BIVARIANT THEORIES IN MOTIVIC STABLE HOMOTOPY

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Abstract. The purpose of this work is to study the notion of bivariant theory introduced by Fulton and MacPherson in the context of motivic stable homotopy theory, and more generally in the broader framework of the Grothendieck six functors formalism. We introduce several kinds of bivariant theories associated with a suitable ring spectrum and we construct a system of orientations (called fundamental classes) for global complete intersection morphisms between arbitrary schemes. These fundamental classes satisfy all the expected properties from classical intersection theory and lead to Gysin morphisms, Riemann-Roch formulas as well as duality statements, valid for general schemes, including singular ones and without need of a base field. Applications are numerous, ranging from classical theories (Betti homology) to generalized theories (algebraic K-theory, algebraic cobordism) and more abstractly to étale sheaves (torsion and ℓ-adic) and mixed motives.

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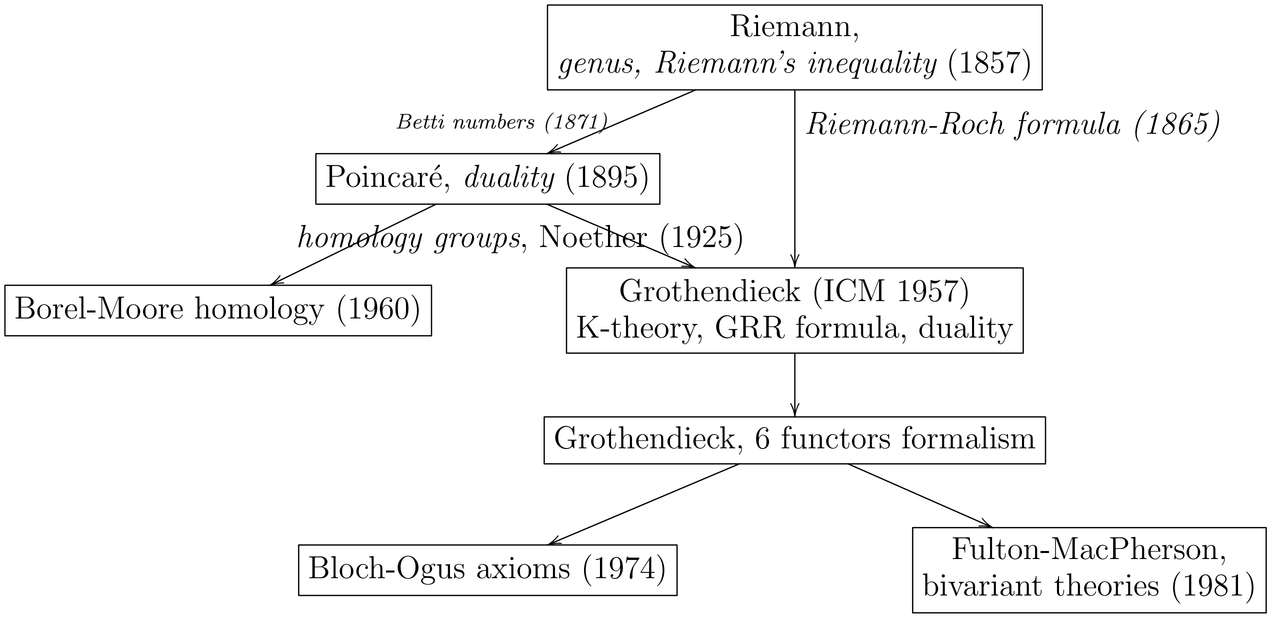
Introduction

***Genealogy* of ideas.** As this work comes after a long line of ideas about cohomology, may be it is worthwhile to draw the following genealogical tree of mathematicians and concepts.



So our starting point is the bivariant formalism of Fulton and MacPherson ([FM81]), which appears in one of the ending point of the above tree. The ambition of this formalism is to unify homology and cohomology into a single theory. More than that: it was slowly observed that Poincaré duality gives a wrong-way functoriality of singular cohomology. The study of this phenomena, which was later discovered to occur also in homology, was developed from many perspectives all along the century. The striking success of the bivariant formalism is to explain all these works by the construction of an element, generically called *orientation* by Fulton and MacPherson, of a suitable bivariant group.

In fact, this theory can be seen as a culminating point of the classical theory of characteristic classes. A brilliant illustration is given by Fulton and MacPherson's interpretation of each of the known Riemann-Roch formulas as a comparison between two given orientations. Let us now detail these general principles from our point of view, based on motivic homotopy theory.

Motivic homotopy theory. Our previous works on Gysin morphisms ([Dég08, Dég14b]) naturally lead to the bivariant language. As we will see, it allows to treat both the fundamental class of an algebraic cycle and the cobordism class of a projective morphism within a single framework. But it was only when the six functors formalism was fully available in motivic homotopy theory, after the work of Ayoub ([Ayo07a]), that we became aware of a plain incorporation of bivariant theories into motivic homotopy theory.

One can already trace back this fact in the work of Fulton and MacPherson as one of their examples of a bivariant theory, in the étale setting, already uses the six functors formalism. In this work we go further, showing that any representable cohomology in $\mathbb{A}^1$-homotopy admits a canonical extension to a bivariant theory. Basically, it applies to any known cohomology theory (in algebraic geometry) which is homotopy invariant.

This result is obtained as a by-product of Morel-Voevodsky's motivic homotopy theory, but more generally, we use the axioms of Ayoub-Voevodsky's cross functors, fundamentally developed by Ayoub in [Ayo07a]. This theory was amplified later by Cisinski and the author in [CD12b] as a general axiomatic of Grothendieck' six functors formalism. Such an axiomatic theory, called a *triangulated motivic category*, is in the first place a triangulated category $\mathscr{T}$ fibered over a fixed suitable category of schemes $\mathscr{S}$ and equipped with the classical six operations, $f^*, f_*, f_!, f^!, \otimes, \underline{\mathrm{Hom}}$. There are many concrete realizations of this formalism in the literature so we will only recall here the specific axioms added by Ayoub and Voevodsky:

- $\mathbb{A}^1$-*homotopy.–* for any scheme X in $\mathscr{S}$, $p : \mathbb{A}^1_X \to X$ being the projection of the affine line, the adjunction map $1 \to p_*p^*$ is an isomorphism;
- $\mathbb{P}^1$-*stability.–* for any scheme X in $\mathscr{S}$, $p : \mathbb{P}^1_X \to X$ being the projection of the projective line and $s : X \to \mathbb{P}^1_X$ the infinite-section, the functor $s^!p^*$ is an equivalence of categories.

