname{ch}(E)$, $\operatorname{Td}(E)$) its Euler characteristic (resp. Chern character, Todd class) with values in singular cohomology. The proof of Hirzebruch, rather technical, uses (and developed) the theory of characteristic classes (Chern, Todd classes,...) and makes use again of Thom cobordism theory.[3]

But the final revolution of Riemann's original problem was imagined by a single man whose ideas were to change completely our conception of it, Grothendieck. Shortly after the proof of Hirzebruch (see [BS58]), Grothendieck gave a new and

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[1]This work is unpublished but see the account of [Die74, VIII. 12.].

[2]In a letter to Kodaira and Spencer, Serre conjectured that the Euler characteristic should be expressed by some polynomial expression on Chern classes

[3]To anticipate the content of this paper, one remarks that the universality of cobordism theory was already fully playing its role here.

meaningful interpretation of the formula, whose first practical interest was the simplicity of its proof and its validity for algebraic varieties over an arbitrary base field. The two main ideas introduced there, which had never been anticipated before, was first a relative formulation (*i.e.* for a morphism rather than a single algebraic varieties) and secondly a purely cohomological interpretation of the (RR)-formula by introducing a generalized cohomology theory that would soon become famous as K-theory. From a conceptual point of view, the Grothendieck-Riemann-Roch formula expresses the defect of functoriality of a natural transformation of cohomology theories with respect to the exceptional covariant functoriality; in the assumptions of Borel-Serre, given a proper morphism $f : Y \to X$ of non singular quasi-projective varieties over any field k, T_X (resp. T_Y) being the tangent bundle of X (resp. Y), one has for any element y, in the K-group $K(Y)$ of *virtual vector bundles* over Y,

$$\text{(GRR)} \qquad\qquad \text{ch}(f_*(y)).\,\text{Td}(T_X) = f_*\big(\text{ch}(y).\,\text{Td}(T_Y)\big)$$

where ch denotes the Chern character from K-theory to rational Chow groups, and Td is the Todd class of a vector bundle.

During the period just described, history tells us that interactions between topology, geometry and algebraic geometry were very strong[4]. Therefore, soon after the appearance of the (GRR)-formula, Atiyah and Hirzebruch introduced topological K-theory, that they immediately understood as a generalized cohomology theory[5] and proved the topological formulation of the (GRR)-formula. It was soon realized that the covariant functoriality involved in the formula should be a consequence of Poincaré duality, on the model of the covariant functoriality discovered by Gysin in his study of sphere bundles (1942). Consequently, a very general (GRR)-formula, in which one considers an arbitrary natural transformation of generalized cohomology theories, each equipped with a *complex orientation* to get the usual theory of characteristic classes, was written by Dyer (see [Dye62]) – and stated as a folklore theorem, only 4 years after the original formulation of Grothendieck !

History must stop at some point. We will end it by two cornerstones of which our work is a direct continuation. The first one is Quillen discover of the universality of Thom complex cobordism theory in terms of formal group laws and oriented cohomology theories: [Qui69]. The second one is Grothendieck's final extension of his Riemann-Roch formula to the arithmetic setting in [SGA6].

Motivic stable homotopy and cohomology theories. The purpose of this work is to extend the arithmetic formulation of the Grothendieck-Riemann-Roch formula of [SGA6] in the same way that by Dyer (again [Dye62]) extends the topological formulation of the Hirzebruch-Riemann-Roch formula following Atiyah-Hirzebruch. To that end, the natural framework is Morel-Voevodsky's stable motivic homotopy theory, as it is defined by a clear analogy with the ordinary stable homotopy category of topological spaces used by Dyer.[6]

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[4]It was after a seminar in Princeton which gathered most of the main characters discussed here that Hirzebruch found his proof.

[5]*i.e.* a cohomology theory that satisfies all the axiom of Eilenberg-Steenrod except the dimension axiom;

[6]When we work over the field of complex numbers, the stable motivic homotopy category can be realized in the ordinary stable homotopy category so that any motivic construction or statement has a realization in the topological world.

The objects of the stable homotopy category, both classical and motivic, are meant to represent cohomology theories. Called spectra, they form a triangulated category whose distinguished triangles correspond to universal long exact sequences in cohomology. Similarly, all structures or properties of the stable homotopy category are reflected in the cohomologies representable by spectra. Probably the most important example of such a structure is the existence of a (symmetric) tensor product, called the *smash product*: a (commutative) monoid[7] on a spectra induces a product structure on its cohomology. These monoids are of primary importance in (motivic) stable homotopy; they are called *(motivic) ring spectra*.

In the motivic setting, we work over a base scheme[8] S; the motivic stable homotopy category of Morel-Voevodsky is denoted by $S\mathcal{H}(S)$. The starting point of this paper is that the representability of a cohomology theory has many interesting consequences. Let us first describe the obvious ones, for a given spectrum $\mathbb{E}$ (see Prop. 1.2.10 for details): the cohomology represented by $\mathbb{E}$ is a contravariant functor $\mathbb{E}^{**}$ from smooth S-schemes to bigraded abelian groups (the first index is the degree and the second one is called the *twist*). It satisfies the homotopy invariance property with respect to the affine line $\mathbb{A}^1$ (the affine line $\mathbb{A}^1$ is contractible), stability property with respect to the projective line $\mathbb{P}^1$ (seen as the analogue of the circle in topology). Moreover, it can be extended to a *cohomology theory with support*: given a smooth S-scheme X and a closed subscheme $Z \subset X$, one can define a bigraded abelian group $\mathbb{E}_Z^{**}(X)$ of cohomology classes with support in Z, in an appropriate functorial way and such that $\mathbb{E}_X^{**}(X) = \mathbb{E}^{**}(X)$ (see Def. 1.2.5 for details). This theory with support satisfies the (Nisnevich) excision property (analogue of the excision property of the Eilenberg-Steenrod axioms in topology, see Sec. 1.4, property (Nis) for details).

In this paper, we will use another important property of $S\mathcal{H}(S)$, its basic functoriality in S: it is a fibered category over the category of schemes. A cartesian section[9] with respect to this fibered structure, eventually restricted to a subcategory $\mathscr{S}$ of the category of schemes, will be called an *absolute spectrum* (see Def. 1.2.1 and Ex. 1.2.3 for examples). Such an absolute spectrum represents a cohomology which is defined over the whole category $\mathscr{S}$, allowing to avoid the restriction to smooth S-schemes.[10] It still satisfies all the properties enumerated above. But moreover, under the presence of a ring structure on the absolute spectrum, we get an important product on cohomology with support which is not commonly used (but see [SGA4$\frac{1}{2}$, IV]). We call it the *refined product*: given closed subschemes $T \subset Z \subset X$, it has the form

$$(*) \qquad \mathbb{E}_T^{**}(Z) \otimes \mathbb{E}_Z^{**}(X) \to \mathbb{E}_T^{**}(X)$$

and will be an essential technical tool to our study of fundamental classes (see Par. 1.2.8).

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[7]In this introduction and in the whole paper, all monoid structures will be assumed to be commutative.

[8]In this introduction and in the whole paper, all schemes will be assumed to be Noetherian of finite dimension.

[9]Explicitly: the data of a ring spectrum $\mathbb{E}_X$ for each scheme X, with a given transition isomorphism $f^*(\mathbb{E}_X) \simeq \mathbb{E}_Y$ for any morphism $f : Y \to X$.

[10]Using a terminology introduced by Beilinson, this is an absolute cohomology; this justifies the terminology *absolute spectrum*.

Note that all the basic properties of the cohomologies representable by an absolute spectrum are gathered in Proposition 1.2.10.

Absolute purity. The absolute purity conjecture of Grothendieck, formulated in [SGA5, I 3.1.4], has been a major problem in étale cohomology, because of its consequences on finiteness and duality as pointed out by Grothendieck (see *loc. cit.*). It was solved by Thomason in [Tho84] under some assumptions (on the coefficients) and in full generality by Gabber in [Fuj02]. Roughly, the conjecture says that the cohomology of a regular scheme X with support in a closed regular scheme Z is isomorphic to the cohomology of Z. The case of smooth schemes over a field (or even over some base) can be treated easily (see [SGA4, XVI, 3.9]). The main problem in this conjecture is to treat the case of regular schemes of unequal characteristics, which are the objects of the so-called *arithmetic geometry*.

This problem was confined in the étale setting until D.C. Cisinski and the author discovered that a similar statement could be formulated and proved in the newly defined setting of rational mixed motives (see [CD12b, 14.4.1])[11]. This naturally raises the question of extending the problem of absolute purity to any representable cohomology theories.

So we introduce here (Def. 1.3.2) the property of *absolute purity* for an absolute spectrum and any closed subscheme $Z \subset X$ whose immersion is regular. The formulation of this property is the new technical ingredient introduced by this work. As in the case of étale cohomology, for smooth schemes over a field (or over some base), the property is always fulfilled according to a fundamental result of Morel-Voevodsky. Thus the interest of this property lies in the case of arithmetic geometry. Fortunately, there are several cohomology theories which satisfies absolute purity in the arithmetic case. The matrix case is integral algebraic K-theory according to the localization theorem of Quillen. The cases of rational motivic cohomology and rational cobordism theory follows (see Ex. 1.3.4).

We think that the problematic of absolute purity is an important question for homotopy theory. A new aspect of our definition of this property is that it is intrinsic and do not depend on the choice of a purity isomorphism.[12] Moreover, it is formulated for any cohomology theory without assuming the existence of an orientation (see below). In particular, we conjecture that this property holds integrally not only for the algebraic cobordism spectrum but also for the sphere spectrum (see conjectures A and B p. 22). This result would have several interesting consequences: see Remark 1.3.5.

An important ingredient in the proof of absolute purity by Gabber is the so-called analytical invariance of étale cohomology with support. While we do not attack the previous conjectures on absolute purity, we nevertheless prove the *analytical invariance property* for any cohomology representable by an absolute spectrum (see Thm. 1.4.6), extending a result already obtained by Wildeshaus [Wil06]. Note in particular that the result can be applied to the absolute spectrum representing *rigid cohomology* over a field of characteristic $p > 0$ – according to Ex. 1.3.4(1). Thus our proposition contains in particular Theorem 1.1 of [Ouw14].

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[11]As in the étale setting, this result has important consequences for rational mixed motives. Note the importance of absolute purity was anticipated by Ayoub in [Ayo07a].

[12]This is due to the use of the deformation space.

Orientation theory: characteristic and fundamental classes At this point, we connect motivic homotopy theory with the two fundamental notions that were developed has a natural evolution of the Riemann-Roch problem: characteristic classes and fundamental classes. Recall the first ones have first been studied in algebraic topology, and evolved naturally into the theory of orientation while the second one was the domain of algebraic geometry and evolved into intersection theory.

The idea of transporting orientation theory from algebraic topology to algebraic geometry is comparatively quite recent as it takes its origin in the first proof of the Milnor conjecture by Voevodsky (see [Voe96]). The theory has grown out of an unpublished work of Morel which was developed by several authors.

Let us recall in this introduction the basics of this theory. An orientation c of a ring spectrum $\mathbb{E}$ over a base scheme S is a cohomological class – in degree $(2,1)$ – of the infinite projective space with coefficients in $\mathbb{E}$ (Def. 2.1.2). Giving such a class allows to derive *canonically* a lot of interesting cohomological invariants and structures. First, we can define Chern classes associated with vector bundles – the first Chern class follows directly from the orientation, and the other ones follow from the computation of the cohomology of any projective bundle according to the classical construction of Grothendieck: see Section 2.1.

The connection with the work of Quillen appears at this point. As in topology, Chern classes of an arbitrary oriented spectrum need not be additive: the first Chern class of a tensor product of line bundles is not the sum of the Chern classes of each bundle. Instead, it is described according to a *formal group law* that is canonically associated with the chosen orientation (see Par. 2.1.19 for more details). This is connected with the theory of *Thom classes* that one derives from Chern classes.[13] Indeed, these classes uniquely define a canonical structure of **MGL**-algebra on $\mathbb{E}$ where **MGL** is the algebraic cobordism spectrum (analogue of the complex cobordism spectrum **MU** in topology). Orientations on $\mathbb{E}$ are in one to one correspondence with structures of **MGL**-algebra and it is widely believed that the formal group law of **MGL** is the universal formal group law.[14]

Secondly, the orientation determines fundamental classes. In fact, assuming the absolute purity property for a regular closed immersion $Z \subset X$, the Thom class of the normal bundle of Z in X gives us directly the *refined* fundamental class of Z in X, as a cohomology class of X with support Z. Using the refined product by this fundamental class – $(*)$ with $T = Z$ – gives us the usual purity isomorphism:

$$\mathbb{E}^{**}(Z) \xrightarrow{\sim} \mathbb{E}^{**}_Z(X).$$

With our formalism, all this follows easily. But note however that this is the first appearance of this form of the purity isomorphism in motivic homotopy theory – and even in étale cohomology.[15]

These generalized fundamental classes are the trace of classical intersection theory, though they can be defined in very general theories such as algebraic K-theory

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[13]The Thom class of a vector bundle E, as a class in the projective completion of E, equals the top Chern class of the universal quotient bundle (see Ex. 2.2.4).

[14]According to works respectively of Levine and of Hoyois, given a field k of exponential characteristic p, this is true for **MGL**$[1/p]$ if one restricts to k-schemes.

[15]We were especially inspired by the formulation of Poincaré duality of Bloch and Ogus in [BO74].

or algebraic cobordism. We illustrate this concretely by proving some of the classical formulas known for Chow groups: the excess intersection formula (Cor. 2.4.4) and associativity (compatibility with composition, Th. 2.4.9). Note that the excess intersection formula allows to get back the classical link between fundamental classes and characteristic classes: when Z can be parametrized by a section of a vector bundle over X, the fundamental class of Z in X equals the top Chern class of the vector bundle (see Cor. 2.4.6 for details). Let us also recall that the compatibility with composition of fundamental classes is a key technical problem in orientation theory. The geometrical tool (the double deformation space) used here is not new but the use of the refined product $(*)$ allows both a finer expression of this result and an easier proof.

Residues and Gysin morphisms. A formal consequence of the purity isomorphism for a regular closed immersion $i : Z \to X$ is the existence of a localization long exact sequence in cohomology, so that the theory described just above for an oriented absolute ring spectrum canonically leads to such a sequence. It is made of two interesting morphisms. The first one is a morphism that we have called the *residue map associated with* (X, Z). Our interest to that kind of maps come from our comparison between cycle modules and homotopy sheaves with transfers ([Dég11]) as our residues on sheaves corresponds to Rost residues on cycle modules. One illustrates this phenomenon by the computation of our residue maps when X is a trait, Z its closed point: for motivic cohomology in bidegree (n, n), our residue map coincides with Milnor residue symbol. More generally, the residue map on symbols in classical cohomologies always agree with Milnor residue map (see Par. 5.4.1 for details). More interestingly, we prove here that the residue we have defined by purely geometrical means agrees with *Tate residue* on De Rham cohomology (Th. 5.4.5) – this is an application of the analytical invariance of our residue map. This implies in particular, that for divisors, the residue map we have defined here – by deformation to the normal bundle and orientation theory – agrees with that of Leray. These classical constructions gets also extended to rigid cohomology (see Ex. 5.4.2).

The second interesting map is the so-called *Gysin morphism*[16] associated with the immersion i; in other words, the covariant functoriality of cohomology. According to our formalism, it is simply equal to the multiplication by the refined fundamental class (which gives a class with support in Z) followed by the obvious map which forgets the support (see (3.1.2.a) for details). Thus the good properties of (refined) fundamental classes give all the basic expected properties of these particular Gysin maps.

Once all this ground work is in place, one can introduce the main construction of orientation theory, that of *Gysin morphisms* for certain projective morphisms. In our work, it comes mainly in two settings: the geometric one, for schemes smooth over some fixed base, and the arithmetic one, for any regular schemes. The way we phrased and use absolute purity allows us to treat these two cases in a single turn, thus building Gysin morphisms associated with any projective morphism between one of these two kind of schemes. In fact, the process to build these Gysin maps is very classical: you need to consider the case of closed immersions and of projections

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[16]This is the usual terminology in algebraic geometry. It is named that way after the pioneering work of Gysin that we have described in the historical part.

of a projective bundle, and then find the sufficient condition so that these two cases can be glued. However, because we deal with general oriented spectra, the case of the projection of a projective bundle is not trivial. As in our previous work on the subject ([Dég08a]), to treat that case, we use a duality argument that can be summarized as follows: because of the projective bundle formula, the cohomology of a projective bundle P/X is a finite free module over the cohomology of the base: thus it is dualizable. Then, one shows that the Gysin morphism associated with the diagonal embedding of P induces a duality pairing on $\mathbb{E}^{**}(P)$. Finally one defines the Gysin morphism of the projection $P \to X$ as the transpose of the pullback with respect to this duality (see Def. 3.2.2). The condition for gluing then follows, using a computation of characteristic classes, as well as the main properties of Gysin morphisms for closed immersions: compatibility with composition, projection formula, compatibility with transversal pullbacks, excess of intersection formula.

One of the basic examples of a representable cohomology theory is *Beilinson motivic cohomology* which is representable by absolutely oriented ring spectrum because of the ground work [CD12b]. Our construction gives Gysin morphisms for this cohomology with respect to any projective morphism between regular schemes. This is an improvement of the constructions of [Sou85, Th. 9]. For more general cohomology theories, such as algebraic K-theory or algebraic cobordism, Gysin morphisms reveal a third kind of characteristic classes, the *cobordism classes* (see Def. 3.2.13). Note these classes are interesting only when the formal group law associated with the considered oriented ring spectrum is non additive – see in particular formula (3.2.14.a). The non triviality of these classes in the case of non additive formal group law explains why the definition of Gysin morphisms in our context is far more difficult than in the ordinary case.

As a prelude to the Riemann-Roch formula, note that, inspired by a result of Panin, we give a uniqueness statement for Gysin morphisms by simple axioms (see Th. 3.3.1). This allows us to compare the Gysin morphisms defined here with more classical constructions (see the examples in 3.3.4).

Riemann-Roch formulas. The beauty of the Grothendieck-Riemann-Roch formula lies in its generality and the simplicity of its proof. The same phenomena happens here and the reader will see that the main technical work was to define Gysin morphisms.

The topological interpretation of the Riemann-Roch formula can be summarized as an answer to the following question: what happens if we *change the choice of orientation* on an absolutely pure oriented spectrum $\mathbb{E}$?
The answer is that all the data defined through the orientation theory, as described above, change and in particular the Gysin morphism: the change of the later is exactly measured by the Riemann-Roch formula (see Section 5.2 for that point of view).

The setting of our general Riemann-Roch formula is to consider two absolute oriented ring spectra $(\mathbb{E}, c)$, $(\mathbb{F}, d)$ and an arbitrary morphism of ring spectra $\varphi : \mathbb{E} \to \mathbb{F}$. Then one realizes that $\varphi(c)$ is an orientation of $\mathbb{F}$, which does not necessarily coincide with the given orientation d. Thus, we come back to the question of changing the orientation on a given spectrum and understanding its effect on Gysin morphisms. In a word, this change of orientation is measured by the Todd class Td_φ associated with the morphism φ (see Prop. 4.1.2). With this definition in hands, we prove the analogue of the classical Grothendieck-Riemann-Roch formula: for

any local complete intersection, any projective morphism $f : Y \to X$ in $\mathscr{S}$ with
virtual tangent bundle τ_f, one has for a cohomology class $y \in \mathbb{E}^{**}(Y)$ (see also Th.
4.3.2):

$$(\mathrm{GRR}_\varphi) \qquad\qquad \varphi_X(f_*(y)) = f_*\big(\mathrm{Td}_\varphi(\tau_f) \cup \varphi_Y(y)\big).$$

At this point, the proof is quite easy and follows the initial proof of Borel and
Serre: one reduces to the case of closed regular embeddings and projective smooth
morphisms. In our situation, the last case follows by duality from the first one
– here the proof differs slightly from that of Borel and Serre. The case of closed
embedding is by reduction to divisors in which case it is tautological.

Inspecting this last case, we fall onto the surprising result that the Riemann-
Roch formula for closed regular embedding $i : Z \to X$ has a companion formula
which was never observed till now and involves the residue map $\partial_{X,Z}$. We call it
the residual Riemann-Roch formula. Using the Todd class of the normal bundle
$N_Z X$ of Z in X, it reads as follows for a cohomology class $u \in \mathbb{E}^{**}(X - Z)$ (see
also Th. 4.2.3):

$$(\partial\mathrm{RR}) \qquad\qquad \partial_{X,Z}\big(\varphi_U(u)\big) = \mathrm{Td}_\varphi(-N_Z X) \cup \varphi_Z\big(\partial_{X,Z}(u)\big).$$

Applications and comparison with the work of Gillet and Soulé. Before
going to the applications, let us explain with more details the crucial situation of
changing between two orientations c and d on an absolute ring spectrum $\mathbb{E}$. As
explained above, the orientations c and d correspond over a base scheme X to
formal group laws F_c and F_d with coefficients in the ring $\mathbb{E}^{**}(X)$. Moreover, it
follows automatically that these formal group laws are isomorphic: say from F_d to
F_c, the isomorphism corresponds to a power series $\Psi(t)$ with coefficients in $\mathbb{E}^{**}(X)$
of the form $\Psi(t) = (t + \ldots)$. Then the Todd class which corresponds to changing
the orientation from c to d is uniquely defined in terms of $\Psi(t)$ (see section 5.2 for
a detailed discussion).

This understanding of Todd classes allows to enlighten the classical case. When
$\mathscr{S}$ is the category of regular schemes, Quillen algebraic K-theory (resp. Beilin-
son motivic cohomology) is representable by an oriented absolute ring spectrum
KGL (resp. $\mathbf{H}_\mathrm{B}$). Then the classical (higher) Chern character as defined by Gillet
corresponds to an isomorphism of absolute ring spectra:

$$\mathrm{ch} : \mathbf{KGL} \to \oplus_{i \in \mathbb{Z}} \mathbf{H}_\mathrm{B}(i)[2i]$$

that was first given in these terms by Riou (in a slightly different form, see [Rio10,
6.2.3.9]). The formal group law associated with **KGL** (resp. $\mathbf{H}_\mathrm{B}$) is the multiplica-
tive (resp. additive) formal group law – see Example 2.1.21. It follows from the
theory of formal group laws that there is only one isomorphism of the form $\Psi(t)$
from the additive to the multiplicative formal group law: this is the exponential
power series and one recovers the classical definition of the Todd power series, with
a conceptual explanation why it has this precise form. The Riemann-Roch theorem
that we get for ch is a cohomological version of the formulas obtained by Gillet,
[Gil81, Th. 4.1]: in fact, one will recognize most of the principles of motivic ho-
motopy theory in the work of Gillet. The main difference is that we have built the
Gysin morphisms appearing in the formula while in *loc. cit.* they are part of the
axioms (of a "duality theory with support"). Therefore, the cohomological formula-
tion of our Riemann-Roch formula for ch is new, valid for any projective morphism

between regular schemes. It extends the classical Grothendieck-Riemann-Roch formula for K_0 to higher degrees. Note this formula was proved by Riou in [Rio10] in the particular case of smooth projective morphisms and was obtained in full generality in [AH10] but the construction of Gysin morphisms in *op. cit.* is merely a reference to [Dég08a] – see Example 2.4 in *op. cit.* Note finally we have given a special attention to the residual Riemann-Roch formula in the case of the Chern character resulting in some purely algebraic results (see Section 5.5).

Our other examples are more concerned with pure orientation theory. First the analysis of the change of orientations shows that on motivic cohomology, as well as on mixed Weil cohomologies (they are representable according to [CD12a]), there is *only one* possible orientation (see 5.1.10 for the general statement). Therefore, on these cohomologies, there is only one possible theory of Chern classes and Gysin morphisms (see also Th. 3.3.1 and Ex 3.3.4). This settles the point of unnecessary questions about signs, uniqueness, that frequently occur in this kind of situation. We also give a new proof, as well as a conceptual explanation, of a formula of Quillen for computing the cobordism class of a non necessarily trivial projective bundle, but only with rational coefficients (see Th. 5.2.4 and Example 5.2.7).[17] The idea of the proof is to look at the Riemann-Roch formula that one gets by changing the formal group law of algebraic cobordism (conjecturally the universal formal group law) to the additive one – this is why we need rational coefficients.

Comparison with the work of Panin and étale cohomology. The last setting to which our work can be compared is the fundamental work of Panin on oriented cohomology theories. In [Pan04], Panin proves our general Riemann-Roch formula for cohomology theories satisfying suitable axioms but defined over the category of smooth k-schemes. So our work should be viewed as an extension of the axiomatic of oriented cohomology theory to the arithmetic case. This is what we prove in the last section of this paper: we have extracted a list of all the properties of representable cohomology theories that we have used in this work, dubbed here *arithmetic cohomology* for short (see Def. 6.1.1).[18] This axiomatic indeed generalizes the one of Panin and the proofs of this paper show that our results still apply to it, yielding Gysin morphisms, residues, characteristic classes and their formulas such as the Riemann-Roch ones.

This point of view is in fact useful because it applies especially to the étale l-adic cohomology of $\mathbb{Z}[1/l]$-schemes. In fact, we do not know if this cohomology is representable by an *absolute* ring spectrum.[19] But however, it satisfies all the axioms of an arithmetic cohomology – in particular because of Thomason result about absolute purity. Thus, the constructions of this paper apply to that cohomology, and its rational version, and in particular give Gysin morphisms. We deduce the classical Riemann-Roch formula for rational ℓ-adic cohomology of regular schemes, as expected by Grothendieck in [SGA6, XIV, 6.1] and even the higher Riemann-Roch formula as well as the residual Riemann-Roch formula (∂RR) (see Cor. 6.2.4).

---

[17]This is enough to get the integral formula as well as the case of schemes of characteristic 0: see Remark 5.2.6(2).

[18]A longer but more precise terminology is also introduced: *absolutely pure oriented ringed cohomology with support.*

[19]According to the results of [CD14], it is representable over any scheme by a ring spectrum but we do not know these ring spectra form a cartesian section of $\mathcal{SH}$ – unless one restricts to schemes over a field.

Note finally that one can also apply the recent work of [CD14] on h-motives, to get that étale motivic cohomology with coefficients in any ring R is an arithmetic cohomology.[20] The Gysin morphisms that one get extend to arbitrary coefficients, in the regular case, the recent construction of Gabber-Riou ([Rio14], see remark 6.2.5 for the comparison).

Outline of the work. The paper is organized as follows. In the first section, we recall the basics on the motivic stable homotopy category give the basic properties of representable cohomologies, and states the absolute purity property. We end-up the section with a discussion about analytical invariance. In section 2, we recall orientation theory in motivic homotopy theory, gives the construction of (refined) fundamental classes and their properties. In section 3, we define the residues and Gysin morphisms and study their properties. Section 4 is centered around the general Riemann-Roch formula and its proof. In section 5, we treat examples and give applications of our formulas. Finally, Section 6 discusses the axiomatic of arithmetic cohomologies and the case of étale cohomology, motivic étale cohomology and continuous étale cohomology with various coefficients.

Notations and conventions

All schemes in this paper are assumed to be noetherian of finite dimension. We will say that an S-scheme X, or equivalently its structure morphism, is *projective* if it admits an S-embedding into $\mathbb{P}_S^n$ for a suitable integer n.[21]

In the whole text, unless stated otherwise, $\mathscr{S}$ stands for a sub-category of the category of such schemes. We will assume that $\mathscr{S}$ is stable by blow-up and contains any open subscheme of (resp. projective bundle over) a scheme in $\mathscr{S}$. The category $\mathscr{S}$ can be the category of all schemes, especially in the examples and definitions which do not deal with absolute purity. On the contrary, when dealing with the absolute purity property, the relevant examples for applications area:

- the category $\mathscr{R}$eg of all regular schemes (*i.e.* its local rings are regular.)
- the category $\mathscr{S}m_S$ of smooth S-schemes, for an arbitrary base scheme S.

By convention, unless explicitly stated, when we speak of the rank of a vector bundle, the dimension of a morphism, the codimension of a closed subscheme (or a closed immersion), it will always be assumed to be *constant*.[22]

Given a projective bundle P/X associated with a vector bundle E/X, we will call *canonical line bundle* on P/X the tautological line bundle λ on P characterized by the property $\lambda \subset P \times_X E$ (it corresponds with the notation $\lambda = \mathcal{O}(-1)$).

The letter $\mathbb{N}$ denotes the set of non negative integers.

1. Absolute cohomology and purity

1.1. **Functoriality in stable homotopy.**

---

[20]Recall that in the case where R is a torsion ring of characteristic exponent invertible on the schemes considered, this later cohomology agrees with the usual étale cohomology with coefficients in R.

[21]For example, if one works with quasi-projective schemes over a noetherian affine scheme (or more generally a noetherian scheme which admits an ample line bundle), then a morphism is proper if and only if it is projective with our convention – use [EGA2, Cor. 5.3.3].

[22]If one wants to avoid this convention, see remarks 2.2.3 and 2.3.2.

1.1.1. Recall that the stable homotopy category of schemes defines a 2-functor from the category of schemes to the category of symmetric monoidal closed triangulated categories. This means that for any morphism of schemes $f : T \to S$, we have a pullback functor

$$f^* : S\mathscr{H}(S) \to S\mathscr{H}(T)$$

which is symmetric monoidal and such that for any composable morphisms of schemes f, g, we have the relation: $f^* g^* = (gf)^*$.

We will use the following properties:

(A1) For any morphism (resp. smooth morphism) f, the functor f^* admits a right (resp. left) adjoint denoted by f_* (resp. $f_\sharp$).

(A2) For any cartesian square:

$$
\begin{array}{ccc}
Y & \xrightarrow{q} & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
T & \xrightarrow{p} & S
\end{array}
$$

such that f is a smooth morphism, the base change map

$$p^* f_\sharp \to g_\sharp q^*$$

is an isomorphism.

(A3) For any smooth morphism $f : Y \to X$ and any spectrum $\mathbb{E}$ over Y (resp. $\mathbb{F}$ over X), the canonical transformation:

$$f_\sharp(\mathbb{E} \wedge f^*(\mathbb{F})) \to f_\sharp(\mathbb{E}) \wedge \mathbb{F}$$

is an isomorphism.

(A4) For any closed immersion $i : Z \to X$ with complementary open immersion j, there exists a unique natural transformation $\partial_i : i_* i^* \to j_\sharp j^*[1]$ which fits in a distinguished triangle of the form:

$$j_\sharp j^* \xrightarrow{ad'} Id \xrightarrow{ad} i_* i^* \xrightarrow{\partial_i} j_\sharp j^*[1]$$

where ad (resp. ad') is the unit (resp. counit) map of the adjunction (i^*, i_*) (resp $(j_\sharp, j^*)$).

Except for the last property, these are easy consequences of the construction of $S\mathscr{H}$ – see [Ayo07b]. Property (A4) is a consequence of [MV99, §3, th. 2.21] – see [Ayo07b, 4.5.47] for details.

Remark 1.1.2. One can deduce from (A4) that i_* is fully faithful. Moreover, when i is a nil-immersion, i^* is fully faithful, so that (i^*, i_*) is an equivalence of categories.

Note also that we can derive from properties (A1) and (A4) the following ones:

(A5) For any closed immersion i, the functor i_* admits a right adjoint denoted by $i^!$.

(A6) For any cartesian square:

(1.1.2.a)
$$
\begin{array}{ccc}
T & \xrightarrow{k} & Y \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
Z & \xrightarrow{i} & X
\end{array}
$$

such that i is a closed immersion, the base change morphism

$$f^* i_* \to k_* g^*$$

is an isomorphism.

(A7) For any closed immersion $i : Z \to X$ and any spectrum $\mathbb{E}$ over Z (resp. $\mathbb{F}$ over X), the canonical transformation:

$$i_*(\mathbb{E} \wedge i^*(\mathbb{F})) \to i_*(\mathbb{E}) \wedge \mathbb{F}$$

is an isomorphism.

For these three last properties, we refer the reader to [Ayo07a] or for a more compact reference to [CD12b], 2.3.3, 2.3.8 and 2.3.15 respectively.

1.1.3. For any smooth S-scheme X, we denote by $\Sigma_S^\infty X$ the infinite suspension spectrum associated with the sheaf represented by X with a base point added. Recall that $\Sigma_S^\infty X = f_\sharp(\mathbb{1}_X)$.

Consider again the notations of axiom (A4). We simply denote by X/U the cokernel of the map induced by j in the category of Nisnevich sheaves of sets over $\mathscr{S}m_S$. Note it is pointed by identifying U with the base point. As j is a monomorphism, we deduce a homotopy cofiber sequence

$$U \to X \to X/U$$

in the $\mathbb{A}^1$-local model category of simplicial sheaves over S ([MV99, Sec. 3.2, p. 105]). According to (A4) and Remark 1.1.2, the canonical map $\mathbb{1}_Z \to i^*\Sigma^\infty(X/U)$ is an isomorphism.

To simplify notations, we will still denote by X/U the infinite suspension spectrum associated with the sheaf X/U. Given any object K of $S\mathscr{H}(X)$, we get a canonical isomorphism: $i^*\big((X/U) \wedge K\big) = i^*(K)$ which, by adjunction, induces a map

(1.1.3.a) $(X/U) \wedge K \to i_*i^*(K).$

In fact, the localization axiom (A4) for i is equivalent to the fact that this map is an isomorphism and i_* is fully faithful – see [CD12b, 2.3.15].

Remark 1.1.4. In what follows, we will consider the isomorphisms listed in the above properties as identities unless it involves a non trivial commutativity statement.

1.2. **Absolute cohomology.** As usual, a ring spectrum over a scheme S will be a commutative monoid of the symmetric monoidal category $S\mathscr{H}(S)$.

Definition 1.2.1. An $\mathscr{S}$-*absolute spectrum* (resp. *ring spectrum*) $\mathbb{E}$ is a collection of spectra (resp. ring spectra) $\mathbb{E}_X$ over X for a scheme X in $\mathscr{S}$ and the data for any morphism $f : Y \to X$ in $\mathscr{S}$ of an isomorphism of spectra (resp. ring spectra) $\epsilon_f : f^*\mathbb{E}_X \to \mathbb{E}_Y$ satisfying the usual cocycle condition.[23]

A *morphism* $\varphi : \mathbb{E} \to \mathbb{F}$ of $\mathscr{S}$-absolute spectra (resp. ring spectra) is a collection of morphisms $\varphi_X : \mathbb{E}_X \to \mathbb{F}_X$ of spectra (resp. ring spectra) indexed by schemes X in $\mathscr{S}$ such that for any morphism $f : Y \to X$ in $\mathscr{S}$, the following diagram commutes:

$$\begin{array}{ccc} f^*\mathbb{E}_X & \xrightarrow{f^*\varphi_X} & f^*\mathbb{F}_X \\ {\scriptstyle \epsilon_f^{\mathbb{E}}}\downarrow & & \downarrow{\scriptstyle \epsilon_f^{\mathbb{F}}} \\ \mathbb{E}_Y & \xrightarrow{\varphi_Y} & \mathbb{F}_Y. \end{array}$$

―――――――――

[23]In other words, this is a cartesian section of the $\mathscr{S}$-fibered category of spectra (resp. ring spectra).

As the category $\mathscr{S}$ is fixed in the entire paper we will abusively say *absolute* for $\mathscr{S}$-absolute. However, when the category $\mathscr{S}$ is the category of all S-schemes, we will say S-absolute for $\mathscr{S}$-absolute.

Remark 1.2.2. In the whole paper, we will be interested only in $\mathscr{S}$-absolute spectra. However, it is also convenient to consider the sections $(\mathbb{E}_X)_{X \in \mathscr{S}}$ over $\mathscr{S}$ of the stable homotopy category which are not necessarily cartesian (*i.e.* the given transition morphisms $\epsilon_f : f^*(\mathbb{E}_X) \to \mathbb{E}_Y$ are not isomorphisms). We will call these objects *weak $\mathscr{S}$-absolute spectra*. As in the above definition, they can also be equiped with a ring structure.

Example 1.2.3. (1) Let S be a fixed scheme.

Then, up to the choice of the isomorphisms ϵ_f in the above definition, an S-absolute ring spectrum $\mathbb{E}$ is determined by its value on S. Reciprocally, given any ring spectrum $\mathbb{E}_S$ over S, we get a canonical S-absolute spectrum $\mathbb{E}$ by putting $\mathbb{E}_X = f^*(\mathbb{E}_S)$ for any $f : X \to S$. In the S-absolute case, we will frequently identify $\mathbb{E}_S$ and $\mathbb{E}$ to simplify notations.

A basic example of this kind of situation is given by the concept of mixed Weil theory. Let us recall the setting. We let k be a field (non necessarily perfect) and K be a field of characteristic 0. A K-linear mixed Weil theory E over k is a presheaf of commutative differential graded K-algebras over the category of smooth affine k-algebras satisfying axioms: homotopy invariance, Nisnevich excision, dimension, stability and Künneth formula (see [CD12a]). To such a theory one canonically associates a ring spectrum $\mathbb{E}$ over k.

We then get an absolute ring spectrum over the category of k-schemes by the preceding procedure (cf. [CD12b, 17.2.5]). The original cohomology defined by E for smooth affine k-schemes gets extended to any k-schemes. This extension is uniquely characterized by the h-descent property (cf. [CD12b, 17.2.6]) and by the commutation with limit property (cf. Lemma 1.2.13 below).

Examples are given by the classical Weil theories: Betti, De Rham, geometric étale and rigid cohomologies.
(2) The 0-sphere spectrum S^0 – unit for the smash product – is obviously an absolute ring spectrum.
(3) For any scheme S, one can consider one of the following ring spectra:
  - The Beilinson motivic cohomology spectrum $\mathbf{H}_{\mathrm{B},S}$ (see [Rio10, CD12b]).
  - The homotopy invariant K-theory ring spectrum $\mathbf{KGL}_S$ (see [Voe98, Rio10]).[24]
  - The cobordism ring spectrum $\mathbf{MGL}_S$ (see [PPR08, 2.1]).
  Then each of these examples defines an absolute ring spectrum denoted respectively by $\mathbf{H}_{\mathrm{B}}$, $\mathbf{KGL}$, $\mathbf{MGL}$ (see the respective reference given above for this assertion).
(4) In [Voe98, §6], Voevodsky defined the *motivic Eilenberg-Mac Lane spectrum* over any smooth k-scheme S where k is a perfect field, representing motivic cohomology with integral coefficients. This construction was generalized to any base scheme S in the work of [DRØ03, Ex. 3.4].

---

[24]Recall it represents *homotopy invariant K-theory* according to [Cis13].

In [CD12b, 11.2.17], D.C. Cisinski and the author gave a general construction of this spectrum relying on other ideas of Voevodsky. Let Λ be a localization of the ring $\mathbb{Z}$ and S be any scheme. Then one defines a ring spectrum $\mathbf{H}\Lambda_S$ such that the abelian group

$$\mathrm{Hom}_{S\mathscr{H}(S)}(S^0, \mathbf{H}\Lambda_S(m)[n]) = H_{\mathcal{M}}^{n,m}(S, \Lambda)$$

is Voevodsky's motivic cohomology of S with coefficients in Λ.[25] According to [CD12b, 11.2.21], for any morphism $f : S' \to S$ of schemes, there exists a canonical morphism of ring spectra:

$$\tau_f : f^*(\mathbf{H}\Lambda_S) \to \mathbf{H}\Lambda_{S'}.$$

In other word, we have a weak absolute ring spectrum (over the category of all schemes) that we will denote $\mathbf{H}\Lambda^w$. A fundamental conjecture of Voevodsky (cf [Voe02, Conj. 17] for the case of integral coefficients) can be reformulated saying that $\mathbf{H}\Lambda^w$ is in fact a (strong) $\mathscr{S}$-absolute ring spectrum over the category of all schemes.

Unfortunately, this conjecture is still not known. In [CD12b, 16.1.7], we prove it when $\Lambda = \mathbb{Q}$ if one restricts to the case of geometrically unibranch schemes.[26]

On the other hand, by construction, the above map τ_f is obviously an isomorphism when f is smooth.[27] Thus, if $\mathscr{S}$ is the category of smooth schemes over an arbitrary base scheme S, then the collection $\mathbf{H}\Lambda_X$ defines an S-absolute ring spectrum simply denoted by $\mathbf{H}\Lambda_{/S}$. Moreover, from the perspective of this paper, we can safely extend $\mathbf{H}\Lambda_{/S}$ as an S-absolute ring spectrum taking the various pullbacks of $\mathbf{H}\Lambda_S$ over any S-scheme as in the first Example above. This is particularly relevant if S is the spectrum of a prime field (we refer the reader to [CD14] for more details).

(5) Recall finally a very interesting construction of Markus Spitzweck. Let Λ be a localization of $\mathbb{Z}$. In [Spi13], Spitzweck defines a ring spectrum $\mathbf{M}\Lambda_{\mathbb{Z}}$ over $\mathbb{Z}$ (*loc. cit.*, Def. 4.27) whose pullbacks to any field $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{Z})$ is Voevodsky's Eilenberg-Mac Lane spectrum (*loc. cit.* 6.7, 9.16 and 9.17). Thus, the absolute ring spectrum $\mathbf{M}\Lambda$ obtained by considering for any scheme $f : X \to \mathrm{Spec}(\mathbb{Z})$ the ring spectrum $f^*(\mathbf{M}\Lambda_{\mathbb{Z}})$ is an interesting candidate for motivic cohomology in general. Note in particular that according to *loc. cit.*, 7.19, for any smooth $\mathbb{Z}$-scheme X,

$$\mathbf{M}\mathbb{Z}^{n,m}(X) = CH_{2m-n}(X, m)$$

where the right hand side is Bloch's higher Chow group as defined by Levine.

1.2.4. In the remainder of this section, we consider an absolute ring spectrum $\mathbb{E}$ (which can be weak in the sense or Remark 1.2.2 until Paragraph 1.2.8) and define structures on its associated cohomology theory.

---

[25]Note that, in the current state of the theory, the consideration of the ring Λ for coefficients is crucial as, given a localization Λ' of Λ, we do not always have:

$$\mathbf{H}(\Lambda')_S \simeq \mathbf{H}\Lambda_S \otimes_\Lambda \Lambda'.$$

See however Proposition 11.2.19 of *loc. cit.* for cases where this identification holds.

[26]In fact, we proved that the ring spectrum $\mathbf{H}\mathbb{Q}_S$ coincides with $\mathbf{H}_{\mathbb{E},S}$ when S is geometrically unibranch.

[27]Using the notations of [CD12b, Par. 11.2.21], when f is smooth, the exchange morphism $f^*\varphi_* \to \varphi_* f^*$ used to construct τ_f is an isomorphism.

As usual, we call closed (resp. open) pair any couple (X, Z) (resp. (X/U)) such that X is a scheme and Z (resp. U) is a closed (resp. open) subscheme of X.

Definition 1.2.5. Given a closed pair (X, Z), corresponding to a closed immersion i, and a couple $(n, m) \in \mathbb{Z}^2$, we define the relative cohomology of (X, Z) in bi-degree (n, m) with coefficients in $\mathbb{E}$ by one of the following equivalent formulas:

$$\mathbb{E}_Z^{n,m}(X) := \mathrm{Hom}_X(X/X - Z, \mathbb{E}_X(m)[n])$$
$$= \mathrm{Hom}_X(i_*(\mathbb{1}_Z), \mathbb{E}_X(m)[n]) = \mathrm{Hom}_Z(\mathbb{1}_Z, i^! \mathbb{E}_X(m)[n]).$$

When it clarifies the notation, we will also put: $\mathbb{E}^{n,m}(X, Z) := \mathbb{E}_Z^{n,m}(X)$.
When $X = Z$ we simply put as usual: $\mathbb{E}^{n,m}(X) := \mathbb{E}_X^{n,m}(X)$.

A morphism $\varphi : \mathbb{E} \to \mathbb{F}$ of absolute ring spectra obviously induces for any closed pair (X, Z) a morphism

(1.2.5.a) $$\varphi_* : \mathbb{E}_Z^{n,m}(X) \to \mathbb{F}_Z^{n,m}(X).$$

1.2.6. *Contravariant functoriality*: Consider a closed pair (X, Z) and a morphism $f : Y \to X$ in $\mathscr{S}$. Let $T = Y \times_X Z$ be the pullback in the category of schemes, considered as a closed subscheme of Y. Then we get a morphism of abelian groups:

$$f^* : \mathbb{E}_Z^{n,m}(X) \to \mathbb{E}_T^{n,m}(Y)$$

by one of the following equivalent definitions:

- According to Remark 1.1.3, $f^*(X/X - Z) = Y/Y - T$. Thus, for a cohomology class $\rho : (X/X - Z) \to \mathbb{E}_X(m)[n]$, the pullback map $f^*(\rho)$ gives the desired map:

$$(Y/Y - T) = f^*(X/X - Z) \to f^* \mathbb{E}_X(m)[n] \xrightarrow{\epsilon_f} \mathbb{E}_Y(m)[n].$$

- Consider the pullback square (1.1.2.a) with f as above and i the immersion of Z in X. Then property (A6) gives a canonical identification: $f^* i_*(\mathbb{1}_Z) = k_* g^*(\mathbb{1}_Z) = k_*(\mathbb{1}_T)$. Thus, taking a cohomology class $\rho : i_*(\mathbb{1}_Z) \to \mathbb{E}_X(m)[n]$, the pullback map $f^*(\rho)$ gives the desired map:

$$k_*(\mathbb{1}_T) = f^* i_*(\mathbb{1}_Z) \to f^* \mathbb{E}_X(m)[n] \xrightarrow{\epsilon_f} \mathbb{E}_Y(m)[n].$$

1.2.7. *Covariant functoriality*: Consider closed immersions $T \xrightarrow{\nu} Z \xrightarrow{i} X$ and put $k = i \circ \nu$.
We define a pushforward in cohomology with support

$$\nu_! : \mathbb{E}_T^{n,m}(X) \to \mathbb{E}_Z^{n,m}(X)$$

by one of the following equivalent definitions:

- By functoriality of homotopy colimits, the immersion $(X - Z) \to (X - T)$ induces a canonical map $\bar{\nu} : (X/X - Z) \to (X/X - T)$ in $S\mathscr{H}(X)$. Then we associate to a cohomology class $\rho : (X/X - T) \to \mathbb{E}_X(m)[n]$ the map $\nu_!(\rho) := \rho \circ \bar{\nu}$.
- The unit map of the adjunction (ν^*, ν_*) gives a morphism

$$ad_\nu : i_*(\mathbb{1}_Z) \to i_* \nu_* \nu^*(\mathbb{1}_Z) = k_*(\mathbb{1}_T)$$

in $S\mathscr{H}(X)$. Then for any cohomology class $\rho : k_*(\mathbb{1}_T) \to \mathbb{E}_X(m)[n]$, we put: $\nu_!(\rho) := \rho \circ ad_\nu$.

1.2.8. *Products*: Consider the assumption of the preceding paragraph, except that now we ask $\mathbb{E}$ is an absolute ring spectrum. We define a *refined product* in cohomology with supports

(1.2.8.a) $$\mathbb{E}_T^{n,m}(Z) \otimes \mathbb{E}_Z^{s,t}(X) \to \mathbb{E}_T^{n+s,m+t}(X), (\lambda, \rho) \mapsto \lambda \cdot \rho$$

as follows. Consider cohomology classes:

$$\lambda : \nu_*(\mathbb{1}_T) \to \mathbb{E}_Z(t)[s], \ \rho : i_*(\mathbb{1}_Z) \to \mathbb{E}_X(m)[n].$$

Applying i_* to λ, we get a map:

$$k_*(\mathbb{1}_T) = i_*\nu_*(\mathbb{1}_T) \to i_*(\mathbb{E}_Z(m)[n]) \xrightarrow{\epsilon_i^{-1}} i_*i^*(\mathbb{E}_X(m)[n]) \simeq \mathbb{E}_X \wedge i_*(\mathbb{1}_Z)(m)[n]$$

where the last identification uses (A7). Let us simply denote by $i_*(\lambda)$ this composite map. We define the product $\lambda \cdot \rho$ as the following composite morphism:

$$k_*(\mathbb{1}_T) \xrightarrow{i_*(\lambda)} \mathbb{E}_X \wedge i_*(\mathbb{1}_Z)(m)[n] \xrightarrow{Id \wedge \rho} \mathbb{E}_X \wedge \mathbb{E}_X(m+s)[n+t] \xrightarrow{\mu_X} \mathbb{E}_X(m+s)[n+t].$$

One can deduce from this product the usual product of cohomology with support as follows. Assume we are given a cartesian square of closed immersions:

$$\begin{array}{ccc} T & \xrightarrow{i'} & Z' \\ \nu \downarrow & & \downarrow \iota \\ Z & \xrightarrow{i} & X \end{array}$$

Then one defines the *cup-products* by the following formula:

(1.2.8.b) $$\mathbb{E}_Z^{n,m}(X) \otimes \mathbb{E}_{Z'}^{s,t}(X) \to \mathbb{E}_T^{n+s,m+t}(X), (\alpha, \beta) \mapsto \alpha \cup \beta = \iota^*(\alpha) \cdot \beta.$$

One can also describe this product using the following identification in $S\mathscr{H}(X)$:

$$(X/X - Z) \wedge (X/X - Z') = (X/X - T).$$

This can be obtained by a direct computation of homotopy colimits or by applying formula (1.1.3.a) and (A6) as follows:

$$(X/X - Z') \wedge (X/X - Z) = \iota_*(\mathbb{1}_{Z'}) \wedge i_*(\mathbb{1}_Z) = \iota_*(\iota^* i_*(\mathbb{1}_Z)) = \iota_* i'_* \nu^*(\mathbb{1}_Z) = (X/X - T).$$

Then given $\alpha : (X/X - Z) \to \mathbb{E}(m)[n]$ and $\beta : (X/X - Z') \to \mathbb{E}_X(t)[s]$, one checks that $\iota^*(\alpha) \cdot \beta$ is equal to the following composite map:

$$(X/X - T) = (X/X - Z) \wedge (X/X - Z') \xrightarrow{\alpha \wedge \beta} \mathbb{E}_X \wedge \mathbb{E}_X(m+s)[n+s]$$

$$\xrightarrow{\mu_X} \mathbb{E}_X(m+s)[n+r].$$

Note that when $Z = Z' = X$ the product (1.2.8.b) describes the usual cup-product in $\mathbb{E}$-cohomology.

Remark 1.2.9. The need of the refined product just defined is the only reason for us to use absolute ring spectra. Indeed, we have not only used the existence of the structural map ϵ_f but also the fact it is an isomorphism.

We have gathered the basic properties of these operations in the following proposition:

Proposition 1.2.10. *Given an absolute ring spectrum $\mathbb{E}$, the following properties hold:*

(E1) $f^*g^* = (gf)^*$, $\nu'_! \nu_! = (\nu'\nu)_!$ *whenever defined.*

(E2) *When ν is a closed nil-immersion, $\nu_!$ is an isomorphism.*

(E3) *Consider the following cartesian squares:*

$$
\begin{array}{ccccc}
T' & \xrightarrow{\nu'} & Z' & \longrightarrow & X' \\
\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
T & \xrightarrow{\nu} & Z & \longrightarrow & X
\end{array}
$$

where horizontal maps are closed immersions. Then for any cohomology class $\rho \in \mathbb{E}_T^{**}(X)$, $f^*\nu_!(\rho) = \nu'_! f^*(\rho)$.

(E4) *Consider closed immersions: $W \xrightarrow{\iota} T \xrightarrow{\nu} Z \xrightarrow{i} X$.*
Then for any triple $(\lambda, \alpha, \beta) \in \mathbb{E}_W^{**}(T) \times \mathbb{E}_T^{**}(Z) \times \mathbb{E}_Z^{**}(X)$, one has:
$\lambda \cdot (\alpha \cdot \beta) = (\lambda \cdot \alpha) \cdot \beta$.

(E5) *Under the assumption of (E3), for any couple $(\lambda, \rho) \in \mathbb{E}_T^{**}(Z) \times \mathbb{E}_Z^{**}(X)$, one has:*
$f^*(\lambda \cdot \rho) = g^*(\lambda) \cdot f^*(\rho)$.

(E6) *Under the assumption of (E4), for any couple $(\lambda, \rho) \in \mathbb{E}_T^{**}(Z) \times \mathbb{E}_Z^{**}(X)$, one has:*
$\nu_!(\lambda \cdot \rho) = \nu_!(\lambda) \cdot \rho$.

(E7) *Consider the following diagram:*

$$
\begin{array}{ccccc}
T' & \xrightarrow{\nu'} & Z' & & \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} & & \\
T & \xrightarrow{\nu} & Z & \xrightarrow{i} & X
\end{array}
$$

made of closed immersions and such that the square is cartesian. Then for any couple $(\lambda, \rho) \in \mathbb{E}_T^{**}(Z) \times \mathbb{E}_{Z'}^{**}(X)$, one has: $h_!(g^*(\lambda) \cdot \rho) = \lambda \cdot g_!(\rho)$.

Proof. (E1) is clear and (E2) follows from Remark 1.1.2.

Consider (E3). Recall that $\nu_!$ (resp. $\nu'_!$) is induced by (pre)composition with the canonical map $\epsilon_X : (X/X - Z) \to (X/X - T)$ (resp. $\epsilon_Y : (Y/Y - Z') \to (Y/Y - T')$). Then (E3) simply follows from the fact $f^*(\epsilon_X) = \epsilon_Y$.

Point (E4) follows easily using the associativity of the product on the ring spectrum $\mathbb{E}_X$. Point (E5) follows easily using (A6) and point (E6) is clear.

Point (E7) is the most difficult one. Let us denote by i' (resp. k, k') the obvious embedding of Z' (resp. T, T') in X. The cohomology classes of the statement to be proved can be written as follows: $\lambda : \nu_*(\mathbb{1}_T) \to \mathbb{E}_Z$, $\rho : i'_*(\mathbb{1}_{Z'}) \to \mathbb{E}_X$.
We let $ad_g : i_*(\mathbb{1}_Z) \to i'_*(\mathbb{1}_{Z'})$ (resp. $ad_h : k_*(\mathbb{1}_T) \to k'_*(\mathbb{1}_{T'})$) be the map induced by the unit of the adjunction (g_*, g^*) (resp. (h^*, h_*)). Then the left hand side of the relation to be proved is equal to:

$$
k_*(\mathbb{1}_T) \xrightarrow{ad_h} k'_*(\mathbb{1}_{T'}) = i'_* g^* \nu_*(\mathbb{1}_Z) \xrightarrow{i'_* g^*(\lambda)} i'_* g^*(\mathbb{E}_Z) = i'_*(\mathbb{E}_{Z'}) = \mathbb{E}_X \wedge i'_*(\mathbb{1}_{Z'})
$$
$$
\xrightarrow{Id \wedge \rho} \mathbb{E}_X \wedge \mathbb{E}_X \xrightarrow{\mu} \mathbb{E}_X,
$$

while the right hand side is:

$$
k_*(\mathbb{1}_T) = i_* \nu_*(\mathbb{1}_Z) \xrightarrow{i_*(\lambda)} i_*(\mathbb{E}_Z) = \mathbb{E}_X \otimes i_*(\mathbb{1}_Z) \xrightarrow{Id \wedge ad_g} \mathbb{E}_X \wedge i'_*(\mathbb{1}_{Z'})
$$
$$
\xrightarrow{Id \wedge \rho} \mathbb{E}_X \wedge \mathbb{E}_X \xrightarrow{\mu} \mathbb{E}_X.
$$

To check the identity, we prove the commutativity of the following diagram:

$$k_*h_*h^*(\mathbb{1}_T) = i_*g_*\nu'_*h^*(\mathbb{1}_T) \xrightarrow{b^{-1}} i_*g_*g^*\nu_*(\mathbb{1}_T) \xrightarrow{\lambda} i_*g_*g^*(\mathbb{E}_Z) = \mathbb{E}_X \wedge i_*g_*g^*(\mathbb{1}_Z)$$

$$\uparrow^{ad_h} \qquad (1) \qquad \uparrow^{ad_g} \qquad \uparrow^{ad_g} \qquad \uparrow^{Id \wedge ad_g}$$

$$k_*(\mathbb{1}_T) = i_*\nu_*(\mathbb{1}_T) \xrightarrow{\lambda} i_*(\mathbb{E}_Z) = \mathbb{E}_X \wedge i_*(\mathbb{1}_Z)$$

where the maps with label ad_g (resp. ad_h) are induced by the unit of the adjunction (g^*, g_*) (resp. (h^*, h_*)) and the map b, whose inverse appears in the above diagram, stands for the base change isomorphism obtained from (A6). Only the commutativity of part (1) is non trivial. It follows from the description of the base change map b as the composite:

$$g^*\nu_* \xrightarrow{ad_h} g^*\nu_*h_*h^* = g^*g_*\nu'_*h^* \xrightarrow{ad'_g} \nu'_*h^*$$

and the relation between the unit and the counit ad'_g of the adjunction (g^*, g_*). □

1.2.11. It is usually convenient to introduce the following notions. We define a *morphism of closed pairs* $\Delta : (Y, T) \to (X, Z)$ as being a commutative diagram

$$\begin{array}{ccc} T & \rightarrowtail & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \rightarrowtail & X \end{array}$$

such that the induced map $T \xrightarrow{\nu} Y \times_X Z$ is a nil-immersion.[28] We say that the morphism Δ is *cartesian* when the above square is cartesian in the category of schemes – *i.e.* ν is an isomorphism. We also use the notation (f, g) for Δ when we want to refer to the morphisms in the above square.

The composition of morphisms of closed pairs is given in categorical terms by the vertical composition of squares.

Using (E2), we associate to Δ the following composite morphism of abelian groups:

$$\Delta^* : \mathbb{E}^{n,m}_Z(X) \xrightarrow{f^*} \mathbb{E}^{n,m}_{Z \times_X Y}(Y) \xrightarrow{\nu_!^{-1}} \mathbb{E}^{n,m}_T(Y).$$

The relation $\Delta^*\Theta^* = (\Theta\Delta)^*$ is clear from (E1) and (E3).

Remark 1.2.12. Given a transformation $\varphi : \mathbb{E} \to \mathbb{F}$, it is clear that the associated morphism (1.2.5.a) is natural with respect to contravariant and covariant functorialities. Moreover, it is compatible with all the products of Paragraph 1.2.8.

The following property is not essential to our purpose so that we do not list it among the fundamental axioms of an absolute cohomology.[29]

Lemma 1.2.13. *Let $(X_\alpha, Z_\alpha)_{\alpha \in A}$ be an essentially affine projective system of closed pairs of $\mathscr{S}$ whose projective limit[30] (X, Z) is still a closed pair in $\mathscr{S}$.*

Then, given any $\mathscr{S}$-absolute ring spectrum $\mathbb{E}$, the canonical map:

$$\varinjlim_{\alpha \in A^{op}} \left(\mathbb{E}^{n,i}_{Z_\alpha}(X_\alpha) \right) \to \mathbb{E}^{n,i}_Z(X)$$

is an isomorphism.

---

[28]In other words, the diagram Δ is topologically cartesian.

[29]It will only be applied in Example 1.3.4 in the case of k-absolute ring spectra.

[30]in the category of pairs of schemes,

Proof. According to [CD12b, 4.3.6], the category $S\mathscr{H}$ is a continuous motivic category in the sense of *loc. cit.*, 4.3.2. Thus the lemma follows by applying *lo. cit.*, 4.3.4 to the projective systems

$$(X_\alpha/X_\alpha - Z_\alpha)_{\alpha \in A}, (\mathbb{E}_{X_\alpha})_{\alpha \in A}$$

given that for any index $\alpha \in A$, the object $(X_\alpha/X_\alpha - Z_\alpha)$ of $S\mathscr{H}(X_\alpha)$ is constructible. $\qquad\square$

1.3. Absolute purity.

1.3.1. Let (X, Z) be a closed pair in $\mathscr{S}$. We say that (X, Z) is *regular* if the inclusion $i : Z \to X$ is a regular embedding.

Assume (X, Z) is regular. We let $N_Z X$ (resp. $B_Z X$) be the normal cone (resp. blow-up) of Z in X. Recall the definition of the *deformation space* of (X, Z) as:

$$D_Z X = B_{0 \times Z}(\mathbb{A}_X^1) - B_Z X$$

(see [Ros96], or [Dég08a, §4.1] for this presentation).[31] This is a flat scheme over $\mathbb{A}^1$ whose fiber over 1 (resp. 0) is X (resp. $N_Z X$). Note also that $D_Z Z = \mathbb{A}_Z^1$ is a closed subscheme of $D_Z X$ so that we finally get a *deformation diagram* of closed pairs:

(1.3.1.a) $$(X, Z) \xrightarrow{\sigma_1} (D_Z X, \mathbb{A}_Z^1) \xleftarrow{\sigma_0} (N_Z X, Z),$$

made of cartesian morphisms. Note this diagram is natural with respect to cartesian morphisms of closed pairs.

Definition 1.3.2. Let $\mathbb{E}$ be a weak $\mathscr{S}$-absolute (ring) spectrum (Remark 1.2.2).

For any closed pair (X, Z), we say that (X, Z) is $\mathbb{E}$-*pure* if (X, Z) is regular and the morphisms

$$\mathbb{E}^{**}(X, Z) \xleftarrow{\sigma_1^*} \mathbb{E}^{**}(D_Z X, \mathbb{A}_Z^1) \xrightarrow{\sigma_0^*} \mathbb{E}^{**}(N_Z X, Z)$$

induced by the above deformation diagram (see notation in Definition 1.2.5) are isomorphisms of bigraded abelian groups.

We say that $\mathbb{E}$ is *absolutely pure* if any regular closed pair in $\mathscr{S}$ is $\mathbb{E}$-pure.

Note the following trivial stability properties of spectra satisfying the absolute purity property with respect to a given closed pair:

Proposition 1.3.3. *Let (X, Z) be a regular closed pair in $\mathscr{S}$.*

Then the category of absolute spectra (resp. weak absolute spectra) $\mathbb{E}$ such that (X, Z) is $\mathbb{E}$-pure is stable by suspension and twists, direct factors, infinite direct sums and distinguished triangles.[32]

Example 1.3.4. We refer the reader to Example 1.2.3 for the absolute ring spectra appearing below:

(1) Let $\mathbb{E}$ be any absolute ring spectrum and S be a scheme. We will say that a closed S-pair (X, Z) is smooth if X and Z are smooth over S.

According to [MV99, §3.2, th. 2.23, p. 115], any smooth closed S-pair (X, Z) is $\mathbb{E}$-pure. Thus, if $\mathscr{S}$ is the category of smooth S-schemes, any $\mathscr{S}$-absolute ring spectrum is absolutely pure. Note in particular this is the case

---

[31]Note that according to our convention on $\mathscr{S}$, this is a scheme in $\mathscr{S}$.

[32]A distinguished triangle of (weak) absolute spectra is the datum of distinguished triangles over each schemes which are compatible with the structural base change maps.

for the $\mathscr{S}m_S$-absolute ring spectrum $\mathbf{H}_\Lambda$, representing motivic cohomology with Λ-coefficients.

(2) Let k be a perfect field.

According to Popescu theorem, any regular closed pair (X, Z) over k can be written as a projective limit of smooth closed pairs over k provided X is regular. Thus, according to the computation of the cohomology with supports of a projective limit (cf. Lemma 1.2.13), we deduce from the previous example that any k-absolute ring spectrum is absolutely pure. This is in particular the case for the k-absolute motivic Eilenberg-MacLane spectrum $\mathbf{H}_\Lambda$.

(3) According to [CD12b], respectively Theorems 14.4.1 and 13.6.3, the absolute ring spectra $\mathbf{H}_{\mathbb{B}}$ and $\mathbf{KGL}$ are absolutely pure.

(4) Recall from [NSØ09b, 10.5] that there is an isomorphism of absolute ring spectra:
$$\mathbf{MGL} \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbf{H}_{\mathbb{B}}[b_1, b_2, \ldots]$$
where b_i is a generator of degree $(2i, i)$. Then the preceding example implies $\mathbf{MGL} \otimes \mathbb{Q}$ is absolutely pure. We deduce from that example that any Landweber spectrum (cf. [NSØ09b, th. 7.3]) with rational coefficients is absolutely pure.

The only integral example of an absolutely pure ring spectrum is given by homotopy invariant K-theory. However, in view of the previous examples, we think it is reasonable to conjecture the following:

Conjecture A. *The absolute ring spectrum $\mathbf{MGL}$ is absolutely pure.*

Conjecture B. *The absolute ring spectrum S^0 is absolutely pure.*

Remark 1.3.5.   (1) Recall the notion of cellular spectra, first introduced in [DI05]. Over a scheme S, one defines the category of cellular spectra as the smallest thick triangulated subcategory of $S\mathscr{H}(S)$ which contains the spheres that we denote here $S^0(m)[n]$ for any integers $(n, m) \in \mathbb{Z}$. According to the previous proposition, Conjecture B implies that any absolute spectrum $\mathbb{E}$ which is cellular over $\mathrm{Spec}(\mathbb{Z})$ is absolutely pure.

In particular, Conjecture B implies Conjecture A because $\mathbf{MGL}_S$ is cellular (see [DI05, 6.4]) – and the former conjecture would reprove the absolute purity for $\mathbf{KGL}$ (see [DI05, 6.4]).

(2) Let S be any scheme and Λ be the localization of $\mathbb{Z}$ at the primes which are not invertible on S. Let $\mathbf{M}\Lambda$ be the absolute ring spectrum of Spitzweck (see Ex. 1.2.3(5)). According to [Spi13, Cor. 11.4], $\mathbf{M}\Lambda_S$ is cellular. Thus, according point (1) above, Conjecture B implies that $\mathbf{M}\Lambda$ is absolutely pure over the category of S-schemes.

(3) Supporting these conjectures:
  - $\mathbf{MGL}$ and S^0 are absolutely pure for closed pairs which are smooth over some base.
  - The conjecture for $\mathbf{MGL} \otimes \mathbb{Q}$ is true according to point (3) of the preceding example.
  - The conjecture for $S^0 \otimes \mathbb{Q}$ is true if one restricts to base schemes over which -1 is a sum of squares. In fact, under this assumption, according to Morel theorem one has $S^0 \otimes \mathbb{Q} \simeq \mathbf{H}_{\mathbb{B}}$ (see [CD12b, Cor. 16.2.14]) and we are reduced to point (3) of the above example.