Recall the first property corresponds to the contractibility of the affine line and the second one to the invertibility of the Tate twist.[1] Note already that this axiomatic is satisfied in the étale setting (torsion and ℓ-adic coefficients). Besides, thanks to the work of the motivic homotopy community, there are now many examples of such triangulated categories.[2]

Absolute ring spectra and bivariant theories. From classical and motivic homotopy theories, we retain the notion of a ring spectrum but use a version adapted to our theoretical context. An *absolute ring $\mathscr{T}$-spectrum* will be a cartesian section of the category of commutative monoids in $\mathscr{T}$, seen as a fibered category over $\mathscr{S}$; concretely, the data of a commutative monoid $\mathbb{E}_S$ of the monoidal category $\mathscr{T}(S)$ for any scheme S in $\mathscr{S}$, with suitable base change isomorphisms $f^*(\mathbb{E}_S) \xrightarrow{\sim} \mathbb{E}_T$ associated with any morphism $f : T \to S$ in $\mathscr{S}$ (see Definition 1.1.1).[3] Our main observation is that to such an object is associated not

[1] While the first property is often remembered, the second one was slightly overlooked at the beginning of the theory but appears to be fundamental in the establishment of the six functors formalism.

[2] Stable homotopy, mixed motives, modules over ring spectra such as K-theory, algebraic cobordism. These examples will appear naturally in the course of the text; see Example 1.1.2 in the first place.

[3] The terminology is inspired by the terminology used by Beilinson in his formulation of the Beilinson's conjectures. In particular, the various absolute cohomologies considered by Beilinson are representable by

only the classical cohomology theory but also a bigraded bivariant theory: to a separated morphism $f : X \to S$ of finite type and integers $(n, m) \in \mathbb{Z}^2$, one associates the abelian group:

$$\mathbb{E}^{BM}_{n,m}\big(X \xrightarrow{f} S\big) = \underbrace{\mathrm{Hom}_{\mathscr{T}(S)}(f_!(\mathbb{1}_X)(n)[m], \mathbb{E}_S)}_{(*)} \simeq \underbrace{\mathrm{Hom}_{\mathscr{T}(S)}(\mathbb{1}_X(n)[m], f^!\mathbb{E}_S)}_{(**)}$$

It usually does not lead to confusions to denote these groups $\mathbb{E}^{BM}_{n,m}(X/S)$. Now the word bivariant roughly means the following two fundamental properties:

- *Functoriality.*– the group $\mathbb{E}^{BM}_{n,m}(X/S)$ is covariant in X with respect to proper S-morphisms and covariant in S with respect to any morphism;
- *Product.*– given Y/X and X/S, there exists a product:
$$\mathbb{E}^{BM}_{n,m}(Y/X) \otimes \mathbb{E}^{BM}_{s,t}(X/S) \to \mathbb{E}^{BM}_{n+s,m+t}(Y/S).$$

Of course, these two structures satisfy various properties; we refer the reader to Section 1.2 for a complete description.[4]

Several paths lead naturally to our definition. First, our initial motivation comes from the case where $f = i : Z \to S$ is a closed immersion. In this case, from the localization triangle attached to i by the six functors formalism, one realizes that the abelian group $\mathbb{E}^{BM}_{**}(Z/X)$ is nothing else than the $\mathbb{E}$-cohomology group of X with support in Z, which naturally receives the refined fundamental classes we had built earlier ([Dég13, 2.3.1]). Secondly, when $\mathscr{T}(X) = D^b_c(X_{\text{ét}}, \Lambda)$, $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$, for ℓ a prime invertible on X, and $\mathbb{E}_X = \Lambda_X$ is the constant sheaf, formulas $(*)$ and $(**)$ agree with that considered by Fulton and MacPherson in [FM81, 7.4.1] for defining a bivariant étale theory. Finally, when $S = \mathrm{Spec}(k)$ is the spectrum of a field and $\mathbb{E}_X = \mathbb{1}_X$, formula $(**)$ gives an interpretation of $\mathbb{E}^{BM}_{**}(X/k)$ as the cohomology with coefficients in the object $f^!(\mathbb{1}_k)$, which is Grothendieck's formula for the dualizing complex.[5] In other words, $\mathbb{E}^{BM}_{**}(X/k)$ is the analogue of Borel-Moore homology defined in [BM60].[6] This last example justifies our notation: the letters "BM" stands for Borel-Moore and we use the terminology *Borel-Moore homology* for the bivariant theory $\mathbb{E}^{BM}_{**}$.

Besides, we can define other bivariant theories from the absolute ring spectrum $\mathbb{E}$. In fact, we remark that one can attach to $\mathbb{E}$ four theories: cohomology as usual, Borel-Moore homology defined by the above formula $(**)$, but also cohomology with compact support and (plain) homology. The two later theories are in fact bivariant theories: they can be defined not only for k-schemes but for any morphism of schemes (in $\mathscr{S}$). We refer the

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absolute spectra in our sense, with $\mathscr{S}$ being the category of all (noetherian, finite dimensional) schemes, or that of schemes over a field (for example in the case of Deligne cohomology).

[4]Note also that we get a particular instance of what Fulton and MacPherson call a bivariant theory. However, in algebraic geometry, this instance is the most common one.

[5]Recall the object $f^!(\mathbb{1}_k)$ is indeed a dualizing object in $\mathscr{T}(X)$ if $\mathscr{T}$ is $\mathbb{Q}$-linear (see [CD12b]) or under suitable assumptions of resolution of singularities (see [Ayo07a]).

[6]In fact we prove in Example 1.3.3(3) that when $\mathscr{T}$ is Morel-Voevodsky's stable homotopy category and $\mathbb{E}$ is the spectrum representing Betti cohomology, this abelian group is precisely Borel-Moore homology – the twists in that case do not change the group up to isomorphism.

reader to Definition 1.3.2 but one can also guess the formulas: they are the variants of formula $(**)$ obtained from the various possibility of combining the functors $f_*, f_!, f^*, f^!$. Note our main result, which will be stated just below, will give new structures for each of the four theories.