(4) Let us consider the Eilenberg-MacLane motivic ring spectrum, as a weak absolute ring spectrum, $\mathbf{H}\mathbb{Z}^w$ – Example 1.2.3(4). Independently of the conjecture of Voevodsky which ask whether $\mathbf{H}\mathbb{Z}^w$ is an absolute ring spectrum, it is interesting to ask if $\mathbf{H}\mathbb{Z}^w$ is absolutely pure. Note that if this was true, then using the coniveau spectral sequence we will get for any regular scheme S an isomorphism

$$H_{\mathcal{M}}^{2n,n}(S, \mathbb{Z}) \simeq CH^n(S)$$

where the right hand side is the group of n-codimensional cycles in S modulo rational equivalence (see [Ful98, Gil05]).[33] The existence of this isomorphism is particularly interesting as there is a well defined product of Voevodsky's motivic cohomology while it is still an open question to define a product on the classical Chow (see [SGA6, XIV, §8]).

(5) We have separated the case of $\mathbf{MGL}$ with that of S^0 because $\mathbf{MGL}$ is oriented (see next section) and S^0 is not. Note however that in our formulation of absolute purity, we do not need any orientation. In the particular, one can see that the Conjecture B is equivalent to ask the following:

For any closed immersion $i : Z \to X$ between regular schemes, with normal bundle $N_Z X$, there exists a canonical isomorphism in $S\mathcal{H}(Z)$:

$$i^!(S^0) \simeq \text{Th}(-N_Z X)$$

– see Par. 2.2.1 for recall on the definition of the right hand side.

The resulting map in $S\mathcal{H}(S)$

$$i_*(\text{Th}(-N_Z X)) \to S^0$$

would be called the (unoriented) fundamental class of i (relative to S^0). This is the unoriented version of Definition 2.3.1. Moreover, one can see that this class would be universal among the fundamental classes constructed in this paper (using that any ring spectrum is an algebra over S^0).

1.4. Analytical invariance.
By construction of the stable homotopy category, an absolute cohomology theory satisfies cohomological descent for the Nisnevich topology. A convenient way to express this property uses the so called *excision property*. Let us start by an elementary geometric fact which will link excision with analytical invariance.

Proposition 1.4.1. *Let $f : Y \to X$ be a morphism locally of finite type, Z a closed subscheme of X, $T = f^{-1}(Z)$. Then the following assertions are equivalent:*

(i) *f is étale at all points of the scheme T and the induced morphism $f|_T : T_{red} \to Z_{red}$ is an isomorphism.*

(ii) *the induced morphism $\hat{f} : \hat{Y}_T \to \hat{X}_Z$ between the respective formal completions of Y at T and X at Z is an isomorphism.*

Proof. Note first that given any point $x \in Z$, the completion of the local ring of $\hat{X}_Z$ at x coincides with the completion of the local ring of X at x and the corresponding isomorphism is natural in (X, Z, x).

---

[33]The absolute purity property allows to compute the E_1-term as the $\mathbf{H}\mathbb{Z}^w$-cohomology of the residue fields for $\mathbf{H}\mathbb{Z}^w$, as S is regular. As the motivic cohomology of fields is known, one gets the mentioned computation after identifying one of the differential with the classical divisor class map.

Thus the equivalence of the assertions follow from [EGA4, 17.6.3] which asserts that f is étale at a point y of T, $x = f(y)$, if and only if the induced morphism $\hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{Y,y}$ between the respective completed local rings is an isomorphism.   □

Definition 1.4.2. Let $\Delta = (f, g) : (Y, T) \to (X, Z)$ be a morphism of closed pairs. One says that Δ is *excisive* if the morphism f and the closed scheme $Z \subset X$ satisfy the equivalent assertions of the preceding proposition.

Thus, generalizing slightly [MV99, Lem. 1.6, p. 98], we get the following property of the absolute cohomology represented by $\mathbb{E}$:

(Nis) For any excisive morphism $\Delta : (Y, T) \to (X, Z)$, the associated pullback $\Delta^* : \mathbb{E}_Z^{**}(X) \to \mathbb{E}_T^{**}(Y)$ is an isomorphism.

Remark 1.4.3. In fact, it is well known since the work of Morel-Voevodsky that this property characterizes Nisnevich descent, though one only needs excisive morphisms which are globally étale (see for example [CD12a, Prop. 1.1.10] for a precise statement).

An interesting corollary of the absolute purity property is the following stronger statement (see the end of this section for a stronger result):

Proposition 1.4.4. *Let $\mathbb{E}$ be an absolutely pure $\mathscr{R}$eg-spectrum. Let X be a regular local scheme with closed point x, $\hat{X}$ be its completion at the point x and consider the canonical map $f : (X, x) \to (\hat{X}, x)$.*
Then the induced morphism of cohomology with support:

$$f^* : \mathbb{E}_x^{**}(\hat{X}) \to \mathbb{E}_x^{**}(X)$$

is an isomorphism.

Remark 1.4.5. This result will be used to compute residues in Proposition 5.4.5.

Using the Artin approximation property, one can give a much stronger result than the preceding proposition according to an initial idea of Wildeshaus (cf. [Wil06, §5], in the case of motives over a field).

Theorem 1.4.6. *Let S be an excellent scheme, (X, Z) and (Y, T) be closed pairs made of S-schemes essentially of finite type.*
Let $\mathbb{E}$ be an S-absolute spectrum.
Assume there exists an isomorphism $\mathfrak{f} : \hat{Y}_T \to \hat{X}_Z$ between the respective formal completions. Then there exists an isomorphism:

$$\mathfrak{f}^* : \mathbb{E}_Z^{**}(X) \to \mathbb{E}_T^{**}(Y)$$

which depends only on $\mathfrak{f}$.

Proof. The following proof follows that of [Wil06, 5.5] using the more advanced theory we now have at our will.

We can assume $Z = T$, seen as a reduced scheme. Let us fix a point $z \in Z$, and let $X_{(z)}, Y_{(z)}$ be the respective local schemes of X, Y at z. According to [Swa98, 2.4], the henselisation of $\mathcal{O}_{X,z}$ satisfies the Artin approximation property. Thus according to [Art69, 2.6], there exists a common Nisnevich neighborhood $W_{(z)}$ of $(X_{(z)}, z)$ and $(Y_{(z)}, z)$. Moreover, one can lift the situation in a neighborhood of z,

both in X and Y: there exists an S-scheme W essentially of finite type, which lifts $W_{(z)}$ and fits into the following commutative diagram:

$$
\begin{array}{ccccc}
 & & W & & \\
 & {}^{f}\swarrow & \uparrow & \searrow^{g} & \\
X' & \leftarrow & Z' & \rightarrow & Y' \\
\uparrow & & \downarrow & & \uparrow \\
X & \leftarrow & Z & \rightarrow & Y
\end{array}
$$

where the squares are cartesian, X', Y', Z' are open neighborhood of z in X, Y, Z respectively, and f (resp. g) is a Nisnevich neighborhood of Z' in X (resp. Y).

Using this construction, one can further find Zariski hypercoverings $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ of X, Y, Z and a simplicial scheme $\mathcal{W}$ which fits into the following commutative diagram:

(1.4.6.a)
$$
\begin{array}{ccccc}
 & & \mathcal{W} & & \\
 & {}^{f}\swarrow & \uparrow & \searrow^{g} & \\
\mathcal{X} & \leftarrow & \mathcal{Z} & \rightarrow & \mathcal{Y} \\
{}^{p}\downarrow & & \downarrow & & \downarrow^{q} \\
X & \leftarrow & Z & \rightarrow & Y
\end{array}
$$

such that the squares are cartesian and for each integer $n \geq 0$, $\mathcal{W}_n$ is a Nisnevich neighborhood of $\mathcal{Z}_n$ in $\mathcal{X}_n$ (resp. $\mathcal{Y}_n$).

To finish the proof, one has to use the fact that the stable homotopy category $S\mathcal{H}(S)$ can be extended to simplicial schemes according to [Ayo07b, Chap. 4] or [CD12b, 3.1]. This implies that the cohomology $\mathbb{E}^{**}$ can be extended to simplicial S-schemes and simplicial Z-pairs. Then the following sequence gives us almost the desired isomorphism

$$
\mathfrak{f}_{\mathcal{W}}^{*} : \mathbb{E}_Z^{**}(X) \xrightarrow{p^*} \mathbb{E}_{\mathcal{Z}}^{**}(\mathcal{X}) \xrightarrow{f^*} \mathbb{E}_{\mathcal{Z}}^{**}(\mathcal{W}) \xrightarrow{(g^*)^{-1}} \mathbb{E}_{\mathcal{Z}}^{**}(\mathcal{Y}) \xrightarrow{(q^*)^{-1}} \mathbb{E}_Z^{**}(Y).
$$

Indeed, according to the Zariski descent property of $S\mathcal{H}$ as formulated in [CD12b, 3.2.7, 3.3.5], the maps p^* and q^* are isomorphisms. Moreover, we can derive from property (Nis) the fact the maps f^* and g^* are isomorphisms (either we apply the Zariski descent spectral sequence or we argue directly in $S\mathcal{H}$).

To finally get $\mathfrak{f}^*$, one has to take the limit of the isomorphisms $\mathfrak{f}_{\mathcal{W}}^*$ over the filtering category of diagrams of the form (1.4.6.a). $\qquad\square$

Remark 1.4.7.   (1) The fact $\mathfrak{f}^*$ depends only on $\mathfrak{f}$ can also be supplemented by the following cocycle condition: given composable isomorphisms $\mathfrak{f}$ and $\mathfrak{g}$ of certain formal completions, one gets: $(\mathfrak{f} \circ \mathfrak{g})^* = \mathfrak{g}^* \circ \mathfrak{f}^*$.
    (2) Under the presence of a ring structure on $\mathbb{E}$, the isomorphism $\mathfrak{f}^*$ is compatible with products as defined in Paragraph 1.2.8.

As a corollary, we get the following reinforcement of Prop. 1.4.4:

Corollary 1.4.8. *Let $\mathbb{E}$ be an absolute spectrum. Let X be a local scheme with closed point x, $\hat{X}$ be its completion at the point x and consider the canonical map $f : (X, x) \to (\hat{X}, x)$. Then the induced morphism of cohomology with support:*

$$
f^* : \mathbb{E}_x^{**}(\hat{X}) \to \mathbb{E}_x^{**}(X)
$$

is an isomorphism.

Example 1.4.9. The preceding theorem and its corollary can be applied in particular to any of the absolute ring spectrum of Example 1.2.3. In particular we get another proof of analytical invariance for De Rham cohomology and a proof in the case of rigid cohomology. The later case was also proved independently by Ouwehand in [Ouw14].

Note that it also holds for K-theory and algebraic cobordism. Even in the first case, this seems to be new.

Remark 1.4.10. The preceding corollary is especially useful in dealing with absolute purity. In order to prove it for a given absolute ring spectrum, one easily reduces to the case of a closed pair (X, x) where X is a local regular scheme with closed point x. According to the preceding corollary, one derives that we can further assume that X is the spectrum of a complete local scheme.

2. Orientation and characteristic classes

2.1. **Orientation theory and Chern classes.** The considerations of this section are well known in motivic homotopy theory (see for example [Bor03, Vez01] – in the case of a base field). They were also studied, with a slightly different formalism, in our paper [Dég08a].

2.1.1. Let S be a scheme. We will assume that the scheme $\mathbb{P}_S^n$ is pointed by the infinite point (of homogeneous coordinates $[0 : ... : 0 : 1]$ to fix ideas). Then we get a tower of pointed S-schemes

$$\mathbb{P}_S^1 \to \mathbb{P}_S^2 \to \ldots \mathbb{P}_S^n \xrightarrow{\iota_n} \mathbb{P}_S^{n+1} \to \ldots$$

where ι_n denotes the embedding of the last n-th coordinates. The colimit of this tower in the category of pointed sheaves defines an object $\mathbb{P}_S^\infty$ of the pointed homotopy category $\mathscr{H}_\bullet(S)$ – see [MV99]. We still denote by $\iota_1 : \mathbb{P}_S^1 \to \mathbb{P}_S^\infty$ the induced map in the homotopy category.

The following definition is now basic in motivic homotopy theory:

Definition 2.1.2. Let $\mathbb{E}$ be an absolute ring spectrum with unit $\eta_S : S^0 \to \mathbb{E}_S$ over a scheme S in $\mathscr{S}$. We can see η_S as a class in the reduced cohomology $\tilde{\mathbb{E}}^{2,1}(\mathbb{P}_S^1)$.[34]

An *orientation* of $\mathbb{E}$ over S is a class c_S in the reduced cohomology $\tilde{\mathbb{E}}^{2,1}(\mathbb{P}_S^\infty)$ such that $\iota_1^*(c_S) = \eta_S$.

An (absolute) *orientation* of $\mathbb{E}$ is a family of classes $c = (c_S)$ for any scheme S in $\mathscr{S}$ such that for any morphism $f : T \to S$, $f^*(c_S) = c_T$. In this situation, we also say that $(\mathbb{E}, c)$ is an absolute oriented ring spectrum.

Remark 2.1.3. To give an orientation of an S-absolute ring spectrum $\mathbb{E}$ it is enough and sufficient to give an orientation of $\mathbb{E}_S$.

Example 2.1.4. Let us review the absolute ring spectra of Example 1.2.3:

 (1) According to [CD12a, 2.2.8], the k-absolute ring spectrum associated with a Mixed Weil theory is canonically oriented.
 (2) Voevodsky's Einlenberg-Mac Lane motivic ring spectrum $\mathbf{H}_\Lambda$, considered as an S-absolute ring spectrum, is oriented according to [CD12b, Sec. 11.3].
 (3) The cobordism ring spectrum $\mathbf{MGL}$ is canonically oriented. This follows from theoretical reasons (see recall in Prop. 2.2.6) or can be directly seen from the construction (see Par. 2.2.5).

---

[34]By definition of the Tate twist, $\mathbb{E}^{2,1}(\mathbb{P}_S^1) = \mathbb{E}^{2,1}(S) \oplus \mathbb{E}^{0,0}(S)$.

(4) The K-theory ring spectrum $\mathbf{KGL}$ is oriented. Let S be a regular scheme ($S = \mathrm{Spec}(\mathbb{Z})$ would be enough). By construction, we have a canonical isomorphism:

$$K_0(S) \simeq \mathbf{KGL}^{0,0}(S).$$

In particular, any line bundle L/S defines an element $[L] \in \mathbf{KGL}^{0,0}(S)$. Moreover, Bott periodicity theorem implies the existence of the following isomorphism:

$$\mathbf{KGL}^{0,0}(S) \simeq K_0(S) \simeq \tilde{K}_0(\mathbb{P}^1_S, \infty) \simeq \widetilde{\mathbf{KGL}}^{0,0}(\mathbb{P}^1_S, \infty).$$

The image of 1 by this isomorphism is an element β in $\mathbf{KGL}^{-2,-1}(S)$ called the *Bott element*. Let λ be the canonical line bundle on $\mathbb{P}^\infty_S$. Then we define the orientation of $\mathbf{KGL}$ as follows:[35]

$$c^{\mathbf{KGL}} := \beta^{-1}.(1 - [\lambda^\vee]) \in \mathbf{KGL}^{2,1}(\mathbb{P}^\infty_S).$$

(5) Beilinson motivic cohomology spectrum $\mathbf{H}_\mathrm{B}$ is canonically oriented (see [CD12b, 14.1.5], [NSØ09b]).

It is worth to point out that we will show in Cor. 5.1.10 and Ex. 5.1.11 that the orientations involved in points (1), (2) and (5) are unique.

Remark 2.1.5. The presence of the Bott element in formulas involving the spectrum $\mathbf{KGL}$ can be explained as follows. By construction (see [Rio10]), for any integer n and any regular scheme S, one has a canonical contravariantly functorial isomorphism:

$$\varphi^n_S : \mathbf{KGL}^{n,0}(S) \to K_{-n}(S)$$

where the right hand side is the n-th Quillen K-theory of S. According to the definition above, multiplication by β on $\mathbf{KGL}^{**}(S)$ induces an isomorphism. Thus, for any couple of integers (r, n), we get a canonical isomorphism:

(2.1.5.a) $\qquad \varphi^{n,i}_S : \mathbf{KGL}^{n,i}(S) \to \mathbf{KGL}^{n-2i,0}(S) \to K_{2i-n}(S), x \mapsto \varphi^{n-2i}_S(\beta^i.x).$

2.1.6. Recall from [MV99, §4, 3.7] there exists a canonical isomorphism in $\mathscr{H}_\bullet(S)$:

$$\mathrm{B}\mathbb{G}_m \simeq \mathbb{P}^\infty_S$$

where $\mathrm{B}\mathbb{G}_m$ is the (Nisnevich) classifying space of $\mathbb{G}_m$. This immediately gives an application
(2.1.6.a)

$$\mathrm{Pic}(S) \overset{(*)}{=} \mathrm{Hom}_{\mathscr{H}^s_\bullet(S)}(S_+, \mathrm{B}\mathbb{G}_m) \to \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(S_+, \mathrm{B}\mathbb{G}_m) \simeq \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(S_+, \mathbb{P}^\infty_S)$$

where $\mathscr{H}^s_\bullet(S)$ denotes the simplicial homotopy category and the first map is induced by the projection functor $\mathscr{H}^s_\bullet(S) \to \mathscr{H}_\bullet(S)$ – the target category being the $\mathbb{A}^1$-localization of the source category. We have used [MV99, §4, 1.15] for the identification $(*)$.[36]

---

[35]This choice of orientation coincides with the one of [LM07, Ex. 1.1.5]. In the literature however, one can find different choices of orientations of the ring spectrum $\mathbf{KGL}$. The present choice is justified by Example 3.2.9 as well as the correct form of the Todd class appearing in the Riemann-Roch theorem 5.3.4.

[36]Note also Morel and Voevodsky proved the map (2.1.6.a) is an isomorphism whenever S is regular ; *op. cit.* Prop. 3.8.

Remark 2.1.7. For any integer n, we let λ_n be the canonical line bundle on $\mathbb{P}^n_S$ (see Notations and conventions, p. 12). Then the family $(\lambda_n)_{n \in \mathbb{N}}$ defines an element λ of $\mathrm{Pic}(\mathbb{P}^\infty_S)$ – which generates this group as a ring of formal power series over $\mathbb{Z}$ as soon as S is local and regular. The map (2.1.6.a) is characterized by the fact it sends λ to the canonical projection $\mathbb{P}^\infty_{S+} \to \mathbb{P}^\infty_S$.

Definition 2.1.8. Let $(\mathbb{E}, c)$ be an absolute oriented ring spectrum.

For any scheme S, we associate to the class c a canonical morphism of sets:

$$c_1 : \mathrm{Pic}(S) \xrightarrow{(2.1.6.a)} \mathrm{Hom}_{\mathscr{H}_\bullet(S)}(S_+, \mathbb{P}^\infty_S) \xrightarrow{\Sigma^\infty} \mathrm{Hom}_{S\mathscr{H}(S)}(\Sigma^\infty S_+, \Sigma^\infty \mathbb{P}^\infty_S)$$

$$\xrightarrow{(c_S)_*} \mathrm{Hom}_{S\mathscr{H}(S)}(\Sigma^\infty S_+, \mathbb{E}_S(1)[2]) = \mathbb{E}^{2,1}(S),$$

called the *first Chern class*.

Example 2.1.9. In the case of the orientation of **KGL** defined in Ex. 2.1.4, one gets for any line bundle over a regular scheme S:

$$c_1^{\mathbf{KGL}}(L) = \beta^{-1}.(1 - [L^\vee]).$$

2.1.10. According to this definition and the preceding remark, we get the following properties:

(a) For any morphism of schemes $f : T \to S$ and any line bundle λ on S, $f^* c_1(\lambda) = c_1(f^{-1}\lambda)$.

(b) Let $n \geq 0$ be an integer, λ_n be the line bundle over $\mathbb{P}^n_S$ considered in the above remark and $\nu_n : \mathbb{P}^n_S \to \mathbb{P}^\infty_S$ be the obvious morphism.

Then $c_1(\lambda_n) = \nu_n^*(c_S)$ as classes in $\mathbb{E}^{2,1}(\mathbb{P}^n_S)$, according to the above remark.

Remark 2.1.11. One must be careful that the relation $c_1(\lambda \otimes \lambda') = c_1(\lambda) + c_1(\lambda')$ does not necessarily hold. This is due to the fact that the second of the three maps considered in the definition of c_1 is not a morphism of abelian groups. This remark will be made more precise latter (see 2.1.22).

2.1.12. Next we recall the projective bundle theorem for an absolute oriented ring spectrum $(\mathbb{E}, c)$.

Let $p : P \to S$ be a projective bundle of rank n with canonical line bundle λ. We define the following morphism:

$$\epsilon_P : \oplus_{i=0}^n \mathbb{E}^{**}(X) \to \mathbb{E}^{**}(P), (x_0, \ldots, x_n) \mapsto \sum_i p^*(x_i).c_1(\lambda)^i.$$

Theorem 2.1.13. *With the above assumptions and notations, the morphism ϵ_P is an isomorphism.*

In other words, $\mathbb{E}^{**}(P)$ is a free graded $\mathbb{E}^{**}(X)$-module with basis $(1, c_1(\lambda), \ldots, c_1(\lambda)^n)$ (as usual).

Proof. The proof, essentially due to Morel, is the same as the proof of Th. 3.1 in [Dég08a]. We recall the main steps for the convenience of the reader.

Using the Mayer-Vietoris long exact sequence associated to an open cover by two open subsets, we reduce to the case where P is trivializable and then to the case $P = \mathbb{P}^n_S$ – this uses only property 2.1.10(a).

Then the proof goes on by induction on n ; the case $n = 0$ is trivial and the case $n = 1$ is an immediate consequence of the definition of the orientation c_S.

The principle of the induction is to use the following facts:

- Let $\mathbb{P}_S^n/\mathbb{P}_S^{n-1}$ be the cokernel of the embedding ι_{n-1} in the category of pointed sheaves. Then the sequence

$$\mathbb{P}_S^{n-1} \xrightarrow{\iota_n} \mathbb{P}_S^n \xrightarrow{\pi_n} \mathbb{P}_S^n/\mathbb{P}_S^{n-1}$$

  is homotopy exact ; in particular, it induces a long exact sequence:

$$\ldots \to \mathbb{E}^{**}(\mathbb{P}_S^n/\mathbb{P}_S^{n-1}) \xrightarrow{\pi_n^*} \mathbb{E}^{**}(\mathbb{P}_S^n) \xrightarrow{\iota_n^*} \mathbb{E}^{**}(\mathbb{P}_S^{n-1}) \to \ldots$$

- There exists a canonical isomorphisms in $\mathscr{H}_\bullet(S)$:

$$\tau_n : \mathbb{P}_S^n/\mathbb{P}_S^{n-1} \to (\mathbb{P}_S^1)^{\wedge,n}.$$

Then we are reduce to prove the following relations:

- $c_1(\lambda_{n-1})^n = 0$.
- Using the isomorphism τ_n, we get an isomorphism $\tau_n^* : \mathbb{E}^{2n,n}(\mathbb{P}_S^n/\mathbb{P}_S^{n-1}) \simeq \mathbb{E}^{0,0}(S)$. Then $\tau_n^*\overline{(c_1(\lambda_n)^n)} = \eta_S$, the unit of the ring spectrum $\mathbb{E}_S$ — where we have put: $\pi_n^*\overline{(c_1(\lambda_n)^n)} = c_1(\lambda_n)^n$.

These relations can easily be deduced from the following lemma:

Lemma 2.1.14 (Morel)**.** *Let $\delta_n : \mathbb{P}_S^n \to (\mathbb{P}_S^n)^{\wedge,n}$ be the n-th diagonal of the pointed scheme $\mathbb{P}_S^n/S$. Then the following square commutes in $\mathscr{H}_\bullet(S)$:*

$$
\begin{array}{ccc}
\mathbb{P}_S^n & \xrightarrow{\delta_n} & (\mathbb{P}_S^n)^{\wedge,n} \\
{\scriptstyle \pi_n}\downarrow & & \uparrow{\scriptstyle (\iota_1)^{\wedge,n}} \\
\mathbb{P}_S^n/\mathbb{P}_S^{n-1} & \xrightarrow{\tau_n} & (\mathbb{P}_S^1)^{\wedge,n}.
\end{array}
$$

For this lemma, we refer the reader to the proof of [Dég08a, lem. 3.3]. $\qquad\square$

As first remarked by Morel, the projective bundle theorem admits the following corollary – whose proof can be easily adapted from [Dég08a, Cor. 3.6].

Corollary 2.1.15. *Let $\mathbb{E}$ be an orientable absolute ring spectrum. Then for any scheme X in $\mathscr{S}$ and any closed subschemes Z, Z' of X, one has the following property:*

$$\forall (x,y) \in \mathbb{E}_Z^{n,p}(X) \times \mathbb{E}_{Z'}^{m,q}(X), \ x \cup y = (-1)^{nm} y \cup x.$$

Following the method of Grothendieck, we can now introduce the following definition.

Definition 2.1.16. Let $(\mathbb{E}, c)$ be an absolute oriented ring spectrum.

Let E/S be a vector bundle of rank n. We let $P = \mathbb{P}(E)$ be the associated projective bundle, with projection p and canonical line bundle λ.

Using the previous theorem, we define the *Chern classes of E/S* with coefficients in $(\mathbb{E}, c)$ as the elements $c_i(E)$ of $\mathbb{E}^{2i,i}(S)$ for an integer $i \in [0, n]$ such that

(2.1.16.a) $$\sum_{i=0}^n p^*(c_i(E)).\big(-c_1(\lambda)\big)^{n-i} = 0$$

and $c_0(E) = 1$. We put $c_i(E) = 0$ for $i \notin [0, n]$.

Indeed, the above theorem guarantees the existence and uniqueness of the Chern classes $c_i(E)$.

2.1.17. We deduce from this definition the following (usual) properties of Chern classes:

(a) For any vector bundle E/S, and any morphism $f : T \to S$, $f^*c_i(E) = c_i(f^{-1}E)$.
(b) For any scheme S and any isomorphism of vector bundles $E \simeq E'$ over S, $c_i(E) = c_i(E')$.
(c) For any scheme S, any exact sequence of vector bundles:

$$0 \to E' \to E \to E'' \to 0$$

and integer $k \geq 0$, $c_k(E) = \sum_{i+j=k} c_i(E').c_j(E'')$.

For details on the proof of formula (c) – *Whitney sum formula* – we refer the reader to [Dég08a, 3.13].

Remark 2.1.18. Let $(\mathbb{E}, c)$ be an absolute oriented ring spectrum.

Let Gr_S be the infinite Grassmannian, seen as a Nisnevich simplicial sheaf of sets over $\mathscr{S}m_S$. According to [MV99, 4.3.7], it is isomorphic in $\mathscr{H}_\bullet(S)$ to the classifying space BGL_S of the infinite general linear group over S. Moreover, when S is regular and for any integer $n \geq 0$, one gets a canonical isomorphism:

$$\mathrm{Hom}_{\mathscr{H}_\bullet(S)}(S^n, \mathbb{Z} \times Gr) \simeq K_n(S),$$

according to [MV99, 4.3.13], where $\mathbb{Z} \times Gr$ is the product of $\mathbb{Z}$-copies of Gr_S. This map is compatible with pullbacks of regular schemes.

Using Chern classes, it is well known how to compute $\mathbb{E}^{**}(Gr_S)$ (see [NSØ09a, 6.2]):

$$\mathbb{E}^{**}(Gr_S) \simeq \mathbb{E}^{**}(S)[[\mathfrak{c}_1, \ldots, \mathfrak{c}_n, \ldots]]$$

where $\mathfrak{c}_i$ is a cohomology class of bidegree $(2i, i)$. According to the above isomorphism, it corresponds to a map in $\mathscr{H}_\bullet(S)$:

$$\mathfrak{c}_i : \mathbb{Z} \times Gr_S \to Gr_S \to \Omega^\infty(\mathbb{E}_S(i)[2i]).$$

Moreover, using the techniques of Riou (cf. [Rio10], Th. 1.1.6 as in the proof of Th. 6.2.1.2), one gets an isomorphism:

$$\mathrm{Hom}_{\mathscr{H}_\bullet(S)}(\mathbb{Z} \times Gr, \Omega^\infty(\mathbb{E}(i)[2i])) \to \mathrm{Hom}(K_0(-), \mathbb{E}^{2i,i}(-))$$

where the right hand side stands for the morphisms of presheaves of sets on $\mathscr{S}m_S$. Under this isomorphism, the map $\mathfrak{c}_i$ corresponds to the natural transformation c_i that we have just defined.

As in [Gil81], this allows to automatically extends Chern classes to higher Chern classes with support in a closed subscheme $Z \subset S$ as follows:

$$c_i^Z : K_n^Z(S) \simeq [S^n \wedge (S/S - Z), \mathbb{Z} \times Gr] \xrightarrow{(\mathfrak{c}_i)_*} [S^n \wedge (S/S - Z), \Omega^\infty(\mathbb{E}(i)[2i])]$$
$$\simeq \mathbb{E}_Z^{2i-n,i}(S).$$

2.1.19. *The associated formal group law.–* The usual Segre embeddings

$$\sigma_{nm} : \mathbb{P}_X^n \times_X \mathbb{P}_X^m \to \mathbb{P}_X^{n+m+nm},$$

indexed by a pair of positive integers (n, m), induce a multiplication map

(2.1.19.a) $\sigma : \mathbb{P}_X^\infty \times_X \mathbb{P}_X^\infty \to \mathbb{P}_X^\infty.$

This gives a structure of an H-group to the object $\mathbb{P}_X^\infty$ of $\mathscr{H}_\bullet(X)$ which in turn induces a group structure on the target of the map (2.1.6.a). We deduce from the previous paragraph and the equality

(2.1.19.b) $$\sigma_{nm}^{-1}(\lambda_{n+m+nm}) = \lambda_n \times_X \lambda_m$$

that (2.1.6.a) is a morphism of abelian groups.

Suppose $(\mathbb{E}, c)$ is an absolute oriented ring spectrum. According to Theorem 2.1.13, the pullback along σ corresponds to a (comultiplication) morphism:

$$\mathbb{E}^{**}(X)[[c]] \xrightarrow{\sigma^*} \mathbb{E}^{**}(X)[[x, y]]$$

where x and y stands for the Chern classes of the two canonical line bundles $p_1^{-1}(\lambda)$ and $p_2^{-1}(\lambda)$. It follows that σ^* is determined by $\sigma^*(c)$ which is a power series of the form:

(2.1.19.c) $$F_X(x, y) = \sum_{i,j \geq 0} a_{ij}^X . x^i y^j .$$

As σ^* is a comultiplication, one deduces that F_X is a commutative formal group law with coefficients in $\mathbb{E}^{**}(X)$ – see [Dég08a, §3.7]. In fact, the classes a_{ij}^S enjoy the following properties:

- a_{ij}^S has bidegree $(2 - 2i - 2j, 1 - i - j)$ in $\mathbb{E}^{**}(S)$,
- $a_{0,1}^S = 1$, $a_{0,i}^S = 0$ if $i \neq 1$,
- for every couple (i, j), $a_{ij}^S = a_{ji}^S$,
- for any morphism of schemes $f : T \to S$, $f^*(a_{ij}^S) = a_{ij}^T$.

Definition 2.1.20. Given the notations above, we will say that F_S is the *formal group law* associated with the oriented ring spectrum $(\mathbb{E}, c)$ above S.

We will say that $(\mathbb{E}, c)$ (or just c) is *additive* (resp. *multiplicative* with parameter u) if for any scheme S, $F_S(x, y) = x + y$ (resp. $F_S(x, y) = x + y + u.x.y$).

Example 2.1.21. Consider the absolute oriented ring spectra of Example 2.1.4:

(1) The k-absolute oriented ring spectrum associated with a Mixed Weil theory is additive (cf. [CD12a, 2.2.10]).

   The absolute oriented ring spectrum $\mathbf{H}_B$ is also additive. This last fact follows from the definition. To be more precise, given any regular scheme S, the canonical bijection:

$$\mathbf{H}_B^{2,1}(S) \simeq Gr_\gamma^1 K_0(S)_\mathbb{Q} \simeq Pic(S)_\mathbb{Q}$$

   is in fact an isomorphism of abelian groups. Note also that, restricting to the category of smooth S-schemes, for an arbitrary base S, the absolute oriented ring spectrum $\mathbf{H}_\Lambda$ is additive according to [CD12b, 11.3.5].

(2) The absolute oriented ring spectrum $\mathbf{KGL}$ is multiplicative with parameter $(-\beta)$ as follows from the easy computation, with $l = [L^\vee]$ and $l' = [L'^\vee]$:

$$c_1^{\mathbf{KGL}}(L \otimes L') = \beta^{-1}.(1 - [(L \otimes L')^\vee]) = \beta^{-1}.(1 - ll')$$
$$= \beta^{-1}((1 - l) + (1 - l') - (1 - l).(1 - l'))$$
$$= c_1^{\mathbf{KGL}}(L) + c_1^{\mathbf{KGL}}(L') - \beta.c_1^{\mathbf{KGL}}(L).c_1^{\mathbf{KGL}}(L')$$

(3) Let k be a field of exponential characteristic p. Let $\mathbf{MGL}[1/p]$ be the absolute ring spectrum obtained from $\mathbf{MGL}$ by inverting p. Then, as a consequence of the Theorem of Hopkins-Morel-Hoyois (cf. [Hoy15, Th. 7.12]),

we know that the formal group law of the absolute oriented ring spectrum
$\mathbf{MGL}[1/p]$, considered over the category of all k-schemes, is isomorphic to
the universal formal group law.

More precisely, if (L, F_{univ}) denotes the Lazard ring equipped with its
canonical formal group law, according to *loc. cit.*, Prop. 8.2, there exists
an isomorphism of formal group laws:

$$(L[1/p], F_{univ}) \to (\mathbf{MGL}_{(2,1)*}(k)[1/p], F_{\mathbf{MGL}}).$$

Proposition 2.1.22. *Consider the notations of the previous definition.*

(1) *For any vector bundle E/X and any integer $i > 0$, the class $c_i(E)$ is nilpotent in $\mathbb{E}^{**}(X)$.*

(2) *For any line bundles L_1, L_2 over X,*

$$c_1(L_1 \otimes L_2) = F_X\big(c_1(L_1), c_1(L_2)\big) \in \mathbb{E}^{2,1}(X).$$

Proof. Point (1) follows from the hypothesis that X is noetherian according to the
proof of [Dég08a, 3.8(1)]. Point (2) is tautological by definition of the first Chern
class c_1 and of the formal group law F_X. $\qquad\square$

Remark 2.1.23. Note that unlike in [Dég08a, Prop. 3.8], to prove Point (2), we
do not need that X admits an ample line bundle. This is because we consider
cohomology theories that are representable in $S\mathscr{H}$: in fact, any line bundle L/X
can be represented by a map $S^0 \to \mathbb{P}^\infty_X$ in $\mathscr{H}_\bullet(X)$ according to the theorem of
Morel-Voevodsky. (If L/X is not generated by its sections, this map cannot be
lifted in the category of schemes.)

2.2. Thom classes and MGL-modules.

2.2.1. Let $(\mathbb{E}, c)$ be an absolute oriented ring spectrum (see Def. 2.1.2).

Let E/X be a vector bundle of rank n, $\mathbb{P}(E)$ (resp. $\bar{E}$, $E^\times$) be the associated
projective bundle (resp. projective completion, complement of the zero section).
Recall from [MV99, §3, 2.16] that one defines the Thom space of E/X as the pointed
sheaf

$$\mathrm{Th}(E) = E/E^\times.$$

According to *loc. cit.*, Prop. 2.17, we get a canonical $\mathbb{A}^1$-equivalence of simplicial
sheaves $\mathrm{Th}(E) \simeq \bar{E}/\mathbb{P}(E)$, the right hand side being the cokernel of the canonical
embedding $\nu : \mathbb{P}(E) \to \bar{E}$ in the category of sheaves, equipped with its obvious base
point. Thus we get a homotopy cofiber sequence for the $\mathbb{A}^1$-local model structure:

$$\mathbb{P}(E) \xrightarrow{\nu} \bar{E} \xrightarrow{\pi} \mathrm{Th}(E)$$

which induces a long exact sequence

$$\ldots \to \mathbb{E}^{**}(\mathrm{Th}(E)) \xrightarrow{\pi^*} \mathbb{E}^{**}(\bar{E}) \xrightarrow{\nu^*} \mathbb{E}^{**}(\mathbb{P}(E)) \to \ldots$$

According to Theorem 2.1.13, ν^* is a split epimorphism of free $\mathbb{E}^{**}(X)$-modules of
respective ranks n and $n-1$. Thus $E^{**}(\mathrm{Th}(X))$ is a free $\mathbb{E}^{**}(X)$-module of rank
1, isomorphic to $\ker(\nu^*)$.

Definition 2.2.2. Consider the notations and assumptions above.

We define the *Thom class* of E/X as the following element of $\mathbb{E}^{2n,n}(\bar{E})$:

$$\mathfrak{t}(E) = \sum_{i=0}^{n} p^*(c_i(E)).\big(-c_1(\lambda)\big)^{n-i}$$

where p is the canonical projection. We define the *refined Thom class* $\bar{\mathfrak{t}}(E)$ of E as the unique element of $\mathbb{E}^{2n,n}(\mathrm{Th}(E))$ such that

$$\pi^*\big(\bar{\mathfrak{t}}(E)\big) = \mathfrak{t}(E).$$

When the base of the vector bundle E is not clear, we indicate it as follows: $\mathbb{P}_X(E)$, $\mathrm{Th}_X(E)$, $\mathfrak{t}(E/X)$, $\bar{\mathfrak{t}}(E/X)$.

Note that $\mathbb{E}^{**}(\mathrm{Th}(E)) = \mathbb{E}_X^{**}(E)$ (see Definition 1.2.5). According to the preceding paragraph, the canonical map:

(2.2.2.a) $$\mathbb{E}^{**}(X) \to \mathbb{E}_X^{**}(E), \lambda \mapsto \lambda.\bar{\mathfrak{t}}(E)$$

defined in Paragraph 1.2.8 is an isomorphism called the *Thom isomorphism*.

We deduce from formulas (a), (b) of 2.1.17 that Thom classes are compatible with base change and invariant under isomorphisms of vector bundles.

Remark 2.2.3. Recall that in general the rank of a vector bundle E/X is Zariski locally constant on X. In other words, it is a function $r : \pi_0(X) \to \mathbb{N}$. On the other hand, $\mathbb{E}$-cohomology of X is additive. Moreover, we can give sense to the formula defining the refined Thom class of $\mathbb{E}$ without requiring E/X is of (constant) rank equal to n. Then $\bar{\mathfrak{t}}(E)$ is an element of $\mathbb{E}^{**}(\mathrm{Th}(E))$, which still induces a Thom isomorphism as above. Then, we can define the bidegree of this class as the function $\pi_0(X) \to \mathbb{Z}^2$ which to a connected component X_i of X associates the couple $(2r(X_i), r(X_i))$. The Thom isomorphism is then Zariski locally homogeneous on X with the same bidegree.

Example 2.2.4. Recall the universal quotient bundle ξ on $\bar{E}$ is defined by the exact sequence

$$0 \to \lambda \to p^{-1}(E \oplus 1) \to \xi \to 0.$$

Thus the Whitney sum formula 2.1.17(c) gives the following formula:

(2.2.4.a) $$\mathfrak{t}(E) = c_n(\xi).$$

2.2.5. The cobordism spectrum $\mathbf{MGL}$ is given by the sequence of Thom spaces $\mathrm{Th}(\gamma_n/\mathrm{BGl}_n)$ for $n > 0$ where γ_n is the tautological rank n vector bundle over the classifying space of GL_n. As $\mathrm{Th}(\gamma_1) = \mathrm{B}\mathbb{G}_m = \mathbb{P}^\infty$, $\mathbf{MGL}_S$ is canonically oriented. We denote by $c^{\mathbf{MGL}}$ this orientation.

Given an absolute oriented ring spectrum $(\mathbb{E}, c)$, the Thom class defined previously allows to define a morphism $\varphi_c : \mathbf{MGL} \to \mathbb{E}$. This is the key observation of the following proposition (see [Vez01, 4.3]):

Proposition 2.2.6. *Let $\mathbb{E}$ be an absolute ring spectrum. Then the following sets correspond bijectively:*

  (i) *orientations c of $\mathbb{E}$;*
  (ii) *maps of absolute ring spectra $\varphi : \mathbf{MGL} \to \mathbb{E}$.*

by the following maps:

$$(ii) \to (i), \varphi \mapsto \varphi_*(c^{\mathbf{MGL}}), \ \varphi_* \ induced \ map \ in \ cohomology,$$
$$(i) \to (ii), c \mapsto \varphi_c.$$

2.2.7. A module over the ring spectrum $\mathbf{MGL}_S$ is an $\mathbf{MGL}_S$-module in $S\mathscr{H}(S)$ in the classical sense: a spectrum $\mathbb{E}$ over S equipped with a multiplication map $\gamma_{\mathbb{E}} : \mathbf{MGL}_S \wedge \mathbb{E} \to \mathbb{E}$ satisfying the usual identities – see [ML98].

Given two $\mathbf{MGL}_S$-modules $\mathbb{E}$ and $\mathbb{F}$, a morphism of $\mathbf{MGL}_S$-modules is a morphism $f : \mathbb{E} \to \mathbb{F}$ in $S\mathscr{H}(S)$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbf{MGL}_S \wedge \mathbb{E} & \xrightarrow{1 \wedge f} & \mathbf{MGL}_S \wedge \mathbb{F} \\
{\scriptstyle \gamma_{\mathbb{E}}}\downarrow & & \downarrow{\scriptstyle \gamma_{\mathbb{F}}} \\
\mathbb{E} & \xrightarrow{\quad f \quad} & \mathbb{F}.
\end{array}
$$

We will denote by $\mathbf{MGL}-mod_S^w$ the additive category of $\mathbf{MGL}_S$-modules.

Given any spectrum $\mathbb{E}$, $\mathbf{MGL}_S \wedge \mathbb{E}$ has an obvious structure of an $\mathbf{MGL}_S$-module. The assignment $L_{\mathbf{MGL}}^w : \mathbb{E} \mapsto \mathbf{MGL}_S \wedge \mathbb{E}$ defines a functor left adjoint to the inclusion functor $\mathcal{O}_{\mathbf{MGL}}^w$ and we get an adjunction of categories:

(2.2.7.a) $$ L_{\mathbf{MGL}}^w : S\mathscr{H}(S) \leftrightarrows \mathbf{MGL}-mod_S^w : \mathcal{O}_{\mathbf{MGL}}^w. $$

The category $\mathbf{MGL}-mod_S^w$ is not well behaved: it has no triangulated monoidal structure. In [Dég08a, Ex. 2.12(2)] and [Dég13, Sec. 2.2], we introduced the homotopy category of *strict* $\mathbf{MGL}_S$-modules. We will denote it by $\mathbf{MGL}-mod_S$ and call it the *homotopy category of $\mathbf{MGL}_S$-modules*. This is an enrichment of the category of $\mathbf{MGL}_S$-modules: one defines a canonical commutative diagram of functors:

(2.2.7.b)
$$
\begin{array}{ccc}
 & S\mathscr{H}(S) & \\
{\scriptstyle L_{\mathbf{MGL}}}\swarrow & & \searrow{\scriptstyle L_{\mathbf{MGL}}^w} \\
\mathbf{MGL}-mod_S & \xrightarrow{\ \mathcal{O}'_{\mathbf{MGL}}\ } & \mathbf{MGL}-mod_S^w
\end{array}
$$

where $L_{\mathbf{MGL}}$ is a triangulated functor and $\mathcal{O}'_{\mathbf{MGL}}$ is a conservative functor. Therefore, using the previous commutative diagram and Proposition 2.2.6, we get the following result.

Proposition 2.2.8. *Let $(\mathbb{E}, c)$ be an absolute oriented ring spectrum.*

Then for any scheme S, the functor $\varphi_{\mathbb{E}} = \mathrm{Hom}_{S\mathscr{H}(S)}(-, \mathbb{E})$ induces a canonical functor $\tilde{\varphi}_{\mathbb{E}}$ which fits in the following commutative diagram:

$$
\begin{array}{ccc}
S\mathscr{H}(S)^{op} & \xrightarrow{\varphi_{\mathbb{E}}} & \\
{\scriptstyle (L_{\mathbf{MGL}})^{op}}\downarrow & \searrow & \mathscr{A}b \\
(\mathbf{MGL}-mod_S)^{op} & \xrightarrow[\tilde{\varphi}_{\mathbb{E}}]{} &
\end{array}
$$

2.2.9. Let S be a scheme. Recall the considerations of [Dég08a, §2.3.2]. For any cartesian square of smooth S-schemes

$$
\begin{array}{ccc}
W & \rightarrow & V \\
\downarrow & \Delta & \downarrow \\
U & \rightarrow & X
\end{array}
$$

made of immersions, we denote by $\frac{X/U}{V/W}$ the colimit of Δ in the category of sheaves over S.[37]

We denote by $\mathbf{MGL}_S\left(\frac{X/U}{V/W}\right)$ the image of the infinite suspension spectrum $\Sigma^\infty\left(\frac{X/U}{V/W}\right)$ by the functor $L_{\mathbf{MGL}}$ defined above. When $V = W = \varnothing$ (resp. $U =$

---

[37]As all maps in this diagram are cofibrations, this is the homotopy colimit of the diagram in the $\mathbb{A}^1$-local model category of simplicial sheaves. It measures the obstruction for the square Δ to be homotopy cocartesian.

$V = W = \varnothing$), we simply denote this object by $\mathbf{MGL}_S(X/U)$ (resp. $\mathbf{MGL}_S(X)$) – according to the notation of Par. 1.1.3.

As explained in [Dég08a, 2.3.2, Ex. 2.12(2)], the category $\mathbf{MGL}{-}mod_S$ together with the canonical orientation $c^{\mathbf{MGL}}$ of $\mathbf{MGL}_S$ satisfies all the axioms of [Dég08a, §2.1]. Therefore we can apply the results of *loc. cit.* to that category.

Then, according to the previous proposition, for any smooth closed S-pair (X, Z), we get:

$$\mathbb{E}_Z^{n,m}(X) = \mathrm{Hom}_{S\mathscr{H}(S)}(X/X - Z, \mathbb{E}(m)[n]) = \tilde{\varphi}_{\mathbb{E}}\left(\mathbf{MGL}_S(X/X - Z)(-m)[-n]\right).$$

The important thing for us is that, given smooth closed S-pairs (X, Z) and (Y, T) and a couple of integers (n, m), any morphism of $\mathbf{MGL}_S$-modules of the form:

$$\mathbf{MGL}_S(X/X - Z) \to \mathbf{MGL}_S(Y/Y - T)(m)[n]$$

induces a canonical homogeneous morphism of bigraded abelian groups

$$\mathbb{E}_T^{**}(Y) \to \mathbb{E}_Z^{*+n,*+m}(X).$$

Obviously, this association is compatible with composition.

Example 2.2.10. Let $f : Y \to X$ be a projective S-morphism between smooth S-schemes. Assume f has dimension d. Then, applying [Dég08a, 5.12], we get the Gysin morphism

$$f^* : \mathbf{MGL}_S(X) \to \mathbf{MGL}_S(Y)(-d)[-2d]$$

which in turn induces a push-forward in cohomology:

$$f_* : \mathbb{E}^{*,*}(Y) \to \mathbb{E}^{*-d,*-2d}(X).$$

The purpose of the next sections is to generalize this pushforward in the case of regular schemes.

Remark 2.2.11. In fact, all the orientation theory exposed here for ring spectra can be done without ring structure by replacing an orientation c by a structure of a module over $\mathbf{MGL}_S$, on a given spectrum $\mathbb{E}$. This is the content of *loc. cit.* in the case $\mathscr{S} = \mathscr{S}m_S$.[38]

Then, instead of having Chern classes in $\mathbb{E}$ cohomology, we get an *action* of Chern classes, through the $\mathbf{MGL}^{**}$-module structure of $\mathbb{E}^{**}$ – see also Example 5.1.3. Similarly, to anticipate what follows, we do not get fundamental classes in $\mathbb{E}$-cohomology but an action of fundamental classes with coefficients in $\mathbf{MGL}$. Note this is enough to get the purity isomorphism (2.3.1.b) and therefore, Gysin long exact sequence (as in Def. 3.1.2), Gysin morphisms for projective lci morphisms (as in 3.2.7).

However, the general case $\mathscr{S} = \mathscr{R}eg$ is not covered by [Dég08a]. In fact, in this case, Conjecture A (of section 1.3) seems very natural.

2.3. **Fundamental classes.** In this section and the following one, we fix an absolute oriented ring spectrum $(\mathbb{E}, c)$.

Definition 2.3.1. Let (X, Z) be a regular closed pair of codimension n and normal bundle $N_Z X$. Let $i : Z \to X$ be the corresponding embedding.

---

[38]The case of weak $\mathbf{MGL}$-modules can be deduced from the case of strict $\mathbf{MGL}$-modules which is treated in *loc. cit.* using the commutative diagram (2.2.7.b).

When (X, Z) is $\mathbb{E}$-pure we define the *refined fundamental class* of Z in X (with coefficients in $\mathbb{E}$) as the image of the Thom class $\bar{\mathfrak{t}}(N_Z X)$ by the isomorphisms (see Definition 1.3.2):

$$(2.3.1.a) \qquad \mathbb{E}^{2n,n}(\mathrm{Th}(N_Z X)) \xrightarrow{(\sigma_0^*)^{-1}} \mathbb{E}^{2n,n}_{\mathbb{A}^1_Z}(D_Z X) \xrightarrow{\sigma_1^*} \mathbb{E}^{2n,n}_Z(X).$$

We denote it by $\bar{\eta}_X(Z)$.

We also define the *fundamental class* of Z in X as the following class of $\mathbb{E}^{2n,n}(X)$:

$$\eta_X(Z) := i_!\left(\bar{\eta}_X(Z)\right).$$

Because σ_0^*, σ_1^* are isomorphisms of $\mathbb{E}^{**}(Z)$-modules, and according to the Thom isomorphism (2.2.2.a), the class $\bar{\eta}_X(Z)$ is in fact a base of the $\mathbb{E}^{**}(Z)$-bigraded module $\mathbb{E}^{**}_Z(X)$. In other words, we get an isomorphism

$$(2.3.1.b) \qquad \mathfrak{p}_{(X,Z)} : \mathbb{E}^{**}(Z) \to \mathbb{E}^{**}_Z(X), z \mapsto z.\bar{\eta}_X(Z)$$

called the *purity isomorphism*.

Remark 2.3.2. For the reader who does not want to restrict to pure codimension pairs, let us indicate that in general, the codimension of a regular closed pair (X, Z) is Zariski local in Z, as well as the rank of its normal bundle $N_Z X$. Then the convention of 2.2.3 will apply both to the refined fundamental class and the purity isomorphism associated with (X, Z).

Example 2.3.3. Let E/X be a vector bundle. Then the closed X-pair (E, X) corresponding to the zero section is $\mathbb{E}$-pure – see Example 1.3.4(1). Then we get from the above definition the equality of classes in $\mathbb{E}^{**}(\mathrm{Th}(E/X))$:

$$(2.3.3.a) \qquad\qquad \bar{\eta}_E(X) = \bar{\mathfrak{t}}(E/X).$$

Indeed, one can identify the deformation space $D_X E$ (resp. normal bundle $N_X E$) with the affine line $\mathbb{A}^1_E$ (resp. the X-vector bundle E itself) and the maps σ_1, σ_0 of (1.3.1.a) with respectively the unit and zero section of $\mathbb{A}^1_E$ (see [Dég08a, Rem. 4.2]) so that the above identification follows from the previous definition. Moreover, if $\bar{E}/X$ denotes the projective completion of E/X, so that $E \subset \bar{E}$ is an open subscheme, one gets by Nisnevich excision an identification $\mathbb{E}^{**}_X(\bar{E}) = \mathbb{E}^{**}_X(E)$ through which $\bar{\eta}_{\bar{E}}(X)$ corresponds to the class $\bar{\eta}_E(X) = \bar{\mathfrak{t}}(E/X)$. In particular, one gets the equality of (non refined) fundamental classes:

$$(2.3.3.b) \qquad\qquad \eta_{\bar{E}}(X) = \mathfrak{t}(E/X).$$

Remark 2.3.4. Consider a smooth closed S-pair (X, Z) of codimension n. Recall from Example 1.3.4(1) that (X, Z) is $\mathbb{E}$-pure. We have defined in [Dég08a, 4.6] a purity isomorphism in the homotopy category of $\mathbf{MGL}_S$-modules:

$$\mathbf{MGL}_S(X/X - Z) \xrightarrow{\sim} \mathbf{MGL}_S(Z)(n)[2n].$$

According to the above example and [Dég08a, 4.3, 4.4], we get that the induced morphism in $\mathbb{E}$-cohomology (see 2.2.9) coincides exactly with the purity isomorphism $\mathfrak{p}_{(X,Z)}$ introduced above. In particular, the class $\bar{\eta}_X(Z)$ in $\mathbb{E}^{**}_Z(X)$ introduced here coincides with that obtained from [Dég08a, 4.14] by considering $\mathbf{MGL}_S$-modules.[39]

---

[39]The terminology here differs slightly from that of *loc. cit.*: we use the adjective "refined" instead of "localized" which seems more classical.

2.4. **Intersection theory.**

Definition 2.4.1. Consider a morphism $\Delta : (Y, T) \to (X, Z)$ of closed pairs (Par. 1.2.11) such that (X, Z) and (Y, T) are $\mathbb{E}$-pure.

Then, according to Definition 2.3.1, there exists a unique class e_Δ in $\mathbb{E}^{**}(T)$ such that

$$\Delta^*(\bar{\eta}_X(Z)) = e_\Delta . \bar{\eta}_Y(T).$$

We call e_Δ the *defect* of the morphism Δ.[40]

The following result generalizes [Dég08a, Prop. 4.16]:

Theorem 2.4.2. *With the notations of the above definition, assume $\Delta = (f, g)$ is cartesian. Then it induces a monomorphism $\nu : N_T Y \to g^{-1}(N_Z X)$ of vector bundles over T. We put*

(2.4.2.a) $$\xi = g^{-1}(N_Z X)/N_T Y$$

and denote by e the rank of this vector bundle.
  Then $e_\Delta = c_e(\xi)$.

Remark 2.4.3. One can apply this theorem in two cases:
- Given a base scheme S, Δ is a morphism of smooth closed S-pairs. This case is already known from [Dég08a, 4.16].
- $\mathbb{E}$ is $\mathscr{R}$eg-absolutely pure and the schemes X, Z, Y, $T = f^{-1}(Z)$ are all regular.

Proof. Recall the deformation diagram (1.3.1.a) is functorial with respect to the morphism Δ, because it is assumed to be cartesian. Thus, going back to the definition of fundamental classes (2.3.1), using the deformation diagrams respectively for the closed pairs (X, Z) and (Y, T) we are reduced to compare Thom classes for the corresponding normal bundles, as summarized in the following cartesian square:

$$
\begin{array}{ccc}
T & \hookrightarrow & N_T Y \\
\downarrow^g & & \downarrow^\nu \\
& & g^{-1}(N_Z X) \\
\downarrow & & \downarrow^{g'} \\
Z & \hookrightarrow & N_Z X,
\end{array}
$$

where ν is a monomorphism of vector bundles over T. This now follows from [Dég08a, Lem. 4.18] (using the considerations of Paragraph 2.2.9). $\square$

In terms of fundamental classes, we get:

Corollary 2.4.4. *Let $(f, g) : (Y, T) \to (X, Z)$ be a cartesian morphism of $\mathbb{E}$-pure closed pairs. Let $f^* : \mathbb{E}_Z^{**}(X) \to \mathbb{E}_T^{**}(Y)$ be the morphism defined in Paragraph 1.2.6.*
  Then, using Definition (2.4.2.a), we get the following formula in $\mathbb{E}_T^{**}(Y)$:

$$f^*(\bar{\eta}_X(Z)) = c_e(\xi) . \bar{\eta}_Y(T).$$

In particular when f is transversal to Z,

(2.4.4.a) $$f^*(\bar{\eta}_X(Z)) = \bar{\eta}_Y(T).$$

---

[40]One could call this class the *defect of transversality*. In fact, there are two kinds of possible defect: excess of intersection, ramification.

Remark 2.4.5. According to formulas (E3) and (E7) of Proposition 1.2.10, we get in the assumptions of the above corollary the even more usual formula in $\mathbb{E}^{**}(Y)$:

$$f^*(\eta_X(Z)) = c_e(\xi).\eta_Y(T).$$

Applying this formula in the case where f is the zero section $s : X \to E$ of a vector bundle of rank n, we get the relation

$$s^*(\eta_E(X)) = c_n(E).\text{[41]}$$

These two relations give the following classical trick, which will be used later, to compute fundamental classes.

Corollary 2.4.6. *Let (X, Z) be an $\mathbb{E}$-pure closed pair of codimension n corresponding to an immersion $i : Z \to X$. Assume there exists a vector bundle E/X with a section s transversal to the zero section s_0 and such that $s_0^{-1}(s) = i$. Then*

$$\eta_X(Z) = c_n(E).$$

An important point of intersection theory is the associativity of the intersection product. In our setting, this can be expressed nicely using the refined product of Paragraph 1.2.8. We begin with a particular case of Theorem 2.4.9 which will be the crucial step.

Proposition 2.4.7. *Consider an exact sequence (σ) of vector bundles over a scheme X:*

$$0 \to E' \xrightarrow{\nu} E \to E'' \to 0.$$

Then the following relation holds in $\mathbb{E}^{**}(\mathrm{Th}(E))$:

$$\bar{\mathfrak{t}}(E/X) = \bar{\mathfrak{t}}(E'/X).\bar{\mathfrak{t}}(E/E')$$

using the pairing $\mathbb{E}_X^{**}(E') \otimes \mathbb{E}_{E'}^{**}(E) \to \mathbb{E}_X^{**}(E)$ (Paragraph 1.2.8).

Proof. Note that according to formula (2.3.3.a), we have to prove the equality:

$$(2.4.7.a) \qquad\qquad \bar{\eta}_E(X) = \bar{\eta}_{E'}(X).\bar{\eta}_E(E').$$

As in the proof of [Rio10, 4.1.1], we can find a torsor T over the X-vector bundle $\underline{\mathrm{Hom}}(E'', E')$ such that the sequence (σ) splits over T. The morphism $T \to X$ is an $\mathbb{A}^1$-weak equivalence ; thus, by compatibility of the Thom class with base change, we can assume the sequence (σ) splits.

Let us consider P' (resp. P'') the projective completion of E'/X (resp. E''/X), $P = P' \times_X P''$, and the following commutative diagram made of cartesian squares:

$$
\begin{array}{ccccc}
X & \longrightarrow & E' & \longrightarrow & E \\
\| & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j} \\
X & \xrightarrow{s'} & P' & \xrightarrow{s''} & P \\
& \underset{s}{\Longrightarrow} & &
\end{array}
$$

where j (resp. s, s', s'') stands for the natural open immersion (resp. canonical sections coming from the obvious zero sections). Remark that if we apply the functor $j^* : \mathbb{E}_X^{**}(P) \to \mathbb{E}_X^{**}(E)$ to the following equality:

$$(2.4.7.b) \qquad\qquad \bar{\eta}_P(X) = \bar{\eta}_{P'}(X).\bar{\eta}_P(P'),$$

---

[41]In other words, given a vector bundle E/X the fundamental class of its zero section coincides up to homotopy with its *Euler class*. This fact justifies our choice of Thom class in Definition 2.2.2.

we get (2.4.7.a) – use formula (E5) of Prop. 1.2.10 and formula (2.4.4.a). Thus we are reduced to prove that equality (2.4.7.b) holds in $\mathbb{E}_X^{**}(P)$.

According to the projective bundle formula, the morphism $s_! : \mathbb{E}_X^{**}(P) \to \mathbb{E}^{**}(P)$ is a split monomorphism (see the end of Par. 2.2.1 for more details). Thus it is sufficient to prove that (2.4.7.b) holds after applying $s_!$. That means we have to prove $\eta_P(X)$ is equal to:

$$s_! \big(\bar{\eta}_{P'}(X).\bar{\eta}_P(P') \big) = s_!'' s_!' \big(\bar{\eta}_{P'}(X).\bar{\eta}_P(P') \big)$$
$$\stackrel{(E6)}{=} s_!'' \big(\eta_{P'}(X).\bar{\eta}_P(P') \big) = s_!'' \big(s''^*[\eta_P(P'')].\bar{\eta}_P(P') \big)$$
$$\stackrel{(E7)}{=} \eta_P(P'').\eta_P(P')$$

using once again the properties of the refined product enumerated in Proposition 1.2.10. This last check follows from a direct computation – see [Dég08a, Lem. 4.25]. $\qquad\square$

Remark 2.4.8. Consider a scheme X. Given vector bundles E' and E'' over X, we get using the product defined in Paragraph 1.2.8 a canonical map of Künneth type:

$$\mathbb{E}^{**}(\mathrm{Th}(E')) \otimes_{\mathbb{E}^{**}(X)} \mathbb{E}^{**}(\mathrm{Th}(E'') \to \mathbb{E}^{**}(\mathrm{Th}(E' \oplus E'')).$$

According to the above proposition, $\bar{\mathfrak{t}}(E' \oplus E'') = \bar{\mathfrak{t}}(E').\bar{\mathfrak{t}}(E'')$. In other words, the above map is an isomorphism (of bigraded $\mathbb{E}^{**}(X)$-modules).

Now, given an exact sequence (σ) of vector bundles as in the above proposition, Riou in the proof of [Rio10, 4.1.1] shows there exists a canonical isomorphism in $S\mathscr{H}(X)$

$$\Sigma^\infty \mathrm{Th}(E') \wedge \Sigma^\infty \mathrm{Th}(E'') \xrightarrow{\epsilon_\sigma} \Sigma^\infty \mathrm{Th}(E).$$

Using this isomorphism, one defines a canonical product of bigraded $\mathbb{E}^{**}(X)$-modules:

$$\mathbb{E}^{**}(\mathrm{Th}(E')) \otimes_{\mathbb{E}^{**}(X)} \mathbb{E}^{**}(\mathrm{Th}(E'')) \longrightarrow \mathbb{E}^{**}(\mathrm{Th}(E') \wedge \mathrm{Th}'(E'')) \xrightarrow{(\epsilon_\sigma^*)^{-1}} \mathbb{E}^{**}(\mathrm{Th}(E)).$$
$$a \otimes b \longmapsto \langle a, b \rangle_\sigma$$

Then the relation obtained in the previous proposition can be rewritten as follows:

$$\bar{\mathfrak{t}}(E) = \langle \bar{\mathfrak{t}}(E'), \bar{\mathfrak{t}}(E'') \rangle_\sigma.$$

Thus, the map $\langle -, - \rangle_\sigma$ is an isomorphism.

Recall that Deligne defines in [Del87, 4.12] the category of *virtual vector bundle* denoted by $\underline{K}(X)$. As remarked by Riou, the isomorphisms of type ϵ_σ show that the functor $\Sigma^\infty \mathrm{Th}_X$ induces a canonical functor

$$\mathrm{Th}_X : \underline{K}(X) \to S\mathscr{H}(X).$$

In fact the preceding considerations show that, for any virtual vector bundle ξ in $\underline{K}(X)$, the bigraded $\mathbb{E}^{**}(X)$-module $\mathbb{E}^{**}(\mathrm{Th}(\xi))$ is free of rank 1. Moreover, it admits a canonical trivialization which we will denote $\bar{\mathfrak{t}}(\xi)$: if $\xi = [E] - [E']$ for two vector bundles E and E', one puts:

$$\bar{\mathfrak{t}}(\xi) := \bar{\mathfrak{t}}(E).\bar{\mathfrak{t}}(E')^*.$$

A conceptual way of stating this result: the canonical functor $\mathbb{E}^{**} \circ \tilde{\mathrm{Th}}_X$ induces a functor of Picard categories:

$$\big(\underline{K}(X)\big)^{op} \to \mathrm{Pic}\big(\mathbb{E}^{**}(X)\big)$$

where the right hand side category is the Picard category of bigraded virtual line bundles over $\mathbb{E}^{**}(X)$.

Theorem 2.4.9. *Consider regular closed immersions $Z \xrightarrow{k} Y \xrightarrow{i} X$ in $\mathscr{S}$ and assume $\mathbb{E}$ is absolutely pure.*

Then we get the following equality in $\mathbb{E}_Z^{**}(X)$:

$$\bar{\eta}_X(Z) = \bar{\eta}_Z(Y).\bar{\eta}_X(Y),$$

using the pairing $\mathbb{E}_Z^{**}(Y) \otimes \mathbb{E}_Y^{**}(X) \to \mathbb{E}_Z^{**}(X)$ (Paragraph 1.2.8) for the right hand side.

Proof. Recall from Paragraph 1.3.1 the deformation space $D_Z X$ associated with the closed pair (X, Z). Following [Ros96, §10], we define the double deformation space by the following formula:

$$D(X, Y, Z) = D(D_Z(X), D_Z(X)|_Y).$$

This scheme is flat over $\mathbb{A}^2$. Its fiber over $(1, 1)$ is X while its fiber over $(0, 0)$ is the normal bundle $E := N(N_Z X, N_Z Y)$. Let us put $D' = D(Y, Y, Z)$ and $E' = N_Z Y$. Then we get the following diagram made of cartesian squares:

$$
\begin{array}{ccccc}
Z & \xrightarrow{s'} & E' & \xrightarrow{\nu} & E \\
{\scriptstyle s_1}\downarrow & & \downarrow & & \downarrow{\scriptstyle d_0} \\
\mathbb{A}_Z^2 & \longrightarrow & D' & \longrightarrow & D \\
{\scriptstyle s_0}\uparrow & & \uparrow & & \uparrow{\scriptstyle d_1} \\
Z & \xrightarrow{k} & Y & \xrightarrow{i} & X
\end{array}
$$

where the first (resp. third) row is seen as the $(0, 0)$-fiber (resp. $(1, 1)$-fiber) of the second row, with respect to its canonical projection to $\mathbb{A}^2$. Because this projection is flat, all the squares in this diagram are transversal.

Thus, one can apply Corollary 2.4.4 to the morphism of closed pairs $(X, Z) \xrightarrow{d_1} (D, \mathbb{A}_Z^2)$ (resp. $(E, Z) \xrightarrow{d_0} (D, \mathbb{A}_Z^2)$): because s_1 (resp. s_0) is a strong $\mathbb{A}^1$-homotopy equivalence, and using the absolute purity property, the pullback morphism $d_1^* : \mathbb{E}_{\mathbb{A}_Z^2}^{**}(D) \to \mathbb{E}_Z^{**}(X)$ (resp. $d_0^* : \mathbb{E}_{\mathbb{A}_Z^2}^{**}(D) \to \mathbb{E}_Z^{**}(E)$) is an isomorphism.