Orientations and fundamental classes. The key idea of this work is that one can recast previously known constructions of Gysin morphisms, based on the orientation theory of motivic ring spectra, in the bivariant framework provided by formula $(*)$. Recall for the sake of notations that an orientation c of an absolute ring $\mathscr{T}$-spectrum $\mathbb{E}$ is roughly a family of classes c_S in $\mathbb{E}^{2,1}(\mathbb{P}^\infty_S)$, cohomology of the infinite projective space, for schemes S in $\mathscr{S}$, satisfying suitable conditions (see Definition 2.2.2 for the precise formulation).

One can attach to the orientation c a complete formalism of Chern classes, and even Chern classes with supports (see Proposition 2.2.6 and 2.4.2). Our main result is the following construction of characteristic classes of morphisms, whose main advantage against previous constructions is that it works for singular schemes and without a base field.

Theorem 1 (see Theorem 2.5.3). *Consider an absolute ring $\mathscr{T}$-spectrum $\mathbb{E}$ with a given orientation c.*

Then for any global complete intersection[7] *morphism $f : X \to S$ of relative dimension d, there exists a class $\bar{\eta}_f$ in $\mathbb{E}^{BM}_{2d,d}(X/S)$, called the* fundamental class *of f associated with $(\mathbb{E}, c)$, with the following properties:*

(1) Normalization.– *If $f = i : D \to X$ is the immersion of a regular divisor then $\bar{\eta}_i = c_1^D(\mathcal{O}(-D))$ (where $\mathcal{O}(-D)$ is the dual of the invertible sheaf parametrizing i; see Example 2.4.7).*
(2) Associativity.– *For any composable morphisms $Y \xrightarrow{g} X \xrightarrow{f} S$, $\bar{\eta}_{f\circ g} = \bar{\eta}_g \,.\, \bar{\eta}_f$.*
(3) Base change.– *Given a morphism $p : T \to S$ which is transversal to the map f, then $p^*(\bar{\eta}_f) = \bar{\eta}_{f\times_S T}$ (see Example 3.1.2).*
(4) Excess intersection.– *Given a morphism $p : T \to S$ such that the base change $f \times_S T$ is a local complete intersection, $p^*(\bar{\eta}_f) = e(\xi).\, \bar{\eta}_{f\times_S T}$ (the class $e(\xi)$ stands for the Euler class of the excess intersection bundle, as in Fulton's classical formula for Chow groups; see Proposition 3.1.1).*
(5) Ramification formula.– *Let $p : Y \to X$ be a dominant morphism of normal schemes, $i : D \to X$ be the immersion of a regular divisor, $(E_j)_{j=1,\dots,r}$ the family of irreducible components of $p^{-1}(D)$ and m_j the ramification index*[8] *of f along E_j. Then one has:*

$$p^*(\bar{\eta}_X(D)) = [m_1]_F.\, \bar{\eta}_Y(E_1) +_F \dots +_F [m_r]_F.\, \bar{\eta}_Y(E_r);$$

[7] We say f is a global complete intersection if it admits a factorization $f = p \circ i$ where p is smooth separated of finite type and i is a regular closed immersion;

[8] or, in other words, the intersection multiplicity of E_j in the pullback of D along p, see [Ful98, 4.3.7]; note the integer m_j is nothing else than the ramification index in the classical sense of the extension $\mathcal{O}_{Y,E_j}/\mathcal{O}_{X,D}$ of discrete valuation rings;

here $([m]_F.-)$ *stands for the power series corresponding to the* m*-th self addition in the sense of the formal group law* F *attached with the orientation* c *(see Corollary 3.1.6).*

(6) Duality.– *When* $f : X \to S$ *is smooth or is the section of a smooth morphism, the multiplication map:*

$$\mathbb{E}^{BM}_{**}(Y/X) \to \mathbb{E}^{BM}_{**}(Y/S), y \mapsto y.\bar{\eta}_f$$

is an isomorphism (see Remark 2.5.4).

Before discussing uniqueness statements of our construction, let us explain the existence part. As mentioned before the statement of the theorem, we use previous methods of construction of Gysin maps. The first one, in the case where f is a regular closed immersion, is a construction whose motivic homotopy formulation is due to Navarro (cf. [Nav16]), based on a method of Gabber (cf. [ILO14, chap. 7]) who treated the case of étale cohomology.[9] The second method, for a smooth quasi-projective morphism f, is given as a corollary of Ayoub's fundamental work on the six functors formalism in motivic homotopy (cf. [Ayo07a]). Actually, the fundamental class $\bar{\eta}_f$ is essentially induced by the purity isomorphism associated with f by the six functors formalism (see Paragraph 2.3.10 for the actual construction).

After realizing that these two methods can actually be formulated in the bivariant language, the main point in the proof of the preceding theorem is to show that "they glue". This can be expressed in a simple equation that we let the reader discover in the key lemma 2.5.1.

From the description of our construction, the uniqueness of the family of classes $\bar{\eta}_f$ is clear: they are the unique formalism that extends the fundamental classes obtained respectively from Navarro's and Ayoub's methods. A more satisfactory statement is obtained if one restricts to quasi-projective local complete intersection morphisms f: then the family of fundamental classes $\bar{\eta}_f$ is uniquely characterized by properties (1), (2), (3), (4) and (6). Actually, we can even replace (4) by the particular case where p is a blow-up: see Theorem 2.6.1.

Riemann-Roch formulas. According to Fulton and MacPherson, one of the motivations for developing bivariant theory was the aim to synthesize several Riemann-Roch formulas (respectively by Grothendieck, by Baum, Fulton and MacPherson, and by Verdier; see [FM81, section 0.1]). The underlying principle is that the classical Chern character corresponds in fact to a natural transformation of bivariant theories, suitably compatible with the structure of a bivariant theory (called a *Grothendieck transformation* in [FM81]). Then the general Riemann-Roch formula essentially comes from the effect of a Grothendieck transformation to a given theory of fundamental classes, as the one given in the above theorem.

[9]The generalization of the method of Gabber to motivic homotopy is non trivial because the Chern classes associated to an oriented ring spectrum are in general non additive: $c_1(L \otimes L') \neq c_1(L) + c_1(L')$. See Prop. 2.2.6.