Applying again Corollary 2.4.4, we deduce that to prove the theorem, it is enough to prove the relation

$$\bar{\eta}_E(Z) = \bar{\eta}_{E'}(Z).\bar{\eta}_E(E').$$

In view of formula (2.3.3.a), this is precisely Proposition 2.4.7.   □

Remark 2.4.10. In the case where $\mathscr{S}$ is the category of smooth S-schemes, for a base scheme S, the above theorem is very close to [Dég08a, 4.30]. The proof given here is considerably simpler, as we use the refined product of 1.2.8 – it uses the localization property of $S\mathscr{H}$, a very strong result.

3. GYSIN MORPHISMS AND LOCALIZATION LONG EXACT SEQUENCE

In this section, we fix an absolute oriented ring spectrum $(\mathbb{E}, c)$ – recall Def. 2.1.2.

3.1. **Residues and the case of closed immersions.**

3.1.1. Consider a closed immersion $i : Z \to X$ with complementary open immersion $j : U \to X$. Property (A4) of Paragraph 1.1.1 implies the existence of the *localization long exact sequence*:

$$(3.1.1.a) \qquad \cdots \mathbb{E}_Z^{n,m}(X) \xrightarrow{i_!} \mathbb{E}^{n,m}(X) \xrightarrow{j^*} \mathbb{E}^{n,m}(U) \xrightarrow{\delta_{X,Z}} \mathbb{E}_Z^{n+1,m}(X) \cdots$$

If we assume that the corresponding closed pair (X, Z) is regular of codimension c and $\mathbb{E}$-pure, the associated purity isomorphism (2.3.1.b) induces a long exact sequence in cohomology from the preceding one:

$$(3.1.1.b) \quad \cdots \mathbb{E}^{n-2c,m-c}(Z) \xrightarrow{i_*} \mathbb{E}^{n,m}(X) \xrightarrow{j^*} \mathbb{E}^{n,m}(U) \xrightarrow{\partial_{X,Z}} \mathbb{E}^{n-2c+1,m-c}(Z) \cdots$$

Definition 3.1.2. Under the notations above, assuming (X, Z) is $\mathbb{E}$-pure of codimension c, we call (3.1.1.b) (resp. i_*, $\partial_{X,Z}$) the Gysin long exact sequence (resp. Gysin morphism, residue morphism) associated with the closed pair (X, Z) (or with the immersion i).

Note that in terms of the refined fundamental class (Def. 2.3.1), the Gysin morphism can be described by the following formula for a cohomology class $z \in \mathbb{E}^{**}(Z)$:

$$(3.1.2.a) \qquad i_*(z) := i_! \big(z . \bar{\eta}_X(Z) \big).$$

Moreover, the residue of a cohomology class $u \in \mathbb{E}^{**}(U)$ is uniquely determined by the following property:

$$(3.1.2.b) \qquad \delta_{X,Z}(u) = \partial_{X,Z}(u) . \bar{\eta}_X(Z).$$

Remark 3.1.3. Note that the Gysin morphism i_* defined above do depend on the orientation c on the absolute ring spectrum $\mathbb{E}$, fixed at the beginning of the section — this dependence is responsible for the Riemann-Roch theorem 4.2.3. Usually, there will be no possible confusion about the chosen orientation, but in case there might, we will denote by i_*^c the corresponding Gysin morphism (see in particular 5.0.1).

Proposition 3.1.4. (1) *Assume $\mathbb{E}$ is absolutely pure. Then for any $\mathbb{E}$-pure closed immersions*
$$T \xrightarrow{k} Z \xrightarrow{i} X,$$
the following equality holds: $i_* k_* = (ik)_*$.
(2) *Consider an $\mathbb{E}$-pure closed immersion $i : Z \to X$. For any couple (x, z) in $\mathbb{E}^{**}(X) \times \mathbb{E}^{**}(Z)$,*
$$i_* \big(i^*(x) . z \big) = x . i_*(z).$$
(3) *Consider a cartesian square*

$$\begin{array}{ccc} T & \xrightarrow{k} & Y \\ g\downarrow & & \downarrow f \\ Z & \xrightarrow{i} & X \end{array}$$

in $\mathscr{S}$ such that i and k are $\mathbb{E}$-pure closed immersions. Let $h : (Y - T) \to (X - Z)$ be the morphism induced by f. Let ξ be the vector bundle over T defined by formula (2.4.2.a) with respect to the morphism of closed pairs (f, g). Let e be the rank of ξ.

Then the following formulas hold:

(3.1.4.a) $$f^* i_*(z) = k_* \big(c_e(\xi).g^*(z) \big).$$

(3.1.4.b) $$\partial_{Y,T} h^*(u) = c_e(\xi).g^* \partial_{X,Z}(u).$$

Proof. Taking care of formulas (3.1.2.a) and (3.1.2.b), points (1), (2), (3) are respectively consequences of Th. 2.4.9, Proposition 1.2.10(E4)+(E7), Corollary 2.4.4. $\square$

Remark 3.1.5. Let $i : Z \to X$ be a closed immersion between smooth S-schemes. According to point (1) of Example 1.3.4, the closed pair (X, Z) is $\mathbb{E}$-pure. Then both Example 2.2.10 and the preceding definition gives a pushforward of the form

$$i_* : \mathbb{E}^{**}(Z) \to \mathbb{E}^{**}(X).$$

Remark 2.3.4 shows that these two constructions coincide.

As we will see in the next examples, residues are connected to the *tame residue symbols* of Milnor in K-theory. This connection can be reduced to the following interesting property:

Proposition 3.1.6. *Let X be a regular scheme and Z be a principal divisor of X parametrized by a regular function $\pi : X \to \mathbb{A}_k^1$. Let us denote by $i : Z \to X$ ($j : U \to X$) be the corresponding closed immersion (resp. complementary open immersion). Let us consider the following composite map:*

$$\gamma_\pi : \mathbb{E}^{n,m}(U) \xrightarrow{(1)} \mathbb{E}^{n,m}(U) \oplus \mathbb{E}^{n+1,m+1}(U) \simeq \mathbb{E}^{n+1,m+1}(\mathbb{G}_m \times U) \xrightarrow{(2)} \mathbb{E}^{n+1,m+1}(U)$$

where (1) is the obvious inclusion and (2) is the pullback by the graph of the induced map $\pi|_U : U \to \mathbb{G}_m$.

Then if (X, Z) is $\mathbb{E}$-pure, the following relation holds:

$$\partial_{X,Z} \circ \gamma_\pi \circ j^* = i^*.$$

Proof. The proof is essentially the same as the one explained in [Dég08b, Prop. 2.6.5]. Let us indicate the main steps.

According to the definition of the purity isomorphism through the deformation diagram (1.3.1.a), and the fact the canonical morphism $\sigma_1 : X \to D_Z X$ is a split monomorphism (cf *loc. cit.*), we restrict to the case of $(N_Z X, Z)$ where $N_Z X$ is the normal bundle of Z in X, while π can be considered as a global trivialization of this line bundle.

Up to isomorphism, we thus are reduced to the case of $(\mathbb{A}_Z^1, Z)$ and the canonical parametrization t of the affine line (given by the identity). Then, the problem can be unfolded to a trivial identity given at the level of schemes (cf. the end of the proof of *loc. cit.*). $\square$

Example 3.1.7. Let R be a discrete valuation ring with valuation v, fraction field K and residue field k.

Recall first that the *tame residue symbol* on Milnor K-theory:

(3.1.7.a) $$\partial_v : K_n^M(K) \to K_{n-1}^M(k)$$

is uniquely characterized by the following properties, where π is a given uniformizing parameter:

(1) $\partial_v(\{\pi\}) = 1$;
(2) for units $u_1, ..., u_r \in R^\times$, $\partial_v(\{\pi, u_1, \ldots, u_r\}) = \{\bar{u}_1, \ldots, \bar{u}_r\}$;
(3) for units $u_1, ..., u_r \in R^\times$, $\partial_v(\{u_1, \ldots, u_r\}) = 0$.

If we assume that $(\mathrm{Spec}(R), \mathrm{Spec}(k))$ is $\mathbb{E}$-pure, we get an abstract residue map with coefficients in $\mathbb{E}$:

$$\partial_v^{\mathbb{E}} : \mathbb{E}^{n,m}(K) \to \mathbb{E}^{n-1,m-1}(k).$$

When R is of equal characteristics, we can apply this construction to the case of Voevodsky's Einlenberg-MacLane motivic ring spectrum $\mathbf{H}_{\mathbb{Z}} - S$ is the spectrum of the prime field of R. Then, the induced map

$$\partial_v : K_n^M(K) \simeq \mathbf{H}_{\mathcal{M}}^{n,n}(K) \to \mathbf{H}_{\mathcal{M}}^{n-1,n-1}(k) \simeq K_{n-1}^M(k)$$

coincides with (3.1.7.a). In fact, relations (1) and (2) follows from the previous proposition while relation (3) follows from the fact $\partial_v \circ j^* = 0$ according to the Gysin long exact sequence (3.1.1.b).

In the unequal characteristics case, we get the same result with rational coefficients by considering Beilinson motivic cohomology ring spectrum $\mathbf{H}_{\mathrm{B}}$.

Remark 3.1.8. Other applications of the previous proposition will be given in Section 5.5.

3.2. **Projective lci morphisms.**

3.2.1. Let X be a scheme, put $P = \mathbb{P}_X^n$ and consider the canonical projection $p : P \to X$. Recall we have introduced in Example 2.2.10 the Gysin morphism: $p_* : \mathbb{E}^{*,*}(\mathbb{P}_X^n) \to \mathbb{E}^{*-2n,*-n}(X)$ based on the construction of [Dég08a]. We give an alternative construction which uses the point of view considered in the present setting. It is based on the following facts which follow directly from the projective bundle theorem (2.1.13). Consider the bigraded ring $A = \mathbb{E}^{**}(X)$. In the paragraph and definition that follow, we work in the category of bigraded A-modules and refer to them simply as A-modules:

- The bigraded group $\mathbb{E}^{**}(P)$ is a free A-module of finite rank.
- The Künneth map

$$\mathbb{E}^{**}(P) \otimes_A \mathbb{E}^{**}(P) \to \mathbb{E}^{**}(P \times_X P), (x, y) \mapsto p_1^*(x) \cup p_2^*(y)$$

is an isomorphism of A-modules.

From the first point, we deduce that the A-module $\mathbb{E}^{**}(P)$ is dualizable. Let $\mathbb{E}^{**}(P)^{\vee}$ be its A-dual and

$$ev : \mathbb{E}^{**}(P)^{\vee} \otimes_A \mathbb{E}^{**}(P) \to A$$

be the evaluation map. Let $\delta : P \to P \times_X P$ be the diagonal immersion of P/X. Using the Gysin morphism and the second point, we get a co-pairing of A-modules:

$$(3.2.1.a) \quad \epsilon_P : A = \mathbb{E}^{**}(X) \xrightarrow{p^*} \mathbb{E}^{**}(P) \xrightarrow{\delta_*} \mathbb{E}^{**}(P \times_X P) \simeq \mathbb{E}^{**}(P) \otimes_A \mathbb{E}^{**}(P).$$

We claim this is a duality co-pairing in the sense that the induced map

$$(3.2.1.b) \quad \mathcal{D}_P : \mathbb{E}^{**}(P)^{\vee} \xrightarrow{1 \otimes \epsilon_P} \mathbb{E}^{**}(P)^{\vee} \otimes_A \mathbb{E}^{**}(P) \otimes_A \mathbb{E}^{**}(P) \xrightarrow{ev \otimes 1} \mathbb{E}^{**}(P)$$

is an isomorphism. Indeed, the computation of the matrix of $\mathcal{D}_P$ in the base given by the projective bundle theorem is precisely the same as that of [Dég08a, Lem. 5.5]: it is a triangular lower matrix with the identity diagonal. Note that $\mathcal{D}_P$ is homogeneous of degree $(2n, n)$.

Definition 3.2.2. Using the above notations, one defines the Gysin morphism associated with p as the following composite morphism:

$$p_* : \mathbb{E}^{**}(P) \xrightarrow{\mathcal{D}_P^{-1}} \mathbb{E}^{**}(P)^\vee \xrightarrow{(p^*)^\vee} \mathbb{E}^{**}(X)^\vee = \mathbb{E}^{**}(X),$$

homogeneous of degree $(-2n, -n)$.

In other words, p_* is the transpose of p^* with respect to the duality given by the co-pairing (3.2.1.a). In view of [Dég08a, Prop. 5.26], the above definition coincides with that of Example 2.2.10.

Remark 3.2.3. Here is a practical way to determine the Gysin map p_* constructed in the preceding definition. Let us again denote by A the cohomology ring $\mathbb{E}^{**}(X)$.

First, one determines the map ϵ_P — see (3.2.1.a). According to the projective bundle theorem, the $\mathbb{E}$-cohomology of $P \times_X P$ is a free A-module with basis $c^i.d^j$, $0 \le i, j \le n$, where c (resp. d) is the first Chern class of the canonical line bundle on the first (resp. second) factor of $P \times_X P$ (see Theorem 2.1.13). The A-linear map ϵ_P is uniquely determined by its value on the multiplicative identity element 1 of A, which can be a priori written as:

$$\epsilon_P(1) = \delta_*(1) = \sum_{0 \le i,j \le n} \pi^*\big(\eta_{ij}^{(n)}\big).c^i d^j.$$

where π is the projection map of $P \times_X P/X$.

Second, the map $\mathcal{D}_P$ — see (3.2.1.b) — is an A-linear map between rank $n+1$ free A-modules. Then the matrix M of $\mathcal{D}_P$ with respect to the basis on the source and targets coming from the projective bundle theorem applied to P/X can be easily computed with the previous notation as:

$$M := \Big(\eta_{ij}^{(n)}\Big)_{\{0 \le i,j \le n\}}.$$

From what was said in the paragraph preceding the above definition, the matrix M is invertible — it is in fact symmetric, lower triangular with 1 on the anti-diagonal; see Example 3.2.14 for a computation of this matrix in terms of the formal group law associated with $(\mathbb{E}, c)$.

According to the preceding definition, the Gysin map p_* is then completely determined by the inverse matrix M^{-1}. In fact, here is a way to compute the image of an element $x \in \mathbb{E}^{**}(P)$ under the Gysin map p_*:

- We write:
$$x = \sum_{0 \le i \le n} p^*(x_i).c^i$$
  where c is the canonical line bundle on P. Let X be the element of $A^{\{0,\ldots,n\}}$ whose value at the i-th place is x_i.
- The map $(p^*)^\vee$ just extract the coefficient of the element $(c^0)^\vee$ with respect to the decomposition of elements of $\mathbb{E}^{**}(P)$ in the dual base $\big((c^i)^\vee\big)$. Therefore, we get that $p_*(x)$ is the 0-th coefficient of $M^{-1}.X$, seen as an element of A^n.

Note that, given the cofactors formula to compute the inverse of a matrix, we get in particular the following formula:

$$(3.2.3.a) \qquad p_*(1) = \frac{1}{\det(M)} . \det\Big(\eta_{ij}^{(n)}\Big)_{\{1 \le i,j \le n\}}.$$

This will be made precise in Example 3.2.14

3.2.4. The Gysin morphism introduced above satisfies the following properties:
- (1) For any integers $n, m \geq 0$, considering the canonical projections

$$
\begin{array}{ccc}
\mathbb{P}_X^n \times_X \mathbb{P}_X^m & \xrightarrow{q'} & \mathbb{P}_X^n \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\
\mathbb{P}_X^m & \xrightarrow{\quad q \quad} & X
\end{array}
$$

one has: $p_* q'_* = q_* p'_*$.
- (2) For any cartesian square

$$
\begin{array}{ccc}
\mathbb{P}_Z^n & \xrightarrow{l} & \mathbb{P}_X^n \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\
Z & \xrightarrow{i} & X
\end{array}
$$

where i is an $\mathbb{E}$-pure closed immersion and p is the canonical projection: $i_* q_* = p_* l_*$.
- (3) For any integer $n \geq 0$ and any section s of the projection $p : \mathbb{P}_X^n \to X$: $p_* s_* = 1$.

For the proof of these properties, we use Remark 3.1.5 and refer the reader to [Dég08a], respectively Lemmas 5.8, 5.9 and 5.10. Then the following lemma follows formally from these properties – see [Dég08a, Lem. 5.11]:

Lemma 3.2.5. *Assume $\mathbb{E}$ is absolutely pure and consider a commutative diagram :*

$$
Y \underset{i}{\overset{k}{\rightrightarrows}} \begin{array}{c} \mathbb{P}_X^n \\ \mathbb{P}_X^m \end{array} \underset{q}{\overset{p}{\rightrightarrows}} X
$$

where i (resp. k) is a closed immersion and p (resp. q) is the canonical projection. Then, using the above definitions, $p_* k_* = q_* i_*$.

3.2.6. Recall that an X-scheme Y is said to be a *local complete intersection* if it admits locally an immersion into an affine space $\mathbb{A}_X^n$ – see [SGA6, VIII, 1.1]. We will say abusively *lci* instead of local complete intersection.
Given a projective lci morphism $f : Y \to X$ in $\mathscr{S}$, it admits a factorization of the form

$$
(3.2.6.a) \qquad\qquad Y \xrightarrow{i} \mathbb{P}_X^n \xrightarrow{p} X
$$

where i is a regular closed immersion and p is the canonical projection. This follows from our convention on projective morphisms and [SGA6, 1.2].

Now, assume that $\mathbb{E}$ is absolutely pure. From the preceding lemma, the composite morphism:

$$
\mathbb{E}^{**}(Y) \xrightarrow{i_*} \mathbb{E}^{**}(\mathbb{P}_X^n) \xrightarrow{p_*} \mathbb{E}^{**}(X)
$$

obtained using Definition 3.1.2 and Paragraph 3.2.1 is independent of the choice of the factorization of f.

Definition 3.2.7. Consider the above assumptions and notations.
We put $f_* = p_* i_*$ and call it the Gysin morphism associated with f.

Note that if f has dimension d, then f_* is homogeneous of bidegree $(-2d, -d)$.

Remark 3.2.8. The warning of Remark 3.1.3 can be equally applied to the preceding Gysin morphism. In fact, it is proved in Theorem 3.3.1 that the preceding Gysin morphisms are uniquely determined by the chosen orientation c on $\mathbb{E}$.

Example 3.2.9.   (1) *Motivic cohomology and Chow groups.*– Let k be a perfect field and consider the k-absolute ring spectrum $\mathbf{H}_\mathbb{Z}$ of Example 2.1.4. It is well known that for any integer $n \geq 0$ and any smooth k-scheme:

$$\mathbf{H}_\mathcal{M}^{2n,n}(X, \mathbb{Z}) \simeq CH^n(X).$$

In fact, we know that the Nisnevich sheaf $H^n(\mathbb{Z}(n))$ is the unramified Milnor K-theory $\mathcal{K}_n^M$, and from the hypercohomology spectral sequence associated with the homotopy t-structure along with some known vanishing[42], one gets:

$$\mathbf{H}_\mathcal{M}^{2n,n}(X, \mathbb{Z}) \simeq H_{\mathrm{Zar}}^n(X, \mathcal{K}_*^M) \simeq CH^n(X).$$

Then, from [Dég11, Prop. 3.16], we get that the Gysin morphism associated with $\mathbf{H}_\mathbb{Z}$ and a projective morphism in the previous definition coincides in degree $(2n, n)$ with the usual pushforward on Chow groups.[43]   (See also Example 3.3.4(2)).

(2) *The spectrum* **KGL** *and algebraic K-theory.*– the Gysin morphism on **KGL** associated with a projective morphism of regular schemes agrees with the usual functoriality: see Example 3.3.4(3).

Proposition 3.2.10. *Assume $\mathbb{E}$ is absolutely pure.*

(1) *For any composable projective lci morphisms f and g: $f_* g_* = (fg)_*$.*
(2) *For any projective lci morphism $f : Y \to X$ and any couple (x, y) in $\mathbb{E}^{**}(X) \times \mathbb{E}^{**}(Y)$,*

$$f_*\big(f^*(x).y\big) = x.f_*(y).$$

Proof. Point (1) follows formally from the preceding facts (for details, see the proof of [Dég08a, Prop. 5.14], using: relations (1), (2) of Paragraph 3.2.4, point (1) of Proposition 3.1.4 and the preceding lemma).

Point (2) follows from relation (2) of Proposition 3.1.4 for the case of a closed immersion together with [Dég08a, Cor. 5.18] for the case of the projection of $\mathbb{P}_X^n/X$. $\square$

3.2.11. Consider a cartesian square

$$\begin{array}{ccc} Y' & \overset{q}{\to} & X' \\ {\scriptstyle g}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ Y & \underset{p}{\to} & X \end{array}$$

in $\mathscr{S}$ such that p is a projective lci morphism.

Choose an X-embedding ν of Y into a suitable projective bundle $\mathbb{P}_X^n$. The preceding square induces a cartesian morphism of closed pairs $\Theta : (\mathbb{P}_{X'}^n, Y') \to (\mathbb{P}_X^n, Y)$ and we denote by ξ the vector bundle over Y' defined by formula (2.4.2.a) with respect to Θ. According to [Ful98, Prop. 6.6(c)] the vector bundle ξ is independent

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[42]namely: $H^q(\mathbb{Z}(n)) = 0$ if $q > n$ and $H^p(X, H^q(\mathbb{Z}(n))) = 0$ if $p > n$, which follows from the Gersten resolution satisfied by the sheaf $H^q(\mathbb{Z}(n))$.

[43]Let us indicate also that one can go from the case of smooth k-schemes to arbitrary regular schemes of equal characteristics using Popescu's desingularization theorem together with Lemma 1.2.13.

up to isomorphism of the choice of ν and we call it the *excess intersection bundle* associated with the square Δ.

Proposition 3.2.12. *Assume $\mathbb{E}$ is absolutely pure and consider the above notations. Let e be the rank of ξ over Y'.*

Then for any $y \in \mathbb{E}^{**}(Y)$, one has: $f^* p_*(y) = q_*\big(c_e(\xi).g^*(y)\big)$.

This follows easily from the definition of ξ and point (3) of Proposition 3.1.4.

As usual, one says the square Δ is transversal if $e = 0 - i.e.$ for any point a in Y', the dimension of q at a is equal to the dimension of p at $g(a)$. In this case, the preceding formula reads simply: $f^* p_* = q_* g^*$.

As a companion of the Gysin morphism, one gets another kind of characteristic classes, the cobordism classes. The definition obtained here is valid in a slightly better generality than that of [Dég08a, §5.27].

Definition 3.2.13. Assume $\mathbb{E}$ is an absolutely pure oriented ring spectrum.

Given any base scheme S in $\mathscr{S}$ and any projective S-scheme X in $\mathscr{S}$ with structural morphism p, we define the *cobordism class* of X/S as the following element of $\mathbb{E}^{**}(S)$:

$$[X/S] := p_*(1).$$

Note that if X/S has constant relative dimension d, the class $[X/S]$ has cohomological degree $(-2d, -d)$.

Example 3.2.14. Let us now recall the explicit computation of the cobordism class of the linear projective n-plane obtained in [Dég08a, 5.31]. In fact, we will actually correct a sign mistake *loc. cit.* that was indicated to us by the referee.

According to Remark 3.2.3, the key point is to compute the *fundamental class of the diagonal δ* of the X-scheme $P = \mathbb{P}^n_X$; in other words, the element $\delta_*(1)$, with the notation of Definition 3.1.2.

The main point used to compute that class is the classical fact that the diagonal subscheme $\delta(P)$ of $P \times_X P$ is the zero locus of a canonical section of the vector bundle

$$\lambda_1^\vee \otimes \xi_2 = \underline{\mathrm{Hom}}(\lambda_1, \xi_2)$$

where λ_1 (resp. ξ_2) is the canonical line (resp. quotient) bundle on the first (resp. second) factor of $P \times_X P$, seen as a vector bundle over $P \times_X P$ (see [Dég08a, Ex. 5.29]). Therefore according to Corollary 2.4.6, one gets:

$$\delta_*(1) = c_n\big(\lambda_1^\vee \otimes \xi_2\big).$$

Then, if $(a_{ij})_{i,j\in\mathbb{N}}$ denotes the coefficients of the formal group law associated with the orientation c of $\mathbb{E}$, one gets the following formula[44]:

$$\delta_*(1) = \sum_{0 \leq i,j \leq n} \pi^*(a_{1,i+j-n}).c^i.d^j.$$

Therefore, with the notations of Remark 3.2.3, the matrix M corresponding to $\delta_*(1)$ is:

$$\begin{pmatrix} 0 & \cdots\cdots & 0 & 1 \\ \vdots & & \diagup & a_{1,1} \\ 0 & & \diagup & \vdots \\ 1 & a_{1,1} & \cdots\cdots & a_{1,n} \end{pmatrix}.$$

---

[44]We refer the reader to [Dég08a, proof of 5.30] for a proof.

Note a basic computation gives that the determinant of this matrix is $(-1)^{\lfloor (n+1)/2 \rfloor}$. Therefore, using formula (3.2.3.a) together with the preceding computation, we get the following expression of the cobordism class of the projective n-plane:

$$(3.2.14.a) \qquad [\mathbb{P}^n/S] = (-1)^{\lfloor (n+1)/2 \rfloor} . \begin{vmatrix} 0 & \cdots & 0 & 1 & a_{1,1} \\ & & & & a_{1,2} \\ 0 & & & & \\ 1 & & & & \\ a_{1,1} & a_{1,2} & \cdots & & a_{1,n} \end{vmatrix} .$$

Beware that the sign written here corrects the sign in [Dég08a, Cor. 5.31]. In fact, this formula coincides with its analogue in topology given by the classical Myschenko formula.

Given an arbitrary projective S-scheme X, we get the following method to compute the cobordism class of X/S: choose an embedding $i : X \to \mathbb{P}^N_S$; compute the fundamental class of i in terms of the canonical basis $(c^i, 0 \le i \le N)$ of $\mathbb{E}^{**}(\mathbb{P}^n_S)$ as a free $\mathbb{E}^{**}(S)$-module, where c is the first Chern class of dual canonical invertible bundle:

$$\eta_{\mathbb{P}^n_S}(X) = \sum_{i=0}^{N} x_i . c^i, x_i \in \mathbb{E}^{**}(S).$$

Then the projection formula yields the following expression:

$$[X/S] = \sum_{i=0}^{N} x_i . [\mathbb{P}^{N-i}/S].$$

Remark 3.2.15. Using the Riemann-Roch formula, we will reprove Quillen formula which computes the cobordism class of an arbitrary projective bundle (see Ex. 5.2.7)

3.3. **Uniqueness.** In our setting, we can extend the uniqueness statement of [Pan09, 4.1.4], to obtain the following characterization of the Gysin morphisms we have introduced.

Theorem 3.3.1. *Assume $\mathbb{E}$ is an absolutely pure ring spectrum over $\mathscr{S}$. For any scheme S, we will denote by η_S the class in $\mathbb{E}^{2,1}(\mathbb{P}^1_S)$ obtained by $\mathbb{P}^1$-suspension of the unit $S^0 \to \mathbb{E}$ of the ring spectrum structure.*

Suppose given for any projective lci morphism $f : Y \to X$ a morphism of bigraded abelian groups (non necessarily homogeneous):

$$f_\star : \mathbb{E}^{**}(Y) \to \mathbb{E}^{**}(X)$$

such that the following properties hold:

  (1) *$(fg)_\star = f_\star g_\star$;*
  (2) *$f_\star$ is $\mathbb{E}^{**}(X)$-linear: $f_\star(f^*(x).y) = x.f_\star(y)$;*
  (3) *for any square Δ as in 3.2.11, which is in addition transversal, $f^* p_\star = q_\star g^*$;*
  (4) *if i is a closed immersion with complementary open immersion i', $\mathrm{Im}(i_\star) = \mathrm{Ker}(i'^*)$.*

For any integer $n > 0$, let λ_n be the canonical line bundle on $\mathbb{P}^n_S$ and s_n be its zero section. Let us put:

$$c_{n,S} = s_n^* s_{n\star}(1).$$

Then the following conditions are equivalent:

(i) *the sequences $c_S = (c_{n,S})_{n>0}$ indexed by a scheme S form an absolute orientation of $\mathbb{E}$;*

(ii) *for any $n > 0$, $c_{n,S}$ has bidegree $(2,1)$ and $c_{1,S} = \eta_S$ in $\mathbb{E}^{2,1}(\mathbb{P}^1_S)$.*

When these equivalent conditions are fulfilled, we get in addition for any scheme S which admits an ample family of line bundles:

(5) *for any section s of a line bundle L/S, $s^* s_\star(1) = c_1(L)$ where the right hand side is the first Chern class associated with the orientation c of $\mathbb{E}$ given by the above condition (i).*

Finally, if we assume that condition (5) holds for any scheme S in $\mathscr{S}$, we get:

- *for any projective lci morphism f, one has: $f_\star = f_*$, the last morphism being the Gysin morphism defined in 3.2.7 with respect to $(\mathbb{E}, c)$.*

Remark 3.3.2. To summarize: the Gysin morphism of an absolutely pure oriented ring spectrum $(\mathbb{E}, c)$ is uniquely characterized by properties (1)-(5).

Proof. The equivalence between (i) and (ii) is obvious from Definition 2.1.2 and the fact that Point (3) implies: $\iota_n^*(c_{n+1}^S) = c_n^S$.

Let us assume these equivalent conditions are satisfied. To prove (5), using homotopy invariance, we can assume s is the zero section of L/S. Because S admits an ample family of line bundles, we can use Jouanolou's trick which reduces us to the case where S is affine. Then L is generated by a finite number of its sections and one can find an immersion $i : S \to \mathbb{P}^n_S$ such that $L = i^{-1}(\lambda_n)$. Using (3), one reduces to the case where S is $\mathbb{P}^n_S$, $L = \lambda_n$ which holds by definition.

Let us now prove the final point. Given any projective lci morphism f, we put $f_\flat = f_* - f_\star$. We have proved previously that the Gysin morphisms f_* satisfies properties (1)-(4) stated above. In particular, $f_\flat$ satisfies properties (1)-(3) and in the situation of (4), we get:

(4') $\operatorname{Im}(i_\flat) \subset \operatorname{Ker}(i'^*)$.

<u>Case of closed immersions</u>: assuming $f = i : Z \to S$ is a closed immersion, we show $i_\flat = 0$. Using (3) for $?_\flat$, the deformation diagram (1.3.1.a) induces a commutative diagram:

$$\begin{array}{ccccc}
\mathbb{E}^{**}(Z) & \xleftarrow{\ s_1^*\ } & \mathbb{E}^{**}(\mathbb{A}^1_Z) & \xrightarrow{\ s_0^*\ } & \mathbb{E}^{**}(Z) \\
{\scriptstyle i_\flat}\downarrow & & {\scriptstyle k_\flat}\downarrow & & \downarrow{\scriptstyle s_\flat} \\
\mathbb{E}^{**}(X) & \xleftarrow{\ \sigma_1^*\ } & \mathbb{E}^{**}(D_Z X) & \xrightarrow{\ \sigma_0^*\ } & \mathbb{E}^{**}(N_Z X).
\end{array}$$

We use the following lemma:

Lemma 3.3.3. *Consider the notations above and let $k' : (D_Z X - \mathbb{A}^1_Z) \to D_Z X$ be the complementary open immersion to k. Then the morphism: (k'^*, σ_0^*) is injective.*

Indeed, let $x \in \mathbb{E}^{**}(D_Z X)$ being an element of the kernel. Because of property (4), we get a cohomology class y such that $x = k_\star(y)$. On the other hand, because of (3), $\sigma_0^* k_\star = s_\star s_0^*$. But $s_\star$ is a split monomorphism (because of (1)), and s_0^* is an isomorphism. Thus we deduce $y = 0$ and this concludes.

With the help of this lemma, and the fact $k'^* k_\flat = 0$ from (4') above, we see that it is sufficient to prove $\sigma_0^* k_\flat = 0$. Thus, we are reduced to show that $s_\flat = 0$, where again s is the zero section of the vector bundle $E = N_Z X$ over Z.

Using the splitting principle applied to the vector bundle E/Z and property (1), we reduce to the case where s is the zero section of a line bundle $p : L \to Z$.

According to (2), $s_\flat(x) = s_\flat(1).p^*(x)$. The fact $s_\flat(1) = 0$ is nothing else than assumption (5).

$\underline{\text{Case of a projective bundle}}$: let $p : \mathbb{P}^n_S \to S$ be the canonical projection, and let us show $p_\flat = 0$. Let δ be the diagonal immersion of $\mathbb{P}^n_S/S$. Let us consider the bigraded ring $A = \mathbb{E}^{**}(S)$. Below any A-module is assumed to be a bigraded A-module, but we do not require morphisms of bigraded A-modules are homogeneous. The symbol $\otimes_A$ means the tensor product of bigraded A-modules. Then, according to the projective bundle theorem, $M = \mathbb{E}^{**}(\mathbb{P}^n_S)$ is a free A-module of finite rank. Thus it is a rigid object of the category of A-modules. On the other hand, one gets using (2) the following morphisms of A-modules:

$$p_\star\delta^* : M \otimes_A M \longrightarrow A$$
$$\delta_\star p^* : A \longrightarrow M \otimes_A M.$$

Using (1), (2) and (3), we get that the following composite is the identity morphism:

$$M \xrightarrow{M\otimes_A \delta_\star p^*} M \otimes_A M \otimes_A M \xrightarrow{p_\star\delta^*\otimes_A M} M.$$

The same is true when replacing $\star$ by $*$. Thus, as we have already proved that $\delta_\star = \delta_*$, we deduce that:

$$(p_\flat\delta^* \otimes_A M) \circ (M \otimes_A \delta_* p^*) = 0.$$

As δ_* and p^* are split monomorphisms, we deduce that $p_\flat\delta^* \otimes_A M = 0$. This allows to conclude because M is faithfully flat over A and δ^* is a split epimorphism. $\qquad\square$

Example 3.3.4. As said before, the case where $\mathscr{S}$ is the category of smooth k-schemes was already obtained by Panin and Smirnov, later published by Panin in [Pan09, 4.1.4].[45]

(1) It is worthwhile to mention that this result applies in particular to any Mixed Weil cohomology theory: example 1.2.3(1), where $\mathscr{S}$ is the category of regular k-schemes. Thus, there is only one way to define pushforwards on any such cohomology satisfying conditions (1)-(5) – and in particular compatible with a well behaved first Chern class.