The same goal has mainly contributed to our choice of framework. Indeed we show how to produce Grothendieck transformations by considering a suitable functor $\varphi^* : \mathscr{T} \to \mathscr{T}'$ — more precisely a premotivic adjunction of motivic triangulated categories in the sense of [CD12b] — an absolute ring $\mathscr{T}$-spectrum $\mathbb{E}$ (resp. $\mathscr{T}'$-spectrum $\mathbb{F}$) and a morphism of absolute ring $\mathscr{T}'$-spectra:

$$\phi : \varphi^*(\mathbb{E}) \to \mathbb{F};$$

see Definition 1.1.4 for details. It is clear from our choice of definition that there is an induced Grothendieck transformation:

$$\phi_* : \mathbb{E}^{BM}_{**}(X/S) \to \mathbb{F}^{BM}_{**}(X/S).$$

Then we can prove the following general Riemann-Roch theorem.

Theorem 2 (see Theorem 3.2.6). *Consider a morphism (φ^*, ϕ) as above and respective orientations c of $\mathbb{E}$ and d of $\mathbb{F}$.*

Then there exists a canonical Todd class *morphism:*

$$\mathrm{Td}_\phi : K_0(X) \to \mathbb{F}^{00}(X)^\times$$

from the Grothendieck group of vector bundles on X to the group of units of $\mathbb{F}$-cohomology classes on X with degree $(0,0)$, natural in X and such that for any global complete intersection morphism $f : X \to S$ with virtual tangent bundle τ_f, the following relation holds:

$$\phi_*(\bar{\eta}^c_f) = \mathrm{Td}_\phi(\tau_f).\, \bar{\eta}^d_f$$

where $\bar{\eta}^c_f$ (resp. $\bar{\eta}^d_f$) is the fundamental class associated with f and $(\mathbb{E}, c)$ (resp. (F, d)) in the above theorem.

At this point of the theory, the proof is straightforward and actually essentially works as the original proof of Grothendieck. But the beauty of our theorem is that it contains all previously known Riemann-Roch formulas as a particular case. We will illustrate this by the concrete applications to come.

Gysin morphisms. Let us quit the realm of general principles now and show the new results that our theory brings. As devised by Fulton and MacPherson, the interest of fundamental classes[10] is that they induce wrong-way morphisms in cohomology, and in fact as we remarked in this paper, in the four theories associated to $\mathbb{E}$ above. We generically call these morphisms *Gysin morphisms*. Here are our main examples of applications.

- *Étale cohomology* (resp. *Borel-Moore étale homology*) with coefficients in a ring Λ is *covariant with respect to proper global complete intersection morphisms* (resp. *contravariant with respect to global complete intersection morphisms*) $f : X \to S$ provided Λ is a torsion ring with exponent invertible on S or $\Lambda = R_\ell, Q_\ell$ where R_ℓ is a complete discrete valuation ring over $\mathbb{Z}_\ell$, $Q_\ell = \mathrm{Frac}(R_\ell)$ and we assume ℓ invertible on S (see respectively Examples 3.3.4 and 3.3.6). This was previously only known for regular closed immersions by [ILO14, chap. 7] or for flat proper morphisms by [SGA4, XVIII, 2.9].

[10] Recall they call them "orientations".

- *Higher Chow groups* are *contravariant with respect to global complete intersection morphisms* $f : X \to S$ provided the residue fields of S have all the same characteristic exponent, say p, and we invert p (see Example 3.3.6). Besides the trivial case of flat morphisms, only the case where X and S are smooth was known.
- *Integral motivic cohomology* in the sense of Spitzweck (cf. [Spi13]) — which implies the case of rational motivic cohomology as defined in [CD12b] — is *covariant with respect to proper global complete intersection morphisms* (see Example 3.3.4). The case of projective morphisms between regular schemes (resp. any scheme) was obtained in [Dég13] (resp. [Nav16]).
- *Betti homology* of complex schemes and its analogue *étale homology* with coefficients in a ring Λ as above (not to be mistaken with Borel-Moore étale homology) are *contravariant with respect to proper global complete intersection morphisms.* This example uses a construction due to A. Khan (see Paragraph 3.3.13).

The general constructions of these Gysin type morphisms are given in Definition 3.3.2 and Paragraph 3.3.13. Note that more than a mere existence theorem we also obtain, as a consequence of the properties of fundamental classes stated in the preceding theorem, all of the expected properties of these Gysin morphisms (see Section 3.3). This also includes Grothendieck-Riemann-Roch formulas for Gysin morphisms (see in particular Proposition 3.3.11). May be it is worth to formulate in this introduction the following new formula, analogue to Verdier's Riemann-Roch formula for homology.

Theorem 3 (See Example 3.3.12)**.** *Let k be a field.*

Let $f : Y \to X$ be a global complete intersection morphisms of k-schemes of finite type. Then we get the following commutative diagram:

$$\begin{array}{ccc} G_n(X) & \xrightarrow{\quad f^* \quad} & G_n(Y) \\ \downarrow{\scriptstyle \mathrm{ch}_X} & & \downarrow{\scriptstyle \mathrm{ch}_Y} \\ \bigoplus_{i\in\mathbb{Z}} CH_i(X,n)_{\mathbb{Q}} & \xrightarrow{\mathrm{Td}(\tau_f).f^*} & \bigoplus_{i\in\mathbb{Z}} CH_i(Y,n)_{\mathbb{Q}} \end{array}$$

where ch *is a Chern character isomorphism,* Td *is the Todd class in rational motivic cohomology (which is acting on higher Chow groups), the upper (resp. lower) map f^* is the Gysin morphism associated with f on Thomason's G-theory, or equivalently Quillen's K'-theory (resp. higher Chow groups).*

Note this theorem makes use of a recent result of Jin (cf. [Jin18]) about the representability of G-theory.

Comparison with previous works in motivic homotopy theory.

Orientation theory in motivic homotopy theory has been developed from two different points of view.

In the first point of view, one works with suitable axioms on certain functors, contravariant (cohomology theories) or covariant (homology theories). Among the axioms are the homotopy invariance property and a suitable orientation property (analogue to the one considered here). Then one deduces properties and constructions from the axioms, among which the construction of Gysin morphisms. This is the approach of Panin ([Pan03, Pan04, Pan09]) on the one hand and Morel and Levine ([LM07]) on the other hand. Note the axiomatic of Panin is cohomological while that of Morel and Levine is homological. Besides, Morel and Levine construct the universal oriented (Borel-Moore) homology theory, the *algebraic cobordism.* Both axiomatic are considered for schemes over a given base field.[11]

In the second point of view, one studies oriented ring spectra in the motivic stable homotopy category, following the classical approach of Adams in algebraic topology. This was suggested by Voevodsky, and first worked out by Vezzosi ([Vez01]) and independently Borghesi ([Bor03]). The construction of Gysin morphisms in this context was done in [Dég08] (see also [Dég02, 8.3, 8.4]). The construction of *loc. cit.* works other an arbitrary base and is internal: for example, it gives Gysin morphisms on the level of Voevodsky motives and allows one to get duality for motives of smooth projective schemes over an arbitrary base ([Dég08, 5.23]), a result that previously uses resolution of singularity (see [VSF00, chap. 5, 4.3.2]). It was later generalized in the works of the author ([Dég14b]) and then in the work of Navarro ([Nav16]).

The two approaches are closely related. On the one hand, the results of Panin, and Levine-Morel, can be applied to the corresponding functor associated with a ring spectrum, giving back Gysin morphisms obtained from the second point of view. They are therefore more general, though most of the known homotopy invariant cohomology (homology) theories in algebraic geometry are known to be representable in the motivic stable homotopy category. A notable exception is the algebraic cobordism of Lowrey and Schürg, a version of Levine-Morel's algebraic cobordism defined even over a field of positive characteristic. In this later case, the corresponding cohomology theory is not known to be representable. On the other hand, the methods used for oriented ring spectra can be adapted to the (cohomological) functorial point of view: see [Dég14b, §6]. Besides the methods are stronger as they produce Gysin morphisms internally, for example on motives or on modules over an oriented ring spectrum.

Compared to these previous works, the contribution of this work is two-fold. Firstly, we associate to any ring spectrum a canonical bivariant theory, called here the Borel-Moore homology — as well as variants with compact support. Secondly, when the ring spectrum is oriented, we associate a system of fundamental classes in the corresponding bivariant theory to a suitable class of local complete intersection morphisms and we show this construction recovers all the previously known Gysin morphisms, both in cohomology and in Borel-Moore homology — it also gives Gysin morphisms for the compactly supported versions. These new Gysin maps are defined for more theories and/or for a larger class of morphisms than before.[12]

[11] More recently, this kind of axiomatic as well as the universal property of algebraic cobordism, has been extended to the framework of derived algebraic geometry by Lowrey and Schürg in [LS16].

[12] In Panin's work, it was defined in cohomology for projective morphisms between smooth schemes over a field. In Levine's work, it was defined in Borel-Moore homology for smooth morphisms between

Note finally that the consideration of bivariant theories in motivic homotopy theory, from the functorial point of view, was also considered by Yokura in [Yok09]. We note here that the bivariant theories associated with an oriented ring spectrum defined in this work (Definition 1.2.2) do satisfies the axioms of [Yok09]. Several definitions of a bivariant algebraic cobordism theory, corresponding to Levine and Morel algebraic cobordism, have been introduced in the literature (see [GK15, Sar15, Yok17]). A comparison of our definition in the case of the ring spectrum **MGL** (see Example 1.2.10) with the definitions of these authors, when restricting to fields of characteristic 0 and to the graded parts $(2n, n)$, is an interesting problem (based on the representability of algebraic cobordism in characteristic 0 proved by Levine in [Lev09]).

Further applications and future works. Almost by definition of the bivariant theory $\mathbb{E}^{BM}_{**}$, there is a categorical incarnation of the fundamental class associated with a global complete intersection morphism $f : X \to S$ in the above Theorem 1, closer to the spirit of Grothendieck six functors formalism; it corresponds to a map:

$$\tilde{\eta}_f : \mathbb{E}_X(d)[2d] \to f^!(\mathbb{E}_S),$$

where d is the relative dimension of f. In the end of this paper, we study conditions under which this map is an isomorphism.[13] In brief, this will always be the case when f is an S-morphism between smooth S-schemes, and in general, it is related to the absolute purity property. Let us also indicate that this property implies several duality statements, in the style of Bloch-Ogus duality giving us finally the link of our work with the last ending point of the historical tree on page 2. We refer the reader to Section 4 for details.

Our main motivation for developing a general theory of fundamental classes, without base field, is the following theorem which we state here using the definitions of [BD17] and [Ros96]:

Theorem. *Let S be a base scheme with a dimension function δ and consider further the following assumptions:*

- *S is any noetherian finite dimensional scheme and Λ is a $\mathbb{Q}$-algebra;*
- *S is a scheme defined over a field of characteristic exponent p and Λ is a $\mathbb{Z}[1/p]$-algebra.*

Then for any motive M in the heart of the δ-homotopy t-structure of $\mathrm{DM}(S, \Lambda)$*, the functor $\hat{H}^\delta_0(M)$ defined in* [BD17] *admits a canonical structure, functorial in M, of a cycle module over S in the sense of Rost* [Ros96].

Indeed, the Gysin morphisms (resp. fundamental classes) constructed here are the essential tool to obtain the corestriction and residue operations of the structure of a cycle module. This work is in progress, and is part of a general strategy to prove an original

quasi-projective schemes over a field. In Navarro's work, it was defined in cohomology for projective local complete intersection morphisms.

[13]Note this condition is stronger than the property of being strong, as defined in [FM81]; see Definition 2.1.6.

conjecture of Ayoub (see [Dég14a]). More generally, this theorem will help us to analyse the Leray-type spectral sequence derived from the δ-homotopy t-structure of [BD17].

Let us finally mention that the techniques of this paper will be exploited in a future work ([DJK18]) whose aim is to define Gysin morphisms in motivic stable homotopy, without requiring the orientation used throughout the present paper. This result is motivated and supported by the fundamental work of Morel which analyses the structure of $\mathbb{A}^1$-homotopy groups ([Mor12]).

Outline of the paper. As a general guideline for the reader, let us mention that Section 1 contains the main definitions and notations used in the paper as well as the examples. We advise the reader to use this section as a reference part. The main part of this work is Section 2 which contains our main results while Section 3 deals with applications.