(2) It can also be applied to the usual Chow groups: in [Dég08a], we have shown that the category of stable motivic complexes $DM(k)$ is endowed with a t-structure, called the homotopy t-structure[46] and whose heart is the category of cycle modules defined by Rost. In particular, the cycle module corresponding to Milnor K-theory defines an object in $DM(k)$ which is nothing else than the 0-th cohomology object of the unit in $DM(k)$ (see *loc. cit.*, Th. 5.11). Using the forgetful functor

$$DM(k) \to SH(k)$$

we can see this object as a ring spectrum in $SH(k)$: by definition, it corresponds to the unramified Milnor-K-theory sheaf $\mathcal{K}^M_*$, seen over the Nisnevich site of smooth k-schemes, and represents the Chow group in $SH(k)$, with its product structure. Note also this is a direct factor of the motivic Eilenberg-MacLane spectrum $\mathbf{H}^{\mathbb{Z}}_{\mathcal{M},k}$ representing motivic cohomology (cf.

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[45]See [PS00] for the original work of Panin and Smirnov. I thank the referee for attracting my attention on the paper of Smirnov [Smi06a].

[46]extending the homotopy t-structure of Voevodsky that he defined on $DM^{eff}_-(k)$.

1.2.3): it corresponds to cut-out the groups of bidegree $(2*, *)$. As usual, one extends $\mathcal{K}_k$ to a k-absolute ring spectrum $\mathcal{K}$ by taking pullbacks within the fibred category SH.

Then the preceding theorem applies to the latter ring spectrum $\mathcal{K}$, when $\mathscr{S}$ is the category of regular k-schemes (because of Lemma 1.2.13 and the fact that usual Chow groups commute with limits of regular k-schemes). Thus, it says that prescribing the orientation on CH^*, there exists a unique way of defining pushforwards satisfying properties (1)-(5).

(3) More interestingly, the proposition applies to $\mathbf{KGL}$, the absolute ring spectrum representing K-theory when $\mathscr{S}$ is the category of regular schemes (Example 1.2.3(3)). Recall from [Qui73] the corresponding cohomology has well defined pushforward that satisfies conditions (1)-(4) of the proposition. Moreover, with the choice of orientation of Example 2.1.4(4), condition (5) of the above Theorem has been proved by Thomason in [Tho93, Th. 3.1]. Thus, the isomorphism (2.1.5.a) is covariantly functorial with respect to projective morphisms between regular schemes, where on the source we consider the pushforward defined above with respect to the absolute oriented ring spectrum $(\mathbf{KGL}, c^{\mathbf{KGL}})$ and on the aim we consider Quillen pushforward.

Note also that we need Thomason excess intersection formula only in the case of line bundles: the machinery developed here gives a proof in the general case. Note also that if we make the "hypothèse paresseuse" (referred to by Thomason in the end of the introduction of *loc. cit.*) that schemes, in addition to being regular, admit an ample line bundle, we even get a new proof of the excess intersection formula for higher K-theory – because (5) is then automatically satisfied by our choice of orientation and we can compare our Gysin morphism with the one defined by Quillen.

4. Riemann-Roch formulas

4.0.1. In this section, we will consider two absolute oriented ring spectra $(\mathbb{E}, c)$, $(\mathbb{F}, d)$ and a morphism of absolute ring spectra (cf Definition 1.2.1):

$$\varphi : \mathbb{E} \to \mathbb{F}.$$

When considering the constructions of orientation theory as described previously for $\mathbb{E}$ (resp. $\mathbb{F}$), we will will put an index $\mathbb{E}$ (resp. $\mathbb{F}$) in the notation. However, when no confusion is possible, we drop this index.

In the particular case $\mathbb{E} = \mathbb{F}$, we will also put an upper-index c (resp. d) in the notation of orientation theory (Chern classes, Gysin morphisms, ...) with respect to the orientation c (resp. d).

4.1. **Todd classes.**

4.1.1. Let S be a scheme (in $\mathscr{S}$). We deduce from φ a morphism of graded rings:

$$\mathbb{E}^{**}(\mathbb{P}_S^\infty) \xrightarrow{\varphi_{\mathbb{P}_S^\infty}} \mathbb{F}^{**}(\mathbb{P}_S^\infty)$$

using the notations of Paragraph 2.1.1. According to the projective bundle theorem (2.1.13), this corresponds to a morphism of ring:

$$\mathbb{E}^{**}(S)[[u]] \to \mathbb{F}^{**}(S)[[t]]$$

and we denote by $\Psi_S(t)$ the image of u by this map. In other words, this formal power series is characterized by the relation:

$$(4.1.1.a) \qquad \varphi_{\mathbb{P}_S^\infty}(c) = \Psi_S(d).$$

Note that the restriction of $\varphi_{\mathbb{P}_S^\infty}(c)$ to $\mathbb{P}_S^0$ (resp. $\mathbb{P}_S^1$) is 0 (resp. the multiplicative identity element of the ring $\mathbb{F}^{**}(\mathbb{P}_S^1)$) because c is an orientation and φ is a morphism of ring spectra. Thus, as d is also an orientation of $\mathbb{F}$, we can write $\Psi_S(t)$ as:

$$\Psi_S(t) = t + \sum_{i>1} \alpha_i^S . t^i$$

where $\alpha_i^S \in \mathbb{F}^{2-2i,1-i}(S)$. Obviously, the power series $\Psi_S(t)/t$ is invertible.

We will also consider the commutative monoid $\mathcal{M}(S)$ generated by the isomorphism classes of vector bundles over S modulo the relations $[E] = [E'] + [E'']$ coming from exact sequences

$$0 \to E' \to E \to E'' \to 0.$$

Then $\mathcal{M}$ is a presheaf of monoids on $\mathscr{S}$ whose presheaf of abelian groups is the functor K_0.

Note that $\mathbb{F}^{00}(S)$, equipped with cup-product, is a commutative monoid. We will denote by $\mathbb{F}^{00\times}(S)$ the group made by its invertible elements.

Proposition 4.1.2. *There exists a unique natural transformation of presheaves of monoids over $\mathscr{S}$*

$$\mathrm{Td}_\varphi : \mathcal{M} \to \mathbb{F}^{00}$$

such that for any line bundle L over a scheme X,

$$(4.1.2.a) \qquad \mathrm{Td}_\varphi(L) = \frac{t}{\Psi_S(t)} (t = d_1(L)).$$

Moreover, it induces a natural transformation of presheaves of abelian groups:

$$\mathrm{Td}_\varphi : K_0 \to \mathbb{F}^{00\times}.$$

Proof. The proof is very classical: the uniqueness statement follows from the *splitting principle* while the existence statement follows from the use of *Chern roots*. Note also that the relation (4.1.2.a) is well defined because $c_1(L)$ is nilpotent (see Proposition 2.1.22). The final assertion follows from the fact $t/\Psi_S(t)$ is an invertible formal power series. $\square$

Remark 4.1.3. According to the construction of the first Chern classes for the oriented ring spectra $(\mathbb{E}, c)$ and $(\mathbb{F}, d)$ together with Relations (4.1.1.a) and (4.1.2.a), we get for any line bundle L/S the following identity in $\mathbb{F}^{2,1}(S)$:

$$(4.1.3.a) \qquad \varphi_S(c_1(L)) = \mathrm{Td}_\varphi(-L) \cup d_1(L).$$

Definition 4.1.4. Consider the context and notations of the previous proposition.

Given any virtual vector bundle e over X, the element $\mathrm{Td}_\varphi(e) \in \mathbb{F}^{00}(X)$ is called the *Todd class* of e over X associated with the morphism of ring spectra φ.

4.2. **The case of closed immersions.**

4.2.1. Consider a regular closed immersion $i : Z \to X$, $U = X - Z$. As by assumption $\mathbb{E}$ (resp. $\mathbb{F}$) is absolutely pure, we can consider the associated refined fundamental class $\bar{\eta}_X^{\mathbb{E}}(Z)$ in $\mathbb{E}_Z^{**}(X)$ (resp. $\bar{\eta}_X^{\mathbb{F}}(Z)$ in $\mathbb{F}_Z^{**}(X)$) – Definition 2.3.1. The morphism φ induces a map in relative cohomology:

$$\mathbb{E}_Z^{**}(X) \xrightarrow{\varphi_{X,Z}} \mathbb{F}_Z^{**}(X)$$

According to the definition of the purity isomorphism (2.3.1.b), we deduce there exists a unique class $\tau_\varphi(X, Z) \in \mathbb{F}^{00}(Z)$ such that

(4.2.1.a) $$\varphi_{X,Z}\big(\bar{\eta}_X^{\mathbb{E}}(Z)\big) = \tau_\varphi(X, Z).\bar{\eta}_X^{\mathbb{F}}(Z).$$

This relation together with the definition of the localization long exact sequence (3.1.1.b) immediately gives the following commutative diagram:

(4.2.1.b)
$$\begin{array}{ccccc}
\mathbb{E}^{**}(U) & \xrightarrow{\partial_{X,Z}} & \mathbb{E}^{**}(Z) & \xrightarrow{i_*} & \mathbb{E}^{**}(X) \\
\varphi_U \downarrow & & \tau_\varphi(X,Z).\varphi_Z \downarrow & & \downarrow \varphi_X \\
\mathbb{F}^{**}(U) & \xrightarrow{\partial_{X,Z}} & \mathbb{F}^{**}(Z) & \xrightarrow{i_*} & \mathbb{F}^{**}(X).
\end{array}$$

The commutativity of the square on the right hand side is a tautological Riemann-Roch formula. Our proof of the actual Riemann-Roch theorem in fact lies in the computation of the class involved in this formula:

Lemma 4.2.2. *In the above assumptions, the following relation holds:*

$$\tau_\varphi(X, Z) = \mathrm{Td}_\varphi(-N_Z X).$$

Proof. As Relation (4.2.1.a) characterizes uniquely the class $\tau_\varphi(X, Z)$, whatever the regular closed pair (X, Z) is, the deformation diagram (2.3.1.a) gives the relation: $\tau_\varphi(X, Z) = \tau_\varphi(N_Z X, Z)$.

Thus, we are reduced to prove that for any scheme X and any vector bundle E/X,

$$\mathrm{Td}_\varphi(-E) = \tau_\varphi(E, X).$$

Using again the characterizing relation (4.2.1.a), which involves refined Thom classes according to (2.3.3.a), we deduce:

- for any morphism $f : Y \to X$ and any vector bundle E/X, $f^* \tau_\varphi(E, X) = \tau_\varphi(f^{-1}(E), Y)$.
- for any scheme X and any exact sequence of vector bundles over X,

$$0 \to E' \to E \to E'' \to 0,$$

one has: $\tau_\varphi(E, X) = \tau_\varphi(E', X)\tau_\varphi(E'', X)$.

More precisely: for the first relation, one uses the compatibility of the refined Thom class with pullback and for the second, one applies Proposition 2.4.7.

In other words, one gets a morphism of monoids

$$M(X) \to F^{00}(X), [E] \mapsto \tau_\varphi(E, X)$$

which is contravariantly natural in X. According to the uniqueness statement of Proposition 4.1.2, we are reduced to the case of line bundles.

Let L/X be a line bundle and P be its projective completion. Let ξ be the universal quotient bundle on P. According to Relation (4.1.3.a), one obtains:

$$\varphi_P\big(c(\xi)\big) = \mathrm{Td}_\varphi(-\xi) \cup d(\xi).$$

Let s be the section of P/X induced by the zero section of L/X. Using the commutativity of the right square of (4.2.1.b) when the closed pair (X, Z) is (P, X), we obtain:

$$\varphi_P\big(s_*(1)\big) = s_*(\tau_\varphi(P, X)) = s_*(\tau_\varphi(L, X))$$

According to (3.1.2.a), (2.3.3.b) and (2.2.4.a): $s_*(1) = c(\xi)$. Thus we obtain:

$$\mathrm{Td}_\varphi(-\xi) \cup s_*(1) = s_*(\tau_\varphi(L, X)).$$

Let $p : P \to X$ be the canonical projection. Of course, $p \circ s = Id$. Thus, it is sufficient to apply p_* to the preceding relation to conclude:

$$p_*\big(\mathrm{Td}_\varphi(-\xi) \cup s_*(1)\big) \overset{(*)}{=} p_* s_*\big(s^* \mathrm{Td}_\varphi(-\xi)\big) = s^*(\mathrm{Td}_\varphi(-\xi)) = \mathrm{Td}_\varphi(-L),$$

where $(*)$ is the projection formula 3.1.4(2). $\square$

Finally the following Riemann-Roch formulas are consequences of the commutative diagram (4.2.1.b) in conjunction with the preceding lemma.

Theorem 4.2.3. *Under the assumptions of the previous paragraph, the following formulas hold:*

$$\varphi_X(i_*(z)) = i_*\big(\mathrm{Td}_\varphi(-N_Z X) \cup \varphi_Z(z)\big),$$
$$\partial_{X,Z}\big(\varphi_U(u)\big) = \mathrm{Td}_\varphi(-N_Z X) \cup \varphi_Z\big(\partial_{X,Z}(u)\big).$$

4.3. The general case.

4.3.1. Consider a projective lci morphism $f : Y \to X$ (Par. 3.2.6). Given a factorization (3.2.6.a) of f, we define the *virtual tangent bundle* of f as the element of $K_0(Y)$:

$$\tau_f := [i^{-1}T_p] - [N_i]$$

where T_p is the tangent space of p and N_i is the normal bundle of i. Standard considerations show this definition is independent of the chosen factorization (see [SGA6, VIII, 2.2]).

Theorem 4.3.2. *Consider the notations above. Then for any element $y \in \mathbb{E}^{**}(Y)$, the following formula holds:*

$$\varphi_X(f_*(y)) = f_*\big(\mathrm{Td}_\varphi(\tau_f) \cup \varphi_Y(y)\big).$$

Proof. According to the multiplicativity of the Todd class and the compatibility of the Gysin morphism with respect to composition, this formula can be divided in two cases according to a factorization (3.2.6.a) of f. The case of a closed immersion was treated above (4.2.3) and it remains to consider the case of the canonical projection p of a projective space $P = \mathbb{P}_X^n/X$.

We first treat the case where $\varphi_{\mathbb{P}_X^\infty}(c) = d$. This implies that $\mathrm{Td}_\varphi = 1$, so we have to prove that for $y \in \mathbb{E}^{**}(P)$, one has:

(4.3.2.a) $$\varphi_X(p_*(y)) = p_*\big(\varphi_Y(y)\big).$$

Consider the notations of Paragraph 3.2.1 with respect to both $\mathbb{E}$ and $\mathbb{F}$. The Riemann-Roch formula for the diagonal embedding δ gives the formula:

$$\varphi_{P \times_X P} \delta_* = \delta_* \varphi_P.$$

This is sufficient to conclude in view of Definition 3.2.2. Let us give more details with the considerations of Remark 3.2.3. The preceding formula implies that the map $\varphi_X : \mathbb{E}^{**}(X) \to \mathbb{F}^{**}(X)$ sends each of the coefficients of the matrix $M^{\mathbb{E}}$ of $\delta_*(1) \in \mathbb{E}^{**}(P \times_X P)$ to the corresponding coefficient of the matrix $M^{\mathbb{F}}$ of $\delta_*(1) \in \mathbb{F}^{**}(P \times_X P)$. Therefore, the same result holds for the inverse matrices of $M^{\mathbb{E}}$ and $M^{\mathbb{F}}$. Thus finally, Formula (4.3.2.a) comes from the algorithm to compute p_* given at the end of Remark 3.2.3.

It remains now to consider the case where $\mathbb{E} = \mathbb{F}$, $\varphi = 1_{\mathbb{E}}$, for which we adopt the convention of the last paragraph of 4.0.1. Thus, we have to prove

$$(4.3.2.b) \qquad \forall \alpha \in \mathbb{E}^{**}(P),\ p_*^c(\alpha) = p_*^d(\mathrm{Td}_\varphi(T_p).\alpha).$$

We consider again the notations of Paragraph 3.2.1 with respect to both $(\mathbb{E}, c)$ and $(\mathbb{E}, d)$. Let $\pi : P \times_X P \to P$ be the projection to the second factor. The Riemann-Roch formula for δ can be read in this case as:

$$\delta_*^c(\alpha) = \delta_*^d\big(\mathrm{Td}_\varphi(-N_\delta)\cup\alpha\big) = \delta_*^d\big(\delta^*\pi^*(\mathrm{Td}_\varphi(-N_\delta))\cup\alpha\big) = \pi^*(\mathrm{Td}_\varphi(-N_\delta))\cup\delta_*^d(\alpha)$$

– we use Prop. 3.2.10 for the last equality. Recall from Paragraph 3.2.1 we have a Künneth isomorphism $\mathbb{E}^{**}(P \times_X P) = \mathbb{E}^{**}(P) \otimes_A \mathbb{E}^{**}(P)$. Through this identification, and the fact $N_\delta = T_p$, the class $\pi^*(\mathrm{Td}_\varphi(-N_\delta))$ corresponds to $1 \otimes_A \mathrm{Td}_\varphi(-T_p)$. Thus the preceding equality leads to the commutativity of the following diagrams:

$$
\begin{array}{ccc}
A \xrightarrow{\ \epsilon_P^c\ } \mathbb{E}^{**}(P) \otimes_A \mathbb{E}^{**}(P) & & \mathbb{E}^{**}(P)^{\vee} \xrightarrow{\ \mathcal{D}_P^c\ } \mathbb{E}^{**}(P) \\
\Vert \qquad\qquad \downarrow{\scriptstyle 1\otimes_A \mathrm{Td}_\varphi(T_p)} & \Rightarrow & \Vert \qquad\qquad \downarrow{\scriptstyle \mathrm{Td}_\varphi(T_p)} \\
A \xrightarrow{\ \epsilon_P^d\ } \mathbb{E}^{**}(P) \otimes_A \mathbb{E}^{**}(P) & & \mathbb{E}^{**}(P)^{\vee} \xrightarrow{\ \mathcal{D}_P^d\ } \mathbb{E}^{**}(P).
\end{array}
$$

Then formula (4.3.2.b) finally follows from Definition 3.2.2. $\qquad\square$

5. Examples and applications

5.0.1. In this section, we will adopt the following notations. Orientations of an absolute ring spectrum (Def. 2.1.2) will be denoted by letters c (or c', d, ...). Then the corresponding Chern classes (Def. 2.1.16) with be denoted by the same letter with a lower index: c_1, c_2,... The corresponding Gysin morphisms (resp. fundamental classes, residue morphisms) will be denoted by p_*^c (resp. $\bar{\eta}_X^c(Z)$, $\partial_{X,Z}^c$).

All spectra in this section will be $\mathscr{S}$-absolutely pure ring spectra, so we simply say spectra for $\mathscr{S}$-absolutely pure ring spectra. Recall that an orientation c of such a spectrum $\mathbb{E}$ is a family of orientation c_S of $\mathbb{E}_S$ indexed by schemes S of $\mathscr{S}$, and stable by pullbacks (Def. 2.1.2). As seen above, such a collection of orientations gives rise to a collection F of formal group laws F_S on the ring $\mathbb{E}^{**}(S)$, which will simply be called the formal group law associated with $(\mathbb{E}, c)$ – or just c when $\mathbb{E}$ is clear.

5.1. Principle of computation.

Definition 5.1.1. Let $(\mathbb{E}, c)$ and $(\mathbb{F}, d)$ be oriented spectra.

A *pseudo-morphism* of oriented spectra $\varphi : (\mathbb{E}, c) \to (\mathbb{F}, d)$ is simply a morphism of ring spectra. We will say φ is a *morphism* of oriented spectra if for any scheme S, one has:

$$\varphi_{\mathbb{P}_S^\infty}(c) = d$$

in $\tilde{\mathbb{F}}^{2,1}(\mathbb{P}_S^\infty)$, the reduced cohomology of the pointed scheme $\mathbb{P}_S^\infty$ pointed by the infinite point of $\mathbb{P}_S^1$.

We will also say that a pseudo-morphism φ is *identical* if, as a morphism of ring spectra, it is the identity.

Note that one immediately deduces from the construction of Chern classes and from the Riemann-Roch formulas 4.2.3 and 4.3.2 the following result:

Proposition 5.1.2. *Let $\varphi : (\mathbb{E}, c) \to (\mathbb{F}, d)$ be a morphism of oriented spectra.*

(1) *For any vector bundle E/S and any integer $n \geq 0$, one has: $\varphi_S(c_n(E)) = d_n(E)$.*

(2) *For any projective lci morphism $f : Y \to X$ one has: $\varphi_X \circ f_*^c = f_*^d \circ \varphi_Y$.*

(3) *For any regular closed immersion $i : Z \to X$, $U = X - Z$, one has: $\varphi_Z \circ \partial_{X,Z}^c = \partial_{X,Z}^d \circ \varphi_U$.*

Example 5.1.3. Let $(\mathbb{E}, c)$ be an oriented spectrum. According to Proposition 2.2.6, the choice of c uniquely corresponds to a morphism of oriented spectra:

$$\varphi : \left(\mathbf{MGL}, c^{\mathbf{MGL}} \right) \to (\mathbb{E}, c)$$

where $c^{\mathbf{MGL}}$ is the canonical orientation of $\mathbf{MGL}$.

Then, according to the preceding proposition, the morphism φ is compatible with all the constructions of orientation theory given in this paper.

5.1.4. Given any pseudo-morphism as in the above definition, we obviously get that $c' = \varphi_{\mathbb{P}_S^\infty}(c)$ is an orientation of $\mathbb{F}$. In particular, the pseudo-morphism φ admits a canonical factorization:

(5.1.4.a) $(\mathbb{E}, c) \xrightarrow{\tilde{\varphi}} (\mathbb{F}, c') \xrightarrow{\psi} (\mathbb{F}, c)$

such that $\tilde{\varphi}$ (resp. ψ) is a morphism (resp. identical pseudo-morphism) of oriented spectra.

According to Definition 4.1.4, the Todd class of φ is equal to the Todd class of ψ. Thus the computation of the Todd class of a pseudo-morphism can always be reduced to the case of an identical pseudo-morphism of the aim, which corresponds to the effect of *changing the orientation*. We will describe this case in more details below (see in particular 5.2.1).

Remark 5.1.5. According to Proposition 2.2.6, oriented (ring) spectra correspond to $\mathbf{MGL}$-algebras. In the light of this analogy, pseudo-morphisms (resp. morphisms) of oriented spectra corresponds to morphisms of rings (resp. $\mathbf{MGL}$-algebras) between $\mathbf{MGL}$-algebras. In light of this analogy, the preceding factorization can be understood as follows: $\tilde{\varphi}$ corresponds to a morphism of $\mathbf{MGL}$-algebras while ψ corresponds to a change of the $\mathbf{MGL}$-algebra structure on $\mathbb{F}$.

5.1.6. Recall that given a ring R and formal group laws F, G with coefficients in R, a morphism $\Phi : (R, F) \to (R, G)$ of formal group laws is a power series $\Phi(t) \in R[[t]]$ of positive valuation such that

$$\Phi(F(x, y)) = G(\Phi(x), \Phi(y))$$

in $R[[x, y]]$. Such a morphism is an isomorphism if and only if Φ admits a composition inverse – equivalently, $\Phi'(0)$ is invertible in R. It is called a *strict* isomorphism if $\Phi'(0) = 1$.

According to the conventions of this section, a morphism of formal group laws arising from orientations of an (absolute) ring spectrum will be a family of morphisms indexed by schemes of $\mathscr{S}$ and stable by pullbacks.

Proposition 5.1.7. *Let $\mathbb{E}$ be a ring spectrum and c be an orientation of $\mathbb{E}$ with associated formal group law F_c. Consider the following sets:*

(1) *the orientations d of $\mathbb{E}$ (equivalently: the identical pseudo-morphisms of ring spectra with source $(\mathbb{E}, c)$);*

(2) *the strict isomorphisms Φ of formal group laws with source F_c such that for any scheme S, $\Phi_S(t)$ can be written as a power series of the form:*

$$t + \sum_{i>1} \alpha_i^S.t^i$$

where α_i^S is an element of $\mathbb{E}^{2-2i,1-i}(S)$.

Then the map

$$(2) \xrightarrow{(*)} (1), \Phi \mapsto \Phi(c)$$

is a well defined bijection.

Proof. We prove the map $(*)$ is well defined. First note that, because of the condition on the degree of the coefficients of Φ_S, the cohomology class $\Phi_S(c)$ is of degree $(2,1)$. The fact it is an orientation of $\mathbb{E}_S$ simply follows from the form of $\Phi_S(t)$ which implies: $\Phi_S(0) = 0$ and $\Phi'_S(0) = 1$.

To prove that $(*)$ is a bijection, we construct its inverse. Let d be an orientation of $\mathbb{E}$ and S be a base scheme. Then d_S is an element of the bigraded algebra

$$\mathbb{E}^{**}(\mathbb{P}_S^\infty) \simeq \mathbb{E}^{**}(S)[[c]].$$

This means there is a unique power series

$$\Phi_S(t) = \sum_{i \geq 0} \alpha_i^S.t^i$$

with coefficients in $\mathbb{E}^{**}(S)$ such that $d_S = \Phi_S(c_S)$. Because d_S has bidegree $(2,1)$, we deduce that necessarily, α_i^S has bidegree $(2-2i, 1-i)$. Moreover, because d_S is an orientation of $\mathbb{E}_S$, we deduce that $\alpha_0^S = 0$ and $\alpha_1^S = 1$. By uniqueness of Φ_S, we deduce that the family $\Phi = (\Phi_S)_{S \in \mathscr{S}}$ is stable by pullbacks.

Let us consider the Segre embedding (Par. 2.1.19):

$$\sigma : \mathbb{P}_S^\infty \to \mathbb{P}_S^\infty \times_S \mathbb{P}_S^\infty.$$

By definition of the formal group law F_c (resp. F_d) associated with c (resp. d) we obtain (dropping the reference to the base S):

$$\sigma^*(c) = F_c(c', c''), \text{ resp. } \sigma^*(d) = F_d(d', d'').$$

where, on the left hand side c', c'' (resp. d', d'') corresponds to the first Chern class associated with the orientation c (resp. d) of the canonical line bundle on the first (resp. second) factor of $\mathbb{P}_S^\infty \times_S \mathbb{P}_S^\infty$. Thus, we obtain that Φ_S is a strict isomorphism from F_c to F_d:

$$\Phi(F_c(c', c'')) = \Phi(\sigma^*(c)) = \sigma^*(\Phi(c)) = \sigma^*(d) = F_d(d', d'') = F_d(\Phi(c'), \Phi(c'')).$$

Therefore $\Phi_S(t)$ is a strict isomorphism $(\mathbb{E}^{**}(S), F) \to (\mathbb{E}^{**}(S), F_d)$ of formal group laws.

The uniqueness of Φ_S shows that we have indeed constructed an inverse map to
$(*)$. □

Remark 5.1.8. This proposition is essentially an elaboration of the construction
done in Paragraph 4.1.1. Note to be more precise that, given an identical pseudo-
morphism $\varphi : (\mathbb{E}, c) \to (\mathbb{E}, d)$, the power series $\Phi(t)$ obtained by applying the
previous proposition is the composition inverse of the power series $\Psi(t)$ obtained
in 4.1.1. Thus, in fact, we get reciprocal strict isomorphisms of formal group laws
on $\mathbb{E}^{**}$:

$$(5.1.8.a) \qquad\qquad\qquad \Phi : F_c \leftrightarrows F_d : \Psi$$

From a categorical perspective, having fixed an oriented spectrum $(\mathbb{E}, c)$, the propo-
sition defines a covariant functor $(*)$ from the category of orientations on $\mathbb{E}$ with
morphisms the identical pseudo-morphisms to the category of ($\mathscr{S}$-families) of for-
mals group laws on the bigraded ring $\mathbb{E}^{**}$.

The functor $(*)$ induces a bijection of groupoids when one restricts morphisms on
the target category to strict isomorphisms satisfying the condition on the degrees
of point (2) above.

Remark 5.1.9. According to this proposition, two orientations on a given (orientable
ring) spectrum $\mathbb{E}$ necessarily yields (strictly) isomorphic formal group laws. Thus,
there is a uniquely defined isomorphism class of formal group law associated with
an orientable ring spectrum $\mathbb{E}$ and we can safely qualify such a ring spectrum as
being additive, multiplicative, etc. Note that it is not known whether any formal
group law can be realized as the formal group law associated with an orientable
ring spectrum. However, interesting new examples are provided in [LYZ13, Th. A].

Corollary 5.1.10. *Let $\mathbb{E}$ be an absolute ring spectrum satisfying the following
property:*

(Ann) *For any integer $i < 0$, and any scheme S, $\mathbb{E}^{2i,i}(S) = 0$.*

Then the following assertions hold:

(i) *If an orientation exists on $\mathbb{E}$ over a scheme S, it is unique. Moreover, if $\mathbb{E}$
   is rational,[47] then the formal group law associated with an orientation on
   $\mathbb{E}$ is necessarily additive.*

(ii) *Assume $\mathbb{E}$ is oriented with orientation c and $(\mathbb{F}, d)$ is an oriented spectrum.
   Then, any morphism of ring spectra $\varphi : \mathbb{F} \to \mathbb{E}$ automatically satisfies
   $\varphi(d) = c$.*

Example 5.1.11. Assumption (Ann) is most common in algebraic geometry. It is
fulfilled in any of the examples (1), (2), (5) of 2.1.4.

Over an algebraically closed field $S = \mathrm{Spec}(k)$, or more generally any scheme
S such that -1 is a sum of squares in all residue fields, we get from a theorem
of Morel (cf [CD12b, 16.2.14]) that any S-absolute rational ring spectrum satis-
fying assumption (Ann) is uniquely oriented and the corresponding orientation is
additive.

---

[47]*i.e.* $\mathbb{E}$ is isomorphic to its rationalization.

5.2. Change of orientation.

5.2.1. Let $\mathbb{E}$ be a ring spectrum equipped with two orientations c, d whose respective associated formal group laws are F_c and F_d. This is pictured by the following identical pseudo-morphism of oriented ring spectra:

$$\varphi : (\mathbb{E}, c) \to (\mathbb{E}, d).$$

Let us consider the reciprocal isomorphism of formal group laws (5.1.8.a) uniquely associated with φ in Proposition 5.1.7. Recall the convention: $d = \Phi(c)$, $c = \Psi(d)$.

Then, from Proposition 4.1.2, we get two expressions of the Todd class of a line bundle L/S associated with φ:

$$\mathrm{Td}_\varphi(L) = \left.\frac{t}{\Psi_S(t)}\right|_{t=d_1(L)} = \left.\frac{\Phi_S(t)}{t}\right|_{t=c_1(L)}.$$

Another way of saying this is the relation:

$$d_1(L) = \mathrm{Td}_\varphi(L).c_1(L).$$

5.2.2. In general, it is not easy to compute the strict isomorphism associated with a change of orientations on a ring spectrum. However, let us recall that for any $\mathbb{Q}$-algebra A, given any formal group law F with coefficients in A, there exists a unique strict isomorphism between F and the additive formal group law, called the *logarithm of F*:

$$\log_F : F \to F_{add}.$$

There exists a well known formula for the logarithm. First, one attach to F a formal differential form

$$\omega_F(x) = \left(\left.\frac{\partial F(x,y)}{y}\right|_{y=0}\right)^{-1}.dx$$

and then one defines the logarithm as the primitive of this differential form:

(5.2.2.a)
$$\log_F(t) = \int \omega_F(x).$$

The composition inverse for this power series is usually called the *exponential of F* and denoted by:

$$\exp_F : F_{add} \to F.$$

As a consequence, we get the well-known fact that any two formal group laws with coefficients in A are uniquely strictly isomorphic.

This gives the following formula for computing the Todd class associated with a change of orientations. One determines the formal group laws F_c and F_d associated with c and d. Then the strict isomorphisms $\Phi : F_c \leftrightarrows F_d : \Psi$ are given by the power series:

$$\Phi(t) = \exp_{F_d} \circ \log_{F_c}(t), \Psi(t) = \exp_{F_c} \circ \log_{F_d}(t),$$

Example 5.2.3. It is easy to compute the Todd class using the splitting principle. Let us consider the abstract case of a rational oriented ring spectrum $(\mathbb{E}, c)$ with an abstract formal group law

$$F(x,y) = \sum_{i,j} a_{ij} x^i y^j,$$

and assume $\bar{c} = \log_F(c)$ is the canonical orientation corresponding to the additive formal group law (cf. the preceding examples).

It is convenient to denote by $p_i = [\mathbb{P}^i]$ the cobordism class of the projective space of dimension i – whose expression in terms of the coefficients $a_{1,i}$ can be found in Example 3.2.14. Indeed, according to the Myschenko formula, one gets:

$$\log_F(t) = \sum_{i=0}^{\infty} \frac{p_i}{i+1}.t^{i+1}.$$

Then, associated with the identical pseudo-morphism $(\mathbb{E}, c) \to (\mathbb{E}, \bar{c})$, the Todd class of an arbitrary vector bundle E/S of rank n can be expressed as a polynomial of the p_i and of the Chern classes $\bar{c}_i := \bar{c}_i(E)$ for $1 \le i \le n$. The first few terms are as follows:

$$\mathrm{Td}(E) = 1 + \left(\frac{1}{2}p_1\right)\bar{c}_1 + \left(\frac{3}{4}p_1^2 - \frac{2}{3}p_2\right)\bar{c}_2$$

$$+ \left(-\frac{1}{4}p_1^2 + \frac{1}{3}p_2\right)\bar{c}_1^2 + \left(-\frac{7}{8}p_1^3 + \frac{5}{3}p_1p_2 - \frac{3}{4}p_3\right)\bar{c}_1\bar{c}_2$$

$$+ \left(-\frac{13}{16}p_1^4 + 2p_1^2p_2 - \frac{5}{4}p_1p_3 - \frac{1}{3}p_2^2 + \frac{2}{5}p_4\right)\bar{c}_2^2$$

$$+ \left(\frac{1}{4}p_1^3 - \frac{1}{2}p_1p_2 + \frac{1}{4}p_3\right)\bar{c}_1^3 + \left(\frac{11}{8}p_1^4 - \frac{43}{12}p_1^2p_2 + \frac{17}{8}p_1p_3 + \frac{8}{9}p_2^2 - \frac{4}{5}p_4\right)\bar{c}_1^2\bar{c}_2 + \cdots$$

Note we can also give an expression of the Todd class in function of the original Chern classes $c_i = c_i(E)$:

$$\mathrm{Td}(E) = 1 + \left(\frac{1}{2}p_1\right)c_1 + \left(\frac{1}{4}p_1^2 - \frac{2}{3}p_2\right)c_2$$

$$+ \left(\frac{1}{3}p_2\right)c_1^2 + \left(\frac{1}{6}p_1p_2 - \frac{3}{4}p_3\right)c_1c_2 + \left(-\frac{1}{4}p_1p_3 + \frac{1}{9}p_2^2 + \frac{2}{5}p_4\right)c_2^2$$

$$+ \left(\frac{1}{4}p_3\right)c_1^3 + \left(\frac{1}{8}p_1p_3 - \frac{4}{5}p_4\right)c_1^2c_2 + \left(-\frac{3}{10}p_1p_4 + \frac{1}{12}p_2p_3\right)c_1c_2^2$$

$$+ \left(-\frac{2}{15}p_2p_4 + \frac{1}{16}p_3^2\right)c_2^3 + \cdots$$

As an illustration of the Riemann-Roch formula in the case of a change of orientations, we give the following simple proof of a formula due to Quillen (in complex cobordism, [Qui69]):

Theorem 5.2.4. *Let E be a vector bundle of rank $n+1$ over a scheme S, $P = \mathbb{P}(E)$ be the associated projective bundle, $p : P \to S$ be the canonical projection and $\lambda = \mathcal{O}(1)$ be the canonical dual line bundle on P.*

Put $A = \mathbf{MGL}^{**}(S)$ and let c be the canonical orientation of $\mathbf{MGL}$ over S. Let $F(x, y) \in A[[x, y]]$ be the formal group law associated with $(\mathbf{MGL}, c)$.

Then for any polynomial $\phi(t) \in A[t]$, the following formula holds in $A \otimes_{\mathbb{Z}} \mathbb{Q}$:

$$p_*\big(\phi(c_1(\lambda))\big) = \mathrm{Res}_t\left(\frac{\phi(t).\omega_F(t)}{\prod_i F(t, l_i)}\right)$$

where l_i are the Chern roots of E with respect to the orientation c.

Proof. [D., Levine, Vishik][48] Let $\log_F(t)$ (resp. $\exp_F(t)$) be the logarithm (resp. exponential) associated with the formal group law F – see above. Then the class

---

[48]I thank a lot M. Levine and A. Vishik for helping me to finish this proof.

$\bar{c} = \log_F(c)$ is an orientation of $\mathbf{MGL}_{\mathbb{Q}}$ whose formal group law is additive. Let us denote by $\bar{p}_*$ the Gysin morphism associated with p with respect to $\bar{c}$.

Then the Riemann-Roch formula for the identical pseudo-morphism

$$(\mathbf{MGL}_{\mathbb{Q}}, c) \to (\mathbf{MGL}_{\mathbb{Q}}, \bar{c})$$

and the morphism p reads:

$$(5.2.4.a) \qquad p_*\big(\phi(d)\big) = \bar{p}_*\big(P(d).\operatorname{Td}(T_p)\big)$$

where we have put $d = c_1(\lambda)$ and T_p denotes the tangent bundle of P/S. The following lemma is a reformulation of a well known formula in the (classical) theory of Chern classes:

Lemma 5.2.5. *Let $(\mathbb{E}, \bar{c})$ be an additive oriented ring spectrum over S. Consider the total Chern class of E:*

$$\bar{c}_t(E) = \sum_{i \geq 0} \bar{c}_i(E).t^i.$$

as an invertible power series with coefficients in $\mathbb{E}^{**}(S)$. Put $\bar{d} = \bar{c}_1(\lambda)$.

Then for any power-series $\psi(t)$ with coefficients in $\mathbb{E}^{**}(S)$, one has the following equality in $\mathbb{E}^{**}(S)$:

$$\bar{p}_*(\psi(\bar{d})) = \operatorname{Res}_t \left(\frac{\psi(t).dt}{t^{n+1}.\bar{c}_{t^{-1}}(E)} \right)$$

where Res_t stands for the residue of the indicated Laurent power-series.[49]

Let us write $\psi(t) = \sum_i \alpha_i.t^i$ and $\bar{c}_t(E)^{-1} = \sum_{i \geq 0} \bar{c}_i(-E).t^i$. Recall that for any element $\alpha \in \mathbb{E}^{**}(P)$, $\bar{p}_*(\alpha)$ is the coefficient of $\bar{d}^n$ in the decomposition of α within the A-basis $(\bar{d}^i)_{0 \leq i \leq n}$ of $\mathbb{E}^{**}(P)$. A classical computation according to the defining relation of Chern classes (2.1.16.a) in the additive case gives us:

$$\bar{p}_*(\bar{d}^i) = \bar{c}_{i-n}(-E).$$

In particular, one obtains:

$$\bar{p}_* \left(\sum_i \alpha_i.\bar{d}^i \right) = \sum_i \alpha_i.\bar{c}_{i-n}(-E).$$

To end the proof of the lemma, one has only to realize that the right hand side is the coefficient of t^n in the following Laurent power-series:

$$\left(\sum_i \alpha_i.t^i \right).c_{t^{-1}}(-E).$$

Let us now compute the right hand side of (5.2.4.a). From the exact sequence of vector bundles over P:

$$0 \to \mathcal{O}_P \to \lambda \otimes p^{-1}(E) \to T_p \to 0$$

we get $\operatorname{Td}(T_p) = \operatorname{Td}(\lambda \otimes p^{-1}(E))$. By assumption, the classes $\bar{l}_i = \log_F(l_i)$ are the Chern roots of $p^{-1}(E)$ with respect to the additive orientation $\bar{c}$. Thus, by

---

[49]The formula inside Res_t is a Laurent power series because the Chern classes of E are nilpotent (2.1.22).

definition of the Todd class (see also the formula of 5.2.1 with $\Psi_S = \exp_F$), one gets:

$$\mathrm{Td}(\lambda \otimes p^{-1}(E)) = \prod_{i \in I} \frac{t}{\exp_F(t)} \left(t = \bar{d} + \bar{l}_i \right).$$

Note that because $\bar{l}_i$ are Chern roots of E with respect to $\bar{c}$, one has:

$$\prod_{i \in I} (\bar{d} + \bar{l}_i) = \sum_i \bar{c}_{n+1-i}(E).\bar{d}^i = \left. \left(t^{n+1}.c_{t^{-1}}(E) \right) \right|_{t=\bar{d}}.$$

Because $F(x,y) = \exp_F(\log_F(x) + \log_F(y))$, one also has:

$$\exp_F(\bar{d} + \bar{l}_i) = F(d, l_i) = F\left(\exp_F(\bar{d}), l_i \right).$$

Thus to compute the right hand side of (5.2.4.a) using the formula of the preceding lemma, we are led to introduce the following power series:

$$\psi(t) = \frac{\phi(\exp_F(t)).t^{n+1}.c_{t^{-1}}(E)}{\prod_{i \in I} F\left(\exp_F(t), l_i \right)}.$$

Applying the preceding lemma:

$$\bar{p}_*\left(\phi(d).\,\mathrm{Td}(T_p) \right) = \bar{p}_*(\psi(\bar{d})) = \mathrm{Res}_t \left(\frac{\phi(\exp_F(t)).dt}{\prod_{i \in I} F\left(\exp_F(t), l_i \right)} \right).$$

Computing this residue with the change of variables $x = \exp_F(t)$, one gets the desired result: by (5.2.2.a), $d\log_F(x) = \omega_F(x)$. $\qquad\square$

Remark 5.2.6.   (1) In view of Remark 5.1.3, the preceding formula is universal: It is valid without any change for any S-absolute oriented ring spectrum $\mathbb{E}$ with associated formal group law F. Anyway, it is clear in the above proof that one can faithfully replace **MGL** by $\mathbb{E}$.

(2) The above proof is particularly simple but it works only with rational coefficients whereas Quillen formula is stated in [Qui69] for complex cobordism with integral coefficients. However, one can at least deduce from this proof the formula with integral coefficients in characteristic 0: indeed, in this case, one reduces to $S = \mathrm{Spec}(\mathbb{Q})$ and we know from [Lev09] that $A = \mathbf{MGL}^{**}(\mathbb{Q})$ is the Lazard ring and thus has no torsion. One gets back the usual Quillen formula by using the complex realization functor.

(3) As a particular case of the preceding formula, one gets the classical Myschenko formula computing the cobordism class of $\mathbb{P}_S^n$ for any integer n: take $E = \mathbb{A}_S^{n+1}$ and $P(t) = 1$. May be it is worth to summarize the proof in this case:

$$p_*(1) = \bar{p}_*(\mathrm{Td}(T_{\mathbb{P}^n})) = \bar{p}_*(\mathrm{Td}(\lambda \otimes \mathbb{A}^{n+1}))$$

$$= \bar{p}_* \left(\left(\frac{\bar{d}}{\exp \bar{d}} \right)^{n+1} \right) = \mathrm{Res}_t \left(\frac{dt}{\exp_F(t)^{n+1}} \right) = \mathrm{Res}_x \left(\frac{\omega_F(x)}{x^{n+1}} \right).$$

Note also that this computation, including the change of variable $x = \exp_F(t)$, was used by Borel and Serre in [BS58] to prove the classical Grothendieck-Riemann-Roch formula for $p : \mathbb{P}_S^n \to S$ – replacing **MGL** by **KGL**: as explained below (5.3.3), the Chern character corresponds to changing the natural orientation on **KGL** to the additive orientation.

(4) The general formula of Quillen has been proved integrally in the context of oriented cohomology theories in [Shi07] and [Vis07, App. B]. The proofs given here are equally valid in our more general context.

Example 5.2.7. An interesting particular case of Quillen formula is the following computation of the cobordism class of a projective bundle P/S associated with a vector bundle E/S:

$$[P/S] = \mathrm{Res}_t \left(\frac{\omega_F(t)}{\prod_i F(t, l_i)} \right)$$

where l_i are the Chern roots of E.

Example 5.2.8. We end-up this series of illustration of the (generalized) Riemann-Roch formula by explaining how one can also compare the Chern classes arising from different orientations.

Let us consider again the general setting of Paragraph 5.2.1. In general, given a vector bundle E/S of rank n, we can use the splitting principle to compute the two different type of Chern classes of E associated respectively with the orientations d and c. For any integer $0 \le r \le n$, one gets:

$$d_r(E) = d_r(\oplus_{i=1}^n L_i) = \sum_\alpha d_1(L_{\alpha_1})...d_1(L_{\alpha_r}) = \sum_\alpha \Phi(c_1(L_{\alpha_1}))...\Phi(c_1(L_{\alpha_r})),$$

where α runs over the r-uple of integers $(\alpha_1, \ldots, \alpha_r)$ such that

$$1 \le \alpha_1 < \alpha_2 < \ldots < \alpha_r \le n.$$

Once again, one has to express the right hand side in terms of the elementary symmetric functions in the $c_1(L_i)$ to obtain an expression in terms of $c_i(E)$.

In general, the formulas are pretty complicated. However, one can remark there is a simple formula when $r = n$. Then, one simply obtains:

$$d_n(E) = \mathrm{Td}(E).c_n(E),$$

generalizing the case of a line bundle (Par. 5.2.1).

Another case that can be computed is the case $r = 1$. Assume the strict isomorphism of formal group law Φ such that $d = \Phi(c)$ has been written as:

$$\Phi(x) = x + \sum_{i>1} \alpha_i.x^i.$$

Recall the following classical relation (determinantal form of the Newton's identity) between the power sum symmetric polynomials p_i and the elementary symmetric polynomials e_j:

$$p_i = \begin{vmatrix} e_1 & 1 & 0 & \cdots & 0 \\ 2e_2 & e_1 & & & \\ & e_2 & & & 0 \\ & & & & 1 \\ i.e_i & e_{i-1} & \cdots & e_2 & e_1 \end{vmatrix}$$

Using the splitting principle together with this relation, one obtains the following relation:

$$d_1(E) = c_1(E) + \sum_{i>1} \alpha_i. \begin{vmatrix} c_1(E) & 1 & 0 & \cdots & 0 \\ 2c_2(E) & c_1(E) & & & \\ & & c_2(E) & & 0 \\ & & & & 1 \\ i.c_i(E) & c_{i-1}(E) & \cdots & c_2(E) & c_1(E) \end{vmatrix}.$$

Note the sum above is finite because Chern classes are always nilpotent (cf. Prop. 2.1.22(1)).

Example 5.2.9. As a last illustration of the change of orientation principle, let us consider an arbitrary oriented ring spectrum $(\mathbb{E}, c)$. One defines a new orientation on E by putting for any line bundle L over a scheme S:

$$c_1'(L) = -c_1(L^\vee),$$

where $L^\vee$ denotes the dual of L/S. Indeed, if one denotes by $\mu(t)$ the formal inverse associated with the formal group law of $(\mathbb{E}, c)$, then the strict isomorphism corresponding to this change of orientation is the composition inverse of the power series $-\mu(t)$.

Using the splitting principle, one easily checks that for any vector bundle E/S, the following relation holds:

$$(5.2.9.a) \qquad\qquad c_i'(E) = (-1)^i.c_i(E^\vee).$$

Let λ be the canonical line bundle on $\mathbb{P}(E)$. Then, from this formula and the relation (2.1.16.a) defining Chern classes with respect to the orientation c', one gets the following formula[50] expressing Chern classes in terms of the dual canonical line bundle $\lambda^\vee$:

$$(5.2.9.b) \qquad\qquad \sum_{i=0}^{n} (-1)^i p^* \big(c_i(E^\vee)\big).c_1\big(\lambda^\vee\big)^{n-i} = 0.$$

5.3. **Universal formulas and the Chern character.** In the next proposition, we work over a fixed base scheme S. Recall we have seen in Example 5.1.11 that there exists a unique orientation $c^{\mathbf{H}_\mathrm{B}}$ on the Beilinson motivic cohomology ring spectrum $\mathbf{H}_{\mathrm{B},S}$ whose formal group law is necessarily additive. Using a fundamental result of [CD12a], we get slightly more:

Proposition 5.3.1. *Let $(\mathbb{E}, c)$ be an additive oriented ring spectrum with rational coefficients over S.*

Then there exists a unique morphism of absolute ring spectra

$$\sigma : \mathbf{H}_{\mathrm{B},S} \to \mathbb{E}$$

which is moreover a morphism of oriented ring spectra: $\sigma(c^{\mathbf{H}_\mathrm{B}}) = c$.

In particular, the morphism σ is compatible with Gysin morphisms, residues, Chern classes and fundamental classes as constructed in the preceding sections.

---

[50]This formula was suggested to me by Alberto Navarro.

Proof. The existence and the uniqueness of σ follow from [CD12b, 14.2.16]. The fact σ is necessarily a morphism of oriented ring spectra follows from Proposition 5.1.7 and the fact that the unique strict automorphism of a formal group law on a rational ring is the identity (fact recalled in Paragraph 5.2.2). The last assertion is then an application of the Riemann-Roch formula (or its extension for Chern classes and fundamental classes) to σ. $\square$

Remark 5.3.2. Another way of stating the previous proposition is that $\mathbf{H}_{\mathrm{B}}$ is the universal absolute *orientable* rational ring spectrum, – whereas $\mathbf{MGL}$ is the universal absolute *oriented* ring spectrum.

Thus, the proposition (as well as its integral counterpart stated in a few paragraphs) answers a *desideratum* raised in the introduction (cf. second paragraph) of Beilinson's fundamental work [Beĭ84].

The morphism σ could be called the *higher cycle class* morphism: in degree $(2n, n)$, it gives a morphism from the Chow group of n-codimensional cycles.[51] Over a field and in the integral case (see below), this fits well with the fact motivic cohomology (of regular schemes) is given by Bloch's higher Chow groups.

5.3.3. Recall that the decomposition of algebraic K-theory according to the eigenvalues of the Adams operation has been lifted by J. Riou to the stable homotopy category of schemes resulting in a canonical isomorphism of ring spectra:

$$\mathrm{ch}_t : \mathbf{KGL}_{\mathbb{Q}} \to \oplus_{i \in \mathbb{Z}} \mathbf{H}_{\mathrm{B}}(i)[2i].$$

This is essentially the results of [Rio10] as explained in [CD12b], Lemma 14.1.4 for the existence of this isomorphism and Corollary 14.2.17 for the fact it is a morphism of ring spectra. By definition, for any regular scheme X and any integer n, this isomorphism induces the canonical decomposition

$$K_n(X)_{\mathbb{Q}} \to \oplus_{i \geq 0} \mathrm{Gr}_{\gamma}^i K_n(X)_{\mathbb{Q}}$$

for the γ-filtration on Quillen K-theory with rational coefficients (cf. [Sou85]). According to [Sou85, §7], the projector on the i-th graded part of this decomposition is given by the i-th Chern character. Thus the morphism of ring spectra ch_t lifts to the category of spectra the usual Chern character in higher K-theory with values in Beilinson motivic cohomology (see again [Sou85]): for any pair $(n, r) \in \mathbb{N}^2$ and any regular scheme, one gets the usual (higher) Chern character:

$$\mathrm{ch}_{r,n} : K_r(X) \to \mathbf{H}_{\mathrm{B}}^{2n-r,n}(X).$$

For $r = 0$, this coincide withs the Chern character of [SGA6]. In particular, it is uniquely characterized by its value on the class of a line bundle L/X:

$$\mathrm{ch}_{0,n}\left([L]\right) = \frac{1}{n!} c_1(L)^n.$$

Using the principle explained in paragraphs 5.1.4 and 5.2.1, we can easily determine the Todd class associated with the morphism of spectra ch_t. Indeed, the formal group law associated with $c^{\mathbf{KGL}}$ is $F_{\mathbf{KGL}}(x, y) = x + y - \beta.xy$ (Ex. 2.1.21). By definition of β, we get that $\mathrm{ch}_t(\beta) = 1$. We deduce that the formal group law associated with the orientation $\mathrm{ch}_t(c^{\mathbf{KGL}})$ on the aim is the following multiplicative formal group law $F_{mult}(x, y) = x + y - xy$. On the other hand $c_{\mathrm{B}}^{\mathbf{H}}$ is additive, thus the pair of reciprocal strict isomorphisms associated with ch_t is

---

[51]Note that, even when $\mathbb{E}$ is the spectrum representing Deligne-Beilinson cohomology (say over $\mathbb{Q}$, see [AH10]), the map σ is not exactly the *regulator map* as described in [Sou86, §3.3].

the logarithm/exponential of the formal group law F_{mult}. Formula (5.2.2.a) easily yields:

$$\Phi(t) = \log_{F_{mult}}(t) = -\ln(1-t),$$
$$\Psi(t) = \exp_{F_{mult}}(t) = 1 - e^{-t}.$$

Thus, from 5.2.1, we obtain that the Todd class associated with ch_t is defined by the power series $\frac{t}{1-e^{-t}}$ (Prop. 4.1.2). In particular, it coincides exactly with the usual Todd class

$$\mathrm{Td} : K_0 \to \mathrm{Gr}_\gamma^* K_0(X)_\mathbb{Q} \simeq \mathbf{H}_\mathrm{B}^{2*,*}(X)$$

of [SGA6].

In the end, taking care about the isomorphisms (2.1.5.a), Theorem 4.3.2 yields the following higher arithmetic Grothendieck-Riemann-Roch formula:

Proposition 5.3.4. *Consider the preceding notations. Let $f : Y \to X$ be a projective morphism between regular schemes and $\tau_f \in K_0(Y)$ be its virtual tangent bundle (see 4.3.1).*

Then for any integer $r \geq 0$, the following diagram is commutative:

$$
\begin{array}{ccc}
K_r(Y)_\mathbb{Q} & \xrightarrow{\quad f_* \quad} & K_r(X)_\mathbb{Q} \\
{\scriptstyle \mathrm{Td}(\tau_f).\,\mathrm{ch}_t} \downarrow & & \downarrow {\scriptstyle \mathrm{ch}_t} \\
\oplus_{n \geq 0} \mathbf{H}_\mathrm{B}^{2n-r,n}(Y) & \xrightarrow{\quad f_* \quad} & \oplus_{n \geq 0} \mathbf{H}_\mathrm{B}^{2n-r,n}(X).
\end{array}
$$

In other words, for any integer $n \geq 0$ and any element $y \in K_r(Y)$, one has:

$$\mathrm{ch}_{r,n}\left(f_*(y)\right) = f_*\left(\sum_{i+j=n} \mathrm{Td}_i(\tau_f).\,\mathrm{ch}_{r,j}(y)\right).$$

Remark 5.3.5. Taking into account Proposition 5.3.1, the preceding formula is universal. Indeed, given any additive oriented rational ring spectrum $\mathbb{E}$, the Chern character and the Todd class with values in $\mathbb{E}$ are induced by those with values in Beilinson motivic cohomology. Then we get the following commutative diagram:

$$
\begin{array}{ccccc}
K_r(Y)_\mathbb{Q} & \xrightarrow{\mathrm{Td}(\tau_f).\,\mathrm{ch}_t} & \oplus_{n \geq 0} \mathbf{H}_\mathrm{B}^{2n-r,n}(Y) & \xrightarrow{\sum \sigma_Y^{2n-r,n}} & \oplus_{n \geq 0} \mathbb{E}^{2n-r,n}(Y) \\
{\scriptstyle f_*} \downarrow & & \downarrow {\scriptstyle f_*} & & \downarrow {\scriptstyle f_*} \\
K_r(X)_\mathbb{Q} & \xrightarrow{\mathrm{ch}_t} & \oplus_{n \geq 0} \mathbf{H}_\mathrm{B}^{2n-r,n}(X) & \xrightarrow{\sum \sigma_X^{2n-r,n}} & \oplus_{n \geq 0} \mathbb{E}^{2n-r,n}(X),
\end{array}
$$

Note that the map ch_t exists over any scheme. On the other hand, it is known from [Cis13] that the spectrum $\mathbf{KGL}_\mathbb{Q}$ represents Weibel homotopy invariant K-theory KH_*. Therefore, because $\mathbf{KGL}_\mathbb{Q}$ and $\mathbf{H}_\mathrm{B}$ are $\mathscr{S}m_S$-absolutely pure for any scheme S, the preceding proposition admits the following version, which is finer when S is singular:

Proposition 5.3.6. *Let S be any scheme and $f : Y \to X$ be a projective morphism between smooth S-schemes with virtual tangent bundle $\tau_f \in K_0(Y)$.*

Then for any integer $r \geq 0$, the following diagram is commutative:

$$
\begin{array}{ccc}
KH_r(Y)_{\mathbb{Q}} & \xrightarrow{\ f_*\ } & KH_r(X)_{\mathbb{Q}} \\
{\scriptstyle \mathrm{Td}(\tau_f).\,\mathrm{ch}_t} \downarrow & & \downarrow {\scriptstyle \mathrm{ch}_t} \\
\oplus_{n\geq 0}\mathbf{H}_{\mathcal{B}}^{2n-r,n}(Y) & \xrightarrow{\ f_*\ } & \oplus_{n\geq 0}\mathbf{H}_{\mathcal{B}}^{2n-r,n}(X).
\end{array}
$$

Remark 5.3.7. The morphism f_* in the above proposition stands *a priori* for the Gysin morphisms we have defined. However, recall from [TT90, 3.16.4], that K-theory is covariant with respect to proper maps of finite Tor-dimension, thus a fortiori with respect to the morphism f in the above proposition. The definition of KH_* makes it clear that this covariant functoriality extends to KH_* in such a way that the following diagram commutes:

$$
\begin{array}{ccc}
K_r(Y) & \xrightarrow{\ f_*\ } & K_r(X) \\
\downarrow & & \downarrow \\
KH_r(Y) & \xrightarrow{\ f_*\ } & KH_r(X).
\end{array}
$$

Applying Theorem 3.3.1 as in Example 3.3.4(3), we get that this last morphism coincides with our Gysin morphism. Therefore, because of the preceding commutative square, the above Riemann-Roch formula is even true when replacing KH_* by Thomason-Trobaugh algebraic K-theory.[52]

5.3.8. Finally, we show how to get an almost integral (weaker) version of Proposition 5.3.1 using the recent Hopkins-Morel-Hoyois theorem (cf. [Hoy15]). Let us restate this theorem for the sake of notations. Consider a field k of exponential characteristic p. The canonical orientation on $\mathbf{MGL}$ induces a formal group law on its ring of coefficients, say over k. By universality of the Lazard ring, we get a canonical morphism of ring:

$$
L \longrightarrow \mathbf{MGL}_{**} := \mathbf{MGL}^{-*,-*}(\mathrm{Spec}(k))
$$

which in fact induces the following isomorphism (cf. [Hoy15, 8.2]) of graded rings:

$$(5.3.8.\text{a}) \qquad\qquad L[1/p] \longrightarrow \mathbf{MGL}_{(2,1)*}[1/p]$$

where $\mathbf{MGL}_{(2,1)*}$ denotes the elements of $\mathbf{MGL}_{**}$ of bidegree $(2n, n)$ for an integer $n \in \mathbb{Z}$. The Lazard ring is in fact a graded polynomial algebra $L = \mathbb{Z}[b_1, b_2, \dots]$ where b_i has degree i. In particular, any element b_i corresponds to an element of $\mathbf{MGL}_{2i,i}$. Following [Hoy15], paragraph after Corollary 6.9, one can define the quotient spectrum

$$\mathbf{MGL}/\{b_i, i > 0\}$$

by killing the elements b_i. In fact, this amounts to kill the coefficients a_{ij} of the formal group law of $\mathbf{MGL}$ — which, after inverting p, corresponds to the universal formal group law on the Lazard ring through the isomorphism (5.3.8.a).

As the formal group law associated with the canonical orientation on the motivic Eilenberg Mac Lane spectrum $\mathbf{H}_{\mathbb{Z}}$ is additive, one gets a canonical map of spectra:

$$(5.3.8.\text{b}) \qquad\qquad \big(\mathbf{MGL}/\{b_i, i > 0\}\big) \to \mathbf{H}_{\mathbb{Z}}$$

which becomes an isomorphism after inverting p (cf. [Hoy15, 7.12]). As a corollary, we get:

---

[52]Recall it coincides with Quillen's definition whenever X admits an ample line bundle.

Proposition 5.3.9. *Let k be a field of exponential characteristic p and $\mathbb{E}$ be a $\mathbb{Z}[1/p]$-linear k-absolute ring spectrum.*

Then the following conditions are equivalent:

- *$\mathbb{E}$ admits an additive orientation c.*
- *There exists a morphism of absolute ring k-spectra:*

$$\sigma : \mathbf{H}_{\mathbb{Z}} \to \mathbb{E}.$$

Moreover, when these conditions are fulfilled, the additive orientation c on $\mathbb{E}$ is unique and is the image under σ of the canonical orientation on $\mathbf{H}_{\mathbb{Z}}$. In particular, σ is compatible with Gysin morphisms, residues, Chern classes and fundamental classes as constructed in the preceding sections.

Proof. The fact (i) implies (ii) is obvious: the orientation of $\mathbf{H}_{\mathbb{Z}}$ induces an orientation of $\mathbb{E}$, which is necessarily additive.

Reciprocally. According to Prop. 2.2.6, the orientation c of $\mathbb{E}$ corresponds to a morphism of ring spectra over $\mathrm{Spec}(k)$

$$\varphi : \mathbf{MGL} \to \mathbb{E}.$$

This map induces a morphism of formal group law. Moreover, according to the commutative diagram:

$$
\begin{array}{ccc}
\mathbf{MGL}^{**}(\mathbb{P}^\infty) & \longrightarrow & \mathbf{MGL}^{**}(\mathbb{P}^\infty \times \mathbb{P}^\infty) \\
{\scriptstyle \varphi_*} \downarrow & & \downarrow {\scriptstyle \varphi_*} \\
\mathbb{E}^{**}(\mathbb{P}^\infty) & \longrightarrow & \mathbb{E}^{**}(\mathbb{P}^\infty \times \mathbb{P}^\infty),
\end{array}
$$

the definition of the formal group law associated with an oriented ring spectrum and the fact $(\mathbb{E}, c)$ is additive, we obtain that $\varphi_* : \mathbf{MGL}_{**}(k) \to \mathbb{E}_{**}(k)$ sends all the elements a_{ij} to 0 as soon as $(i, j) \neq (1, 0), (0, 1)$. Thus, φ induces the morphism of spectra σ using the isomorphism induced by (5.3.8.b) and the assumption that $\mathbb{E}$ is $\mathbb{Z}[1/p]$-linear.

Having fixed an additive orientation c of $\mathbb{E}$, from [Vez01, Th. 4.3], the orientations d on $\mathbb{E}$ are in one-to-one correspondence with the isomorphisms of formal group laws on $\mathbb{E}_{(2,1)*}(k)$ from the additive formal group to the formal group law F_d associated with d. As there is only one automorphism of the additive formal group law, the uniqueness statement follows. The remaining assertion is clear from what we have seen. $\qquad\square$

Remark 5.3.10. Another way of stating the preceding proposition is that given a representable $\mathbb{Z}[|1/p]$-linear cohomology theory $\mathbb{E}^{**}(-)$ over (smooth) k-schemes, there is only one possible way to define Chern classes which are additive. Moreover the other structures on the cohomology, that is residues, Gysin morphisms and fundamental classes, are unique. Similarly, the cycle class map, from classical Chow groups, is unique.

5.4. **Residues and symbols.**

5.4.1. Let us consider one of the following two cases:

- $S = \mathrm{Spec}(\mathbb{Z})$, $\Lambda = \mathbb{Q}$;
- $S = \mathrm{Spec}(k)$, k a field of exponential characteristic p, $\Lambda = \mathbb{Z}[1/p]$.

Let us denote simply by $\mathbf{H}_{\Lambda}$ either the Beilinson motivic cohomology spectrum in the first case or the Eilenberg-Mac Lane motivic ring spectrum with coefficients in Λ in the second case.

According to the preceding section, any S-absolute oriented ring spectrum $\mathbb{E}$ with additive formal group law has a structure of $\mathbf{H}_\Lambda$-algebra with structural morphism σ.

In particular, given any field K over S, we get a canonical symbol map:

$$K_n^M(K) \otimes_{\mathbb{Z}} \Lambda \simeq \mathbf{H}_\Lambda^{n,n}(K) \xrightarrow{\sigma_*} \mathbb{E}^{n,n}(K), \{f_1, \ldots, f_n\} \mapsto \{f_1, \ldots, f_n\}_{\mathbb{E}}.$$

If K admits a discrete valuation v such that its ring of integers is an S-scheme, with residue field k, we have obtained (Ex. 3.1.7) a residue map with coefficients in $\mathbb{E}$:

$$\partial_v^{\mathbb{E}} : \mathbb{E}^{n,m}(K) \to \mathbb{E}^{n-1,m-1}(k).$$

As an easy application of the Riemann-Roch residual theorem, we get the following computation of this residue: given any elements $f_1, ..., f_n$ in $K^\times$,

$$\partial_v^{\mathbb{E}}(\{f_1, \ldots, f_n\}_{\mathbb{E}}) = \sigma_*\big(\partial_v(\{f_1, \ldots, f_n\})\big)$$

where ∂_v is Milnor tame residue symbols (cf Example 3.1.7). In fact, this follows from Theorem 4.2.3 applied to σ: the Todd class is equal to 1 in this case as σ is a morphism of oriented ring spectra.[53]

Example 5.4.2. (1) *De Rham cohomology.*– Assume $S = \mathrm{Spec}(k)$ where k is a field of characteristic 0 and consider $\mathbb{E}_{dR}$ the k-absolute ring spectrum representing De Rham cohomology (cf. 1.2.3(1)). Recall the twist on that cohomology is just given by the tensor product with the 1-dimensional k-vector space $k(1) := H_{dR}^1(\mathbb{G}_m)^\vee$.