Let us review the content in more detail. In section 1.1, we settle our framework by introducing absolute ring $\mathscr{T}$-spectra as explained in the above introduction, as well as morphisms between them. In the style of algebraic topology, we also consider modules over ring spectra. Then (section 1.2), we associate to an absolute ring $\mathscr{T}$-spectrum its canonical bivariant theory, called *Borel-Moore homology* as explained in the introduction and show various of its basic properties — note that a variant of the theory is explained for modules over ring spectra. Finally (section 1.3), we define the other bivariant theories associated to absolute ring spectra, cohomology with compact support and homology. Examples are given throughout Section 1.

Our main theorem is developed in Section 2. In sections 2.1 and 2.2, we recall the basic theories of *orientations* for bivariant theories, called here *fundamental classes*, and that of *orientation* for ring spectra, modelled on algebraic topology and giving the first characteristic classes (Chern class and Thom classes). In sections 2.3 and 2.4, we settled the particular cases of fundamental classes that will be used in our main theorem, respectively the case of smooth morphisms and that of regular closed immersions. Our main result (Theorem 1 above) is proved in section 2.5. Section 2.6 deals with finer uniqueness results when restricted to quasi-projective morphisms.

Then Sections 3 and 4 gives applications and properties of the fundamental classes obtained in Section 2. Sections 3.1 and 3.2 are concerned with the main properties of fundamental classes such as behaviour with respects to pullbacks and compatibility with morphisms of ring spectra. Sections 3.3 applies the theory to the construction of Gysin morphisms with an emphasis on concrete examples. Finally in Section 4, we treat the questions of purity (or equivalently *absolute purity* following the classical terminology of the étale formalism) and its relation with duality statements. Again, many examples where purity, and therefore duality, holds are given all along.

Notations and conventions

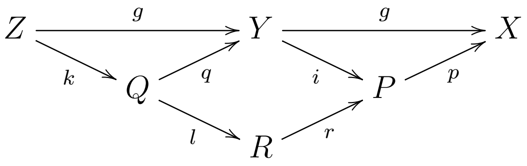
All schemes in this paper are assumed to be noetherian of finite dimension.[14]

[14]Note however that one could work with arbitrary quasi-compact and quasi-separated schemes. Indeed, the stable homotopy category $S\mathscr{H}(S)$ and its six operations, which provides many of our examples, has

We will say that an S-scheme X, or equivalently its structure morphism, is *quasi-projective* (resp. *projective*) if it admits an S-embedding (resp. a closed S-embedding) into $\mathbb{P}^n_S$ for a suitable integer n;[15]

We will also fix a class of morphisms, called *gci*[16] such that the following properties hold:

- any gci morphism admits a factorization into a regular closed immersion followed by a smooth morphism.
- gci morphisms are stable under base change.
- For any composable gci morphisms f and g, one can find a commutative diagram:



  such that i, k, l are regular closed immersions and p, q, r are smooth morphisms.

The main example of such a class is provided by quasi-projective local complete intersection morphisms. One can also fix some base scheme S, restrict to S-schemes which admits an embedding into a smooth S-scheme and work with local complete intersection S-morphisms of such S-schemes.

For short, we will use the term *s-morphism* for separated morphism.

Given a closed (resp. regular closed) subscheme Z of a scheme X, we will denote by $B_Z X$ (resp. $N_Z X$) the blow-up (resp. normal bundle) of Z in X.

In the whole text, $\mathscr{S}$ stands for a sub-category of the category of (noetherian finite dimensional) schemes such that:

- $\mathscr{S}$ is closed under finite sums and pullback along morphisms of finite type.
- For any scheme S in $\mathscr{S}$, any quasi-projective (resp. smooth) S-scheme belongs to $\mathscr{S}$.

The main examples we have in mind are either the category of all schemes or the category of F-schemes for a prime field F.

We will use the axiomatic of Grothendieck six functors formalism and more specifically the richer axioms of motivic triangulated categories introduced in [CD12b].[17] All motivic triangulated categories introduced here will be assumed to be defined over the above fixed category $\mathscr{S}$.

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been recently extended in [Hoy15b, Appendix C] or [Kha16] to the case S quasi-compact and quasi-separated.

[15]For example, if one works with quasi-projective schemes over a noetherian affine scheme (or more generally a noetherian scheme which admits an ample line bundle), then a morphism is proper if and only if it is projective with our convention – use [EGA2, Cor. 5.3.3].

[16]A short for *global complete intersection*.

[17]Recall this axiomatic amounts, up to some minor changes, to the axioms of crossed functors of Ayoub-Voevodsky, [Ayo07a].

From Section 2.2 to the end of the paper, we will make for simplification the assumption that all motivic triangulated categories $\mathscr{T}$ are equipped with a premotivic adjunction:

$$\tau^* : S\mathscr{H} \leftrightarrows \mathscr{T} : \tau_*$$

where $S\mathscr{H}$ is Morel-Voevodsky's stable homotopy category. In fact, this assumption occurs in satisfied in all our main examples and is justified by a suitable universal property (see Remark 1.1.7).

When we will consider the codimension of a regular closed immersion $Z \to X$, the rank of a virtual vector bundle over X or the relative dimension of a gci morphism $X \to Y$, we will understand it as a Zariski locally constant function $d : X \to \mathbb{Z}$. In other words, d is a function which to a connected component X_i of X associates an integer $d_i \in \mathbb{Z}$. To such a function d and to any motivic triangulated category $\mathscr{T}$, we can associate a twist $-(d)$ (resp. shift $-[d]$) on the triangulated category $\mathscr{T}(X)$ by taking twist $-(d_i)$ (resp. shift $_{[}d_i])$ over the component $\mathscr{T}(X_i)$. In that way, we avoid to artificially assume codimensions, ranks or relative dimensions are constant.

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1. Absolute spectra and associated bivariant theories

1.1. **Definition of absolute spectra.** The following notion is a simple extension of [Dég14b, 1.2.1].

Definition 1.1.1. An *absolute spectrum* over $\mathscr{S}$ is a pair $(\mathscr{T}, \mathbb{E})$ where $\mathscr{T}$ is a triangulated motivic category over $\mathscr{S}$ and $\mathbb{E}$ is a cartesian section of the fibered category $\mathscr{T}$ *i.e.* the data:

- for any scheme S in $\mathscr{S}$, of an object $\mathbb{E}_S$ of $\mathscr{T}(S)$,
- for any morphism $f : T \to S$, of an isomorphism $\tau_f : f^*(\mathbb{E}_S) \to \mathbb{E}_T$, called *base change isomorphism*,

and we require that base change isomorphisms are compatible with composition in f as usual. We will also say that $\mathbb{E}$ is an *absolute $\mathscr{T}$-spectrum* and sometimes just absolute spectrum when this does not lead to confusion.

A *ring structure* on $(\mathscr{T}, \mathbb{E})$ will be on each $\mathbb{E}_S$ such that the base change isomorphisms are isomorphisms of monoids. We will say $\mathbb{E}$ is an *absolute ring $\mathscr{T}$-spectrum.*

Given a triangulated motivic category $\mathscr{T}$, the pair $(\mathscr{T}, \mathbb{1})$ where $\mathbb{1}$ is the cartesian section corresponding to the unit $\mathbb{1}_S$ for all schemes S will be called the *canonical absolute $\mathscr{T}$-spectrum*. It obviously admits a ring structure. We will sometime denote it by $\mathbb{H}_{\mathscr{T}}$.

When $\mathscr{T}$ is a Λ-linear category, we will also say *absolute Λ-spectrum*.

Example 1.1.2. (1) Let $\mathscr{S}$ be the category of $\mathbb{Z}[P^{-1}]$-schemes for a set of primes P. Assume $\Lambda = \mathbb{Z}/n\mathbb{Z}$ for n a product of primes in P, or $\Lambda = \mathbb{Z}_\ell, \mathbb{Q}_\ell$ and $P = \{\ell\}$. Then we get the *étale absolute Λ-spectrum* as the canonical absolute spectrum associated with the motivic triangulated category $S \mapsto D^b_c(S_{\text{ét}}, \Lambda)$ — the bounded derived category of Λ-sheaves on the small étale site of S (cf. [SGA4], [Eke90]).

Let now $\mathscr{S}$ be the category of all schemes.

(2) Let Λ be a $\mathbb{Q}$-algebra. The *motivic (Eilenberg-MacLane) absolute Λ-spectrum* $\mathbb{H}\Lambda$ can be defined as the canonical absolute spectrum associated with one of the equivalent version of the triangulated category of rational motives (see [CD12b]). Following a notation of Riou, when $\Lambda = \mathbb{Q}$, this ring spectrum is sometimes denoted by $\mathbf{H}_{\mathrm{B}}$ and called the Beilinson motivic ring spectrum.[18]

(3) Let Λ be any ring. The *étale motivic absolute Λ-spectrum* $\mathbb{H}_{\text{ét}}\Lambda$ can be defined as the canonical absolute spectrum associated with the triangulated motivic category of h-motives of Voevodsky (see [CD15]). When 2 is invertible in Λ, one can also use the étale-local $\mathbb{A}^1$-derived category as defined in [Ayo07b].

When $\Lambda = \mathbb{Z}_\ell$ (resp. $\mathbb{Q}_\ell$) the ring of ℓ-adic integers (resp. rational integers), we will adopt the usual abuse of notations and denote by $\mathbb{H}_{\text{ét}}\Lambda$ the canonical absolute spectrum associated with the homotopy ℓ-completion (resp. rational part of the homotopy l-completion) of the triangulated motivic category of h-motives of Voevodsky (see [CD15, 7.2.1]).

(4) Let $S\mathscr{H}$ be Morel-Voevodsky's stable homotopy category. Then an absolute spectrum $\mathbb{E}$ in the sense of [Dég14b] is an absolute spectrum of the form $(S\mathscr{H}, \mathbb{E})$. This includes in particular the following absolute spectra:
- *algebraic cobordism* **MGL**,
- *Weibel K-theory* **KGL**,
- when Λ is a localization of $\mathbb{Z}$, *motivic cohomology* $\mathbb{H}\Lambda$ *with Λ-coefficients* as defined by Spitzweck (cf. [Spi13]).

We refer the reader to [Dég14b, Ex. 1.2.3 (4,5)] for more details.

When $\mathscr{S}$ is the category of S-schemes for a given scheme S, and $\mathscr{T} = S\mathscr{H}$, any spectrum (resp. ring spectrum) $\mathbb{E}_S$ of $S\mathscr{H}(S)$ gives rise to an absolute spectrum (resp. ring spectrum) $(\mathscr{T}, \mathbb{E})$ by putting for any $f : T \to S$, $\mathbb{E}_T = f^*(\mathbb{E}_S)$. This gives the following classical examples of absolute ring spectra over S-schemes:

(5) $S = \operatorname{Spec}(k)$ for a field k, any mixed Weil cohomology E over k, in the sense of [CD12a];

[18] According to [CD12b, 14.2.14], it represents, over regular schemes, rational motivic cohomology as first defined by Beilinson in terms of rational Quillen K-theory.

(6) $S = \mathrm{Spec}(K)$ for a p-adic field K, the syntomic cohomology with coefficients in K (cf. [DN15]);
(7) $S = \mathrm{Spec}(V)$ for a complete discrete valuation ring V, the rigid syntomic cohomology, with coefficients in K (cf. [DM15]).

Remark 1.1.3. It is not absolutely clear from the literature that the bounded derived category of mixed Hodge modules as defined by Saito satisfies the complete set of axioms of a motivic triangulated category. However, the faithful reader can then consider the associated absolute ring spectrum, an object that should be called the *Deligne absolute ring spectrum.*[19]

Recall that an *adjunction* of motivic triangulated categories (or equivalently a *premotivic adjunction*) is a functor

$$\varphi^* : \mathscr{T} \to \mathscr{T}'$$

of triangulated categories such that $\varphi^*_S : \mathscr{T}(S) \to \mathscr{T}'(S)$ is monoidal and commutes with pullback functors (see [CD12b, 1.4.2]). In particular, given a cartesian section $\mathbb{E}$ of $\mathscr{T}$, we get a cartesian section $\mathbb{F} := \varphi^*(\mathbb{E})$ of $\mathscr{T}'$ by putting $\mathbb{F}_S = \varphi^*_S(\mathbb{E}_S)$.

Definition 1.1.4. A morphism of absolute spectra

$$(\varphi, \phi) : (\mathscr{T}, \mathbb{E}) \to (\mathscr{T}', \mathbb{F})$$

is a premotivic adjunction $\varphi^* : \mathscr{T} \to \mathscr{T}'$ together with a morphism of cartesian sections $\phi : \varphi^*(\mathbb{E}) \to \mathbb{F}$, *i.e.* a family of morphisms $\phi_S : \varphi^*(\mathbb{E}_S) \to \mathbb{F}_S$ compatible with the base change isomorphisms.