Note that $H_{dR}^1(\mathbb{G}_m) = k.d\log(t)$, where $\mathbb{G}_m = \mathrm{Spec}(k[t, t^{-1}])$. The choice of the generator $d\log(t)$ determines an isomorphism:

$$\mathbb{E}_{dR}^{n,i}(X) \simeq H_{dR}^n(X/k)$$

functorial in any smooth k-scheme X. As already mentioned, the fact $\mathbb{E}_{dR}$ is a k-absolute ring spectrum extends De Rham cohomology to any k-scheme. In the particular case of an extension field K/k, the choice of $d\log(t)$ gives a canonical isomorphism:

$$\mathbb{E}_{dR}^{n,i}(K) \simeq H_{dR}^n(K/k).$$

Through this isomorphism (case $i = n$), the *de Rham symbol* associated with a family of units $f_1, ..., f_n$ in an extension field K/k is given by the classical formula:

$$\{f_1, \ldots, f_n\}_{dR} = d\log(f_1) \wedge \ldots \wedge d\log(f_n) \in H_{dR}^n(K/k).$$

Given now a discrete valuation ring (K, v) over k, according to the previous paragraph one gets a residue map:

$$\partial_v^{dR} : H_{dR}^1(K/k) \simeq \mathbb{E}_{dR}^{1,1}(K) \to \mathbb{E}_{dR}^{0,0}(\kappa(v)) \simeq k.$$

According to the previous computation of residues on symbols, one gets:

(5.4.2.a) $$\partial_v^{dR}(d\log(f)) = v(f).$$

---

[53]One could also obtained this computation directly as in Example 3.1.7 using Proposition 3.1.6.

(2) *Rigid cohomology.–* Let V be a complete discrete valuation ring with fraction field E and residue field k. Let $\mathbb{E}_{rig}$ be the k-absolute ring spectrum representing rigid cohomology $H_{rig}(-/E)$.

The situation is analogous to the previous one though the constructions are less concrete. By definition, $H^1_{rig}(\mathbb{G}_m/E)$ is the rational part of the first cohomology group of the weakly complete De Rham complex associated with the weakly complete V-algebra $V\{t, t^{-1}\}$. In particular, it is generated by the differential form $d\log(t)$ of $V\{t, t^{-1}\}$. For smooth (or even singular) k-scheme X, the choice of this differential gives a canonical isomorphism

$$\mathbb{E}^{n,i}_{rig}(X) \simeq H^n_{rig}(X/E)$$

Given a unit $f : X \to \mathbb{G}_m$, this isomorphism in the case $n = i = 1$ sends the symbol $\{f\}_{rig}$ to the element $d\log(f) := f^*(d\log(t))$. Similarly, $\{f_1, \dots, f_n\}_{rig}$ corresponds to $d\log(f_1) \wedge \dots \wedge d\log(f_n)$.

Given an extension field K/k, one gets:

$$\mathbb{E}^{n,i}_{rig}(K) = \varinjlim_{A \subset K} H^n_{rig}(\operatorname{Spec}(A)/E) =: H^n_{rig}(K/E)$$

where A runs over the sub-rings of K which are smooth of finite type over the inseparable closure of k in K. And for any discrete valuation v on K, we get a tame residue symbol:

$$\partial^{rig}_v : H^1_{rig}(K/E) \to H^0_{rig}(K/E) \simeq E$$

satisfying the expected property on symbols.

Remark 5.4.3. Symbols in differential calculus have a beautiful history starting from van der Kallen formula ([vdK71]) and going to the Bloch-Kato conjecture modulo p ([BK86, 2.1]).

5.4.4. Consider again the notations of point (1) of the preceding example. Let C be a proper connected regular curve over k with function field K. Let us fix a closed point of C, in other words a discrete valuation v on K trivial on k. In [Tat68], Tate gives a purely algebraic definition of a residue map: $\partial^{Tate}_v : H^1_{dR}(K/k) \to k$. In view of the preceding example, the reader should not be surprised by the following comparison result:

Proposition 5.4.5. *Using the above notations,* $\partial^{dR}_v = \partial^{Tate}_v$.

Proof. Both definitions are invariant under completion with respect to the valuation v (see Prop. 1.4.4 for ∂^{dR}_v). Thus we can replace K by $\hat{K} \simeq k((t))$ and we are reduced to identify the two residues on differential forms of the form $\omega = f(t)dt$ when $f(t)$ is a power series with coefficients in k. By continuity and additivity of residues, we can assume $\omega = t^i.dt$.

The case $i \geq 0$ is easy because then ω can be extended to the valuation ring of $k[[t]]$ and therefore its image by ∂^{dR}_v is 0 according to the Gysin long exact sequence (3.1.1.b). The case $i = -1$ follows from the residual Riemann-Roch formula as explained in the previous example. For the remaining case, we use the reciprocity formula for $\mathbb{P}^1_k$ and ∂^{dR}_v (according to [Dég08b, 5.2.1] and [Ros96, 2.2]). According to this formula, we get:

$$\sum_{x \in \mathbb{P}^1_{k(0)}} \partial_x(\omega) = 0.$$

Because $\omega = t^i.dt$, the above equality gives:

$$\partial_0^{dR}(t^i.dt) = -\partial_\infty^{dR}(t^i.dt) \overset{(1)}{=} -\partial_\infty^{dR}(-u^{-(i+2)}.du) \overset{(2)}{=} 0$$

where equality (1) is given by the substitution $u = t^{-1}$, and (2) is true because $i + 2 \leq 0$. This concludes. $\qquad\square$

Remark 5.4.6. The above proof also gives a tool to compute residues in rigid cohomology. Given any cohomology class ω in $H_{rig}^1(K/E)$, we can extend it to the completion of K thus it corresponds to an element in $H_{rig}^1(k((t))/E)$, in other words an overconverging differential $\hat{\omega}$ form over $V((t))$. Then $\partial_v(\omega)$ is the residue in the usual sense: write $\omega = f(t).dt$ with $f(t) \in V\{t\}$, then

$$\partial_v(\omega) = \mathrm{res}_t(f(t))$$

seen as an element of K. More generally, our construction of residues should be linked with that of [Ber74, VII, 1.2].

Example 5.4.7. *Functoriality of coniveau spectral sequences.–* Assume that $\mathscr{S}$ is the category of excellent regular schemes. Let $\mathbb{E}$ be any absolute oriented ring spectrum.

Using the method of Bloch-Ogus in [BO74] and Gysin long exact sequences of the form (3.1.1.b), we get for any regular excellent scheme X and any integer $n \in \mathbb{Z}$, a *coniveau* spectral sequence of the form:

$$E_1^{p,q} = \bigoplus_{x \in X^{(n)}} \mathbb{E}^{q-p,n-p}(\kappa(x)) \Rightarrow \mathbb{E}^{p+q,n}(X)$$

which converges to the coniveau filtration on $\mathbb{E}^{**}(X)$.

Recall that this spectral sequence can be defined using the exact couple:

$$D^{p,q} = \varinjlim_{Z^*} \mathbb{E}^{p+q,n}(X - Z^{p+1}), E^{p,q} = \varinjlim_{Z^*} \mathbb{E}^{q-p,n-p}(Z^p - Z^{p+1})$$

where the limit is taken over the sequences $(Z^p)_{p \in \mathbb{N}}$ such that Z^p is a closed subscheme of codimension $\geq p$ in X satisfying the condition that $(Z^p - Z^{p+1})$ is regular. Indeed, using the fact X is excellent, we obtain that the corresponding set, ordered by term-wise inclusion, is filtering.

Then the maps of the exact couples are given by considering localisation long exact in sequences in cohomology — which come from Gysin triangles:

$$
\begin{array}{ccc}
E^{p,q+1} \leftrightsquigarrow \mathbb{E}^{q-p+1,n-p}(Z^p - Z^{p+1}) & & \mathbb{E}^{q-p,n-p}(Z^p - Z^{p+1}) \leftrightsquigarrow E^{p,q} \\
\partial_{X-Z^{p+1},Z^p-Z^{p+1}} \uparrow \qquad\qquad & & \qquad\qquad \downarrow i_{p*} \\
D^{p,q} \leftrightsquigarrow \mathbb{E}^{p+q,n}(X - Z^{p+1}) \xleftarrow{\quad j_p^* \quad} \mathbb{E}^{p+q,n}(X - Z^p) \leftrightsquigarrow D^{p-1,q+1}
\end{array}
$$

(see also the presentation of [Dég14]). According to this presentation, the fact that this spectral sequence, and especially the differentials in the E_1-term, is functorial with respect to any pseudo-morphism $\mathbb{E} \to \mathbb{F}$ of oriented ring spectra follows from the Riemann-Roch formula applied to the morphisms in the above diagram and the fact that it is enough to consider sequences Z^* as above and such that the normal bundle of $Z^p - Z^{p+1}$ in $X - Z^{p+1}$ is trivial.

Note that this phenomena was already observed in a particular case in [Gil05, Th. 3.9].

5.5. Residual Riemann-Roch formula.

5.5.1. So far, we have only worked out the trivial form of the residual Riemann-Roch formula, when the Todd class involved is 1. Let us express the general residual Riemann-Roch formula in the case of the usual Chern character, as introduced in Paragraph 5.3.3.

Theorem 5.5.2. *Consider a closed regular pair (X, Z) of codimension c. Let $N_Z X$ be the normal bundle of Z in X and put $U = X - Z$.*

Then the following diagram is commutative:

$$
\begin{array}{ccc}
K_r(U) & \xrightarrow{\ \partial_{X,Z}\ } & K_{r-1}(Z) \\
{\scriptstyle \mathrm{ch}^U_{r,n}}\downarrow & & \downarrow{\scriptstyle \sum_{i+j=n-c}\mathrm{Td}_i(-N_Z X)\cdot\mathrm{ch}^Z_{r-1,j}} \\
\mathbf{H}^{2n-r,n}_{\mathrm{B}}(U) & \xrightarrow{\ \partial_{X,Z}\ } & \mathbf{H}^{2(n-c)-r+1,n-c}_{\mathrm{B}}(Z)
\end{array}
$$

Remark 5.5.3. Once again, using the universality of Beilinson motivic cohomology among the absolute oriented cohomology with additive formal group law (Prop. 5.3.1), the preceding formula gives also a formula for the classical (mixed Weil) cohomologies.

Example 5.5.4. Let us first consider the case $r = 1$. Then one gets an explicit description of the residue morphism for K-theory, when $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$, assuming A and A/I are regular rings.

Indeed, one knows that $K_1(A_I) = \mathrm{GL}(A_I)^{ab}$, the abelianization of the group of invertible matrices of arbitrary dimensions. Assume we are given an endomorphism $u : A^r \to A^r$ such that $u \otimes_A A_I$ is an automorphism of A_I^r. We will denote by $[u]$ the class of this isomorphism in $K_1(A)$. By assumption, u is a monomorphism whose cokernel is supported on I. We denote by $[\mathrm{coKer}(u)]$ the class of the corresponding (finitely presented) A/I-module in $K_0(A/I)$. With these notations, one has the following formula:

$$\partial_{X,Z}([u]) = [\mathrm{coKer}(u)].$$

Assume furthermore $n = c = 1$. Recall that the following part of the higher Chern character $\mathrm{ch}_{1,1} : K_1(A) \to \mathbf{H}^{1,1}_{\mathrm{B}}(A) = A^\times \otimes \mathbb{Q}$ sends any matrix of $GL(A)$ to its determinant.

Assume Z is connected. By assumption, $I = (\pi)$ for a prime divisor π: A_I is a discrete valuation ring. We let v_π denote its valuation. Then, giving the above notations, the residual Riemann-Roch formula lands in $\mathbf{H}^{0,0}_{\mathrm{B}}(Z) = \mathbb{Q}$ and reads:

$$\boxed{v_\pi\big(\det(u)\big) = \mathrm{rk}_{A/I}\big([\mathrm{coKer}(u)]\big).}$$

Note that in fact, it is an integral formula as all the members are integers.

Consider now the case $n > 1$, r arbitrary: According to the coniveau spectral sequence of 5.4.7 applied for $\mathbb{E} = \mathbf{H}_{\mathrm{B}}$, we get that the group $\mathbf{H}^{2n-1,n}_{\mathrm{B}}(U)$ is the cohomology, tensored with $\mathbb{Q}$, in the middle of the following complex:

$$
\bigoplus_{y \in U^{(n-2)}} K_2^M(\kappa(y)) \longrightarrow \bigoplus_{x \in U^{(n-1)}} \kappa(x)^\times \xrightarrow{\ \mathrm{div}\ } Z^n(U)
$$

where the last map is the usual divisor class map (computation of Quillen). Thus, any element f of $\mathbf{H}_{\mathbb{E}}^{2n-1,n}(U)$ can be described as the class of a finite sum:

$$\sum_{x \in U^{(n-1)}} f_x$$

where f_x is a unit of $\kappa(x)$, which is the identity for almost all x, and such that the following n-codimensional cycle of U is zero:

$$\sum_{x \in U^{(n-1)}} \operatorname{div}_U(f_x) = 0.$$

Moreover, using this description of the group $\mathbf{H}_{\mathbb{E}}^{2n-1,n}(U)$, the residue map $\partial_{X,Z} :$ $\mathbf{H}_{\mathbb{E}}^{2n-1,n}(U) \to CH^{n-c}(Z)_{\mathbb{Q}}$ can be described as follows:

$$\partial_{X,Z}(f) = \sum_{x \in U^{(n-1)}} \operatorname{div}_X(f_x)$$

where div_X denotes the divisor of the rational function of f_x seen as a cycle in X. Indeed, by assumption on (f_x), this cycle has support in Z.

Thus, we will represent the element $\operatorname{ch}_{1,n}([u]) \in \mathbf{H}_{\mathbb{E}}^{2n-1,n}(U)$ as the class of a sum: $\sum_{x \in U^{(n-1)}} f_{u,x}$ satisfying the conditions above.[54]

Assume $n = c$ and Z is connected with generic point η. Then the residual Riemann-Roch formula lands again in $\mathbf{H}_{\mathbb{E}}^{0,0}(Z) = \mathbb{Q}$ and reads:

$$\boxed{\sum_{x \in U^{(n-1)}, \eta < x} \operatorname{ord}_\eta^x(f_{u,x}) = \operatorname{rk}_{A/I}\big([\operatorname{coKer}(u)]\big)}$$

where $\operatorname{ord}_\eta^x$ denotes the order of the rational function at x. Observe this is again an integral formula.

Let us finally consider the case where the codimension c of Z in X is arbitrary less than n. Then the residual formula can be stated as the following equality of cycles in $CH^{n-c}(Z)_{\mathbb{Q}}$:

$$\boxed{\sum_{x \in U^{(n-1)}} \operatorname{div}_X(f_{u,x}) = \sum_{i+j=n-c} \operatorname{Td}_i(-N_Z X).\operatorname{ch}_{0,j}\big([\operatorname{coKer}(u)]\big).}$$

(Recall that by assumption, the cycle on the left has support in Z.)

6. The axiomatic of Panin revisited

6.1. **Axioms for (arithmetic) cohomologies.** Our theory is obviously linked with the more classical theory of *oriented cohomology theory* developed by Panin and Smirnov (see [Pan03, Pan04, Pan09] and [Smi06a, Smi06b]). Our axioms are more restrictive as we ask for representability of a cohomology theory. Nevertheless, one can extract from Section 1 the following generalization of the axioms used by Panin:

Definition 6.1.1. A *ringed cohomology theory (with supports)* $\mathbb{E}$ on $\mathscr{S}$ is the datum for each closed pair (X, Z) in $\mathscr{S}$ of a bigraded abelian group $\mathbb{E}_Z^{**}(X)$ equipped with the following structures:

- contravariant functoriality as described in 1.2.6,

---

[54]It is rather delicate to give a formula for the $f_{u,x}$. One can only say that they describe the part of the element $[u] \in K_1(U)$ (topologically) supported in codimension $(n-1)$.

- covariant functoriality as described in 1.2.7,
- refined products as described in 1.2.8,
- for each closed pair (X, Z), a *boundary morphism*

$$\mathbb{E}^{n,m}(X - Z) \xrightarrow{\delta_{X,Z}} \mathbb{E}^{n+1,m}_Z(X)$$

  contravariantly natural and fitting in a Gysin long exact sequence of the form (3.1.1.a),

which satisfies the axioms (E1)-(E7) described in Prop. 1.2.10 together with the following additional properties:

- *Homotopy*: for any scheme X, $\mathbb{E}^{**}(X) \to \mathbb{E}^{**}(\mathbb{A}^1_X)$ is an isomorphism,
- *Stability*: For any scheme S, let $\tilde{\mathbb{E}}^{2,1}(\mathbb{P}^1_S) := \mathbb{E}^{2,1}(\mathbb{P}^1_S)/\mathbb{E}^{2,1}(\{\infty\})$. There exists a family of classes $\eta_S \in \tilde{\mathbb{E}}^{2,1}(\mathbb{P}^1_S)$ indexed by schemes in $\mathscr{S}$ which is stable by pullbacks and such that for any scheme S and any integers (n, m) the following map is an isomorphism:

$$\mathbb{E}^{n,m}(S) \to \tilde{\mathbb{E}}^{n+2,m+1}(\mathbb{P}^1_S), x \mapsto \eta_S.p^*(x).$$

- *Excision*: for any excisive morphism of closed pairs $f : (Y, T) \to (X, Z)$ (see Def. 1.4.2), the pullback $f^* : \mathbb{E}^{**}_Z(X) \to \mathbb{E}^{**}_T(Y)$ is an isomorphism.

A morphism of ringed cohomology theories with support is a natural transformation compatible with contravariant and covariant functorialities, with refined products and with the operator $\delta_{X,Z}$ for closed pairs (X, Z) in $\mathscr{S}$.

We will say that $\mathbb{E}$ is *oriented* if there exists a natural transformation of presheaves of sets on $\mathscr{S}$:

$$c : \operatorname{Pic} \to \mathbb{E}^{2,1}$$

such that for any scheme S, $c_{\mathbb{P}^1_S}(\lambda) = \eta_S$ where $\lambda = \mathcal{O}(-1)$ is the canonical line bundle on $\mathbb{P}^1_S$.

We will say that a closed pair (X, Z) is $\mathbb{E}$-pure if the morphisms

$$\mathbb{E}^{**}(X, Z) \xleftarrow{\sigma_1^*} \mathbb{E}^{**}(D_Z X, \mathbb{A}^1_Z) \xrightarrow{\sigma_0^*} \mathbb{E}^{**}(N_Z, Z)$$

induced by the deformation diagram (1.3.1.a) are isomorphisms. We say that $\mathbb{E}$ is *absolutely pure* if any regular closed pair in $\mathscr{S}$ is $\mathbb{E}$-pure.

For short, we will say *arithmetic cohomology* for a ringed cohomology with support which is oriented and absolutely pure. A morphism of arithmetic cohomology is defined likewise, but beware we do not require the compatibility with the given orientations.

6.1.2. With this definition, we can extend all the results of sections 2, 3 and 4 as follows:

(1) One has to pay a special attention to the projective bundle formula (2.1.13) and realize that the lemma of Morel 2.1.14 can be stated and proven using cohomology with support (instead of working in the unstable homotopy category). Then one gets the working theory of Chern classes and formal group laws as established in Section 2.1.

(2) For sections 2.3, 2.4, 3 and 4, the arguments just go through as we have been careful to rely only on the axiomatic described in section 1 and restated in the previous definition.

The results obtained here cover the one proved earlier by Panin.

Remark 6.1.3. Note that the axiomatic described here differs especially with that of Panin because of two points:

- we have devised another axiom, *absolute purity*, especially relevant in the arithmetic case;
- we asked for the existence of a refined product.

Both properties are very strong and the natural examples are given by representable cohomology theories – but see also the next section. Note however that in the case of algebraic K-theory, they should be obtained without using representability: the case of absolute purity is of course the localization theorem of Quillen but the case of refined products is less obvious.

6.2. **Étale cohomology.**

Example 6.2.1. (1) Let Λ be a torsion ring of characteristic exponent N. Let $\mathscr{S}$ be the category of regular schemes on which N is invertible. Then it follows from [SGA4], using the method described in Section 1.2 together with the functoriality of étale sheaves established in [SGA4], that for any closed immersion $i : Z \to X$, the bigraded cohomology groups:

$$H_Z^n(X_{\text{ét}}, \Lambda(m))$$

of the twisted sheaf $i^!\Lambda(m)$, computed in the small étale site of X, is a ringed cohomology with support over $\mathscr{S}$ in the sense of the above definition. Recall that according to [SGA4, IX, Th. 3.3], this cohomology theory is oriented with an additive formal group law. Moreover, it is absolutely pure over $\mathscr{R}\text{eg}$ according to the (absolute purity) theorem of Gabber.

Thus our constructions apply to this cohomology, which has an additive formal group law. In particular, we get maps for projective morphisms of regular $\mathbb{Z}[1/N]$-schemes on étale cohomology with coefficients in Λ.[55]

(2) Let l be a prime number and $\mathscr{S}$ be the category of $\mathbb{Z}[1/l]$-regular schemes.

Then we can apply the construction of [Jan88] to get that *continuous l-adic étale cohomology with support*:

$$H_{\text{cont},Z}^n(X, \mathbb{Z}_l(m))$$

defined in *loc. cit.*, Section 3, (after Remark 3.5) is an arithmetic cohomology in the previous sense. In fact, homotopy, stability and excision follow from the known results of [SGA4]. The refined product can be defined using the method of Section 6 given that we have a pairing

$$\Gamma_T(Z, -) \otimes \Gamma_Z(X, -) \to \Gamma_T(X, -)$$

of the functors of global sections with support for torsion sheaves (as in the above example). Axioms (E1)-(E7) then follow. Note this theory is oriented: this is *loc. cit.* (3.26). Finally, using the absolute purity theorem of Gabber in the form of the computation of $\Gamma_Z(X, -)$ for $Z \subset X$ regular schemes, we get that this cohomology theory with support is absolutely $\mathscr{R}\text{eg}$-pure. The same construction works for $\mathbb{Q}_l$-coefficients.

Thus we can also apply the constructions of this paper to l-adic étale cohomology (integral and rational).

---

[55]This Gysin morphisms agree with the one constructed by Gabber-Riou: see Remark 6.2.5.

6.2.2. Using the more sophisticated theory of [CD12b], we can get many examples as follows.

Let $\mathscr{T}$ be a motivic triangulated category over $\mathscr{S}$ in the sense of [CD12b, Def. 2.4.45]: in other words, this is a category fibered over the category $\mathscr{S}$ which satisfies the axioms (A1)-(A4) of Par. 1.1.1 together with the homotopy and stability property (in fact, we will not use the adjoint properties of *loc. cit.*). Then we can associate with $\mathscr{T}$ a ringed cohomology theory with support: for any closed immersion $i : Z \to X$ in $\mathscr{S}$, we put:

$$H_Z^{n,m}(X, \mathscr{T}) := \operatorname{Hom}_{\mathscr{T}(X)}(i_*(\mathbb{1}_Z), \mathbb{1}_X(m)[n])$$

where $\mathbb{1}_?$ is the cartesian section of $\mathscr{T}$ made by the unit for the tensor product and $\mathbb{1}_X(m)$ denotes the m-th Tate twist ([CD12b, 2.4.17]). Then exactly the same arguments as in the proof of Prop. 1.2.10 shows that this theory satisfies axioms (E1)-(E7).

Assume moreover that one has a premotivic adjunction ([CD12b, Def. 1.4.2])

$$\varphi^* : \mathscr{T} \to \mathscr{T}'$$

where $\mathscr{T}$ and $\mathscr{T}'$ are motivic categories. Then, according to [CD12b, 2.3.11 or 2.4.53], for any closed immersion i, $\varphi^* i_* \simeq i_* \varphi^*$ through a canonical isomorphism (called an *exchange isomorphism*). Thus, given any closed pair (X, Z) in $\mathscr{S}$, one gets by applying φ^* a morphism:

$$H_Z^{n,m}(X, \mathscr{T}) \xrightarrow{\varphi} H_Z^{n,m}(X, \mathscr{T}')$$

which is compatible with contravariant functoriality (resp. covariant functoriality, refined product, boundary) because φ^* commutes with f^* for any morphism f (resp. i_* for any closed immersion i, tensor product, localization triangle).

Example 6.2.3. (1) *(Motivic) étale cohomology.*– Let R be any ring. For any scheme S, Cisinski and the author have introduced in [CD14, 5.1.3], following Voevodsky, the category $\mathrm{DM}_\mathrm{h}(S, R)$ of h-motives. We proved in *loc. cit.*, Th. 5.6.2, that it forms, for various S, a motivic triangulated category. In particular, the cohomology theory

$$H_{\text{ét},Z}^{n,m}(X, R) := \operatorname{Hom}_{\mathrm{DM}_\mathrm{h}(X,R)}(i_*(\mathbb{1}_Z), \mathbb{1}_X(m)[n])$$

is a ringed cohomology with support, defined over the category of all schemes. According to *loc. cit.*, 5.6.2, it is even an arithmetic cohomology. According to Voevodsky, this cohomology theory is called the *étale motivic cohomology* with coefficients in R. In view of the second computation below, we think that it should simply be called the *étale cohomology* with coefficients in R.

According to the fundamental results of *loc. cit.*, one gets for any regular scheme X:

- if R is a $\mathbb{Q}$-algebra,

$$H_{\text{ét}}^{n,m}(X, R) = H_\mathrm{B}^{n,m}(X, R) = K_{2m-n}^{(m)}(X)$$

  is Beilinson motivic cohomology;

- if R is a torsion ring with characteristic exponent N, for any scheme X,

$$H_{\text{ét}}^{n,m}(X, R) = H_{\text{ét}}^n(X[1/N], R(m)),$$

where $X[1/N]$ is the open part of X where N is invertible, and the right hand side is the usual étale cohomology of $X[1/N]$ with coefficients in R twisted m-times.

Moreover, for any ring extension R'/R, there is a premotivic adjunction:

$$\varphi : \mathrm{DM}_\mathrm{h}(S, R) \to \mathrm{DM}_\mathrm{h}(S, R')$$

so that we get a morphism of arithmetic cohomologies:

(6.2.3.a) $$\varphi : H_\mathrm{ét}^{**}(-, R) \to H_\mathrm{ét}^{**}(-, R').$$

(2) *Continuous étale cohomology.*– Let R be any valuation ring with parameter ℓ. Then, according to *loc. cit.*, 7.2.11, the homotopy ℓ-adic completion of $\mathrm{DM}_\mathrm{h}(-, R)$ gives a motivic triangulated category $\mathrm{DM}_\mathrm{h}(-, \hat{R}_\ell)$ and in particular a ringed cohomology theory with support, defined over the category of all schemes:

$$H_{\mathrm{cont}, Z}^{n, m}(X, \hat{R}_\ell) := \mathrm{Hom}_{\mathrm{DM}_\mathrm{h}(X, \hat{R}_\ell)}(i_*(\mathbb{1}_Z), \mathbb{1}_X(m)[n]).$$

According to *loc. cit.*, this is an arithmetic cohomology,.

Note that when R is a discrete valuation ring, and X a scheme such that the exponent characteristic of R/ℓ is invertible on X, according to *loc. cit.*, 7.2.21, the triangulated category $\mathrm{DM}_\mathrm{h}(X, \hat{R}_\ell)$ agree with Ekedahl category of ℓ-adic complexes. Thus the cohomology $H_{\mathrm{cont}, Z}^{**}(X, \hat{R}_\ell)$ is Jannsen continuous étale ℓ-adic cohomology and deserves the name of *continuous étale cohomology* with coefficients in $\hat{R}_l$.

From the obvious premotivic adjunction $\hat{\rho}_\ell^* : \mathrm{DM}_\mathrm{h}(S, R) \to \mathrm{DM}_\mathrm{h}(S, \hat{R}_\ell)$ (see [CD14, 7.2.4]), we get a morphism of arithmetic cohomologies:

(6.2.3.b) $$\rho_\ell : H_\mathrm{ét}^{**}(-, R) \to H_\mathrm{cont}^{**}(-, \hat{R}_\ell).$$

Let Q be the fraction field of R. We now easily get the rational version of continuous étale cohomology by taking tensor product by Q over R:

$$H_{\mathrm{cont}, Z}^{n, m}(X, Q_\ell) := H_{\mathrm{cont}, Z}^{n, m}(X, \hat{R}_\ell) \otimes_R Q.$$

which is again an arithmetic cohomology theory. And finally, a rational version of the previous morphism:

(6.2.3.c) $$\rho_\ell : H_\mathrm{ét}^{**}(-, Q) \to H_\mathrm{cont}^{**}(-, Q_\ell).$$

(3) For a prime number ℓ, combining (6.2.3.a) and (6.2.3.c), we get a morphism of ringed cohomologies with support on the category of regular $\mathbb{Z}[1/\ell]$-schemes:

(6.2.3.d) $$\rho_\ell : H_\mathrm{B}^{**}(-) \to H_\mathrm{cont}^{**}(-, \mathbb{Q}_\ell)$$

from Beilinson motivic cohomology to continuous rational ℓ-adic cohomology.

As a corollary of the preceding examples and the constructions of this paper, we thus obtain:

Corollary 6.2.4. *Assume $\mathscr{S}$ is one of the following categories of schemes:*

(a) *regular noetherian schemes of finite dimension;*

(b) *smooth schemes over a noetherian (singular) scheme of finite dimension.*

Let R be a ring (resp. discrete valuation ring with parameter ℓ). In the respective case, we also denote by Q_l the fraction field of $\hat{R}_\ell$.

(1) *The ring cohomology theory $H_{ét}^{**}(-, R)$ (resp. $H_{cont}^{**}(-, \hat{R}_\ell)$, $H_{cont}^{**}(-, Q_\ell)$) admits Chern classes, Gysin morphisms for any projective morphism of schemes in $\mathscr{S}$, and residue morphisms associated with a closed immersion $i : Z \to S$ of schemes in $\mathscr{S}$ which fit into the usual localization long exact sequence. These residues and Gysin morphisms satisfy the following properties: compatibility with transversal pullback, excess of intersection, projection formula.*

(2) *The five natural transformations of Example 6.2.3 are functorial with respect to Gysin morphisms and localization long exact sequences.*

(3) *For any integer $r \geq 0$, there exists a well defined higher Chern character:*

$$\mathrm{ch}_r : K_r(X) \to \oplus_{n \geq 0} H_{ét}^{2n-r,n}(Y, Q_\ell)$$

from Quillen (resp. Thomason-Trobaugh in case (b)) algebraic K-theory such that for any projective morphism $f : Y \to X$ in $\mathscr{S}$, the following diagram commutes:

$$
\begin{array}{ccc}
K_r(Y)_{\mathbb{Q}} & \xrightarrow{\ f_*\ } & K_r(X)_{\mathbb{Q}} \\
{\scriptstyle \mathrm{Td}(\tau_f).\,\mathrm{ch}_t}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_t} \\
\oplus_{n \geq 0} H_{ét}^{2n-r,n}(Y, Q_\ell) & \xrightarrow{\ f_*\ } & \oplus_{n \geq 0} H_{ét}^{2n-r,n}(X, Q_\ell)
\end{array}
$$

where on the top line, f_* is the usual covariant functoriality of algebraic K-theory. Moreover for any closed immersion $i : Z \to X$ in $\mathscr{S}$, one gets:

$$
\begin{array}{ccc}
K_r(X - Z)_{\mathbb{Q}} & \xrightarrow{\ \partial_{X,Z}\ } & K_r(Z)_{\mathbb{Q}} \\
{\scriptstyle \mathrm{Td}(-N_Z X).\,\mathrm{ch}_t}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_t} \\
\oplus_{n \geq 0} H_{ét}^{2n-r,n}(Y, Q_\ell) & \xrightarrow{\ \partial_{X,Z}\ } & \oplus_{n \geq 0} H_{ét}^{2(n-c)-r+1,n-c}(X, Q_\ell),
\end{array}
$$

where $N_Z X$ is the normal bundle of Z in X.

   Under assumption (b), the functor K_r can be replaced by Weibel homotopy invariant K-theory KH_r in the two previous diagram.

As explained in 6.1.2, Point (1) is a compact form of the results of sections 2, 3 (recall excess of intersection: 3.2.12, projection formula: 3.2.10(b)). Point (2) follows from Th. 4.2.3 and Th. 4.3.2 because all theories have additive formal group law and there is only one strict isomorphism of formal group law: this implies the Todd class involved in each formulas is necessarily equal to 1 (see Section 5.1). Point (3) finally follows from Prop. 5.3.4 and Prop. 5.3.6.

Remark 6.2.5. When R is a torsion ring with characteristic exponent N, in [Rio14], Riou following a construction of Gabber has defined Gysin morphisms on étale cohomology of $\mathbb{Z}[1/N]$-schemes with coefficients in R, with respect to all lci projective maps between any noetherian schemes. If one restricts to regular schemes, we obtain using Th. 3.3.1 that our Gysin maps coincide with the construction of Gabber-Riou.

Let us be more precise. First, let us compare our conventions with that of *op. cit.* Let X be a scheme and $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module. Then the vector bundle associated with $\mathcal{E}$ is $E = V(\mathcal{E}^\vee)$, the spectrum over X of the symmetric algebra induced by the *dual* of $\mathcal{E}$. Because of this convention one relates Chern classes used

in [Rio14] with ours by the formula:

$$c_r(\mathcal{E}) = (-1)^r . c_r(E)$$

(compare with relation (5.2.9.a)).

Once this convention is settled, one can apply Theorem 3.3.1 as Riou proved the excess intersection formula in [Rio14, Prop. 2.3.2]. Note also that, because of Cor. 5.1.10, Chern classes are uniquely determined by the choice of a stability isomorphism:

$$H^2_{\text{ét}}(\mathbb{P}^1_S, R(1)) \simeq H^0_{\text{ét}}(S, R(0)) = R$$

(which necessarily appears as a particular case of the projective bundle formula).

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