A *morphism of absolute ring spectra* is a pair (φ, ϕ) as above such that for any scheme S in $\mathscr{S}$, ϕ_S is a morphism of commutative monoids. In that case, we also say that $\mathbb{F}$ is an $\mathbb{E}$-*algebra*.

Obviously, these morphisms can be composed. Moreover we will say that (φ, ϕ) is an *isomorphism* if φ^* is fully faithful and for all schemes S in $\mathscr{S}$, ϕ_S is an isomorphism. Finally we will say that the isomorphism (φ, ϕ) is strong if the functor φ^* commutes with $f^!$ for any s-morphism f between excellent schemes.

Remark 1.1.5. Note that it usually happens that a motivic triangulated category $\mathscr{T}$ admits a distinguished motivic triangulated subcategory $\mathscr{T}_c$ of constructible objects (see [CD12b, Def. 4.2.1]). According to our definitions, the absolute spectrum associated with $\mathscr{T}$ is then canonically isomorphic to that associated with $\mathscr{T}_c$ because for any scheme S, $\mathbb{1}_S$ is constructible, thus belongs to $\mathscr{T}_c$. It frequently happens that the corresponding isomorphism is strong (see for example [CD12b, 4.2.28], [CD16, 6.2.14], [CD15, 6.4]).

Example 1.1.6. (1) It follows from the previous remark and the rigidity theorems of [Ayo14, CD15] that when $\mathscr{S}$ is the category of $\mathbb{Z}[P^{-1}]$-schemes for a set of primes P and $\Lambda = \mathbb{Z}/n\mathbb{Z}$ where n is a product of primes in P, the étale absolute Λ-spectrum and the étale motivic absolute Λ-spectrum are isomorphic.

[19] A short for: the *absolute ring spectrum representing Deligne cohomology.*

(2) Any adjunction of motivic triangulated categories $\varphi^* : \mathscr{T} \to \mathscr{T}'$ trivially induces a morphism of absolute ring spectra

$$(\varphi, Id) : \mathbb{H}_{\mathscr{T}} \to \mathbb{H}_{\mathscr{T}'}$$

because by definition, φ^*_S is monoidal.

This immediately gives several examples of morphisms of absolute spectra:

- when $\mathscr{S}$ is the category of all schemes, for a prime ℓ, we get

$$\mathbb{H}\mathbb{Q} \to \mathbb{H}_{\text{ét}}\mathbb{Q}_\ell$$

associated to the étale realization functor $\rho_\ell : \mathrm{DM}_{\mathbb{Q}} \to \mathrm{DM}_{\mathrm{h}}(-, \mathbb{Q}_\ell)$ defined in [CD16, 7.2.24].
- when $\mathscr{S}$ is the category of k-schemes for a field k of characteristic p, we get

$$\mathbb{H}\mathbb{Z}[1/p] \to \mathbb{H}_{\text{ét}}\mathbb{Z}_\ell$$

associated to the integral étale realization functors

$$\rho_\ell : \mathrm{DM}_{\mathrm{cdh}}(-, \mathbb{Z}[1/p]) \to D(-_{\text{ét}}, \mathbb{Z}_\ell)$$

defined in [CD15, Rem. 9.6].

(3) Following Riou, we have the Chern character:

$$\mathrm{ch} : \mathbf{KGL}_{\mathbb{Q}} \xrightarrow{\simeq} \bigoplus_{i \in \mathbb{Z}} \mathbb{H}\mathbb{Q}(i)[2i],$$

which is an isomorphism of absolute ring $\mathscr{SH}$-spectra (cf. [Dég14b, 5.3.3]).

Remark 1.1.7. The triangulated motivic category $\mathscr{SH}$ is almost initial. In fact, as soon as a triangulated motivic category is the homotopy category of a combinatorial model stable category, there exists an essentially unique premotivic adjunction[20]:

$$\nu^* : \mathscr{SH} \leftrightarrows \mathscr{T} : \nu_*$$

In this case, for any scheme S, we get a ring spectrum[21] $\mathbb{H}^{\mathscr{T}}_S := \nu_*(\mathbb{1}_S)$ in $\mathscr{SH}(S)$ which represents the cohomology associated with the canonical absolute $\mathscr{T}$-spectrum. The collection $\mathbb{H}^{\mathscr{T}}_S$ indexed by schemes S in $\mathscr{S}$ define a section of the fibered category $\mathscr{SH}$ as for any morphism $f : T \to S$, we have natural maps:

$$\tau^{\mathscr{T}}_f : f^*(\mathbb{H}^{\mathscr{T}}_S) = f^*\nu_*(\mathbb{1}_S) \to \nu_*(f^*\mathbb{1}_S) \simeq \nu_*(\mathbb{1}_S) = \mathbb{H}^{\mathscr{T}}_S$$

compatible with the monoid structure. In general, these maps are not isomorphisms *i.e.* $\mathbb{H}^{\mathscr{T}}$ does not form an absolute $\mathscr{SH}$-spectrum.

Note however that $\tau^{\mathscr{T}}_f$ is an isomorphism when the functor φ_* commutes with pullback functors f^*. Most of the examples given in 1.1.2 will go into this case except for one example, that of the (motivic) étale Λ-spectrum for $\Lambda = \mathbb{Z}_\ell, \mathbb{Q}_\ell$. In fact, in this case we do not know whether the right adjoint of the ℓ-adic realization functor commutes with f^*. This latter case justifies the generality chosen in this paper.

[20] The same result holds in the framework of ∞-category according to [Rob15, 1.2].

[21] Indeed, recall that ν_*, as the right adjoint of a monoidal functor, is weakly monoidal.

Definition 1.1.8. Let $(\mathbb{E}, \mathscr{T})$ be an absolute ring spectrum.

A *module over* $(\mathbb{E}, \mathscr{T})$ will be an absolute spectrum $(\mathbb{F}, \mathscr{T})$ and for any scheme S in $\mathscr{S}$ an associative and unital action

$$\phi_S : \mathbb{E}_S \otimes \mathbb{F}_S \to \mathbb{F}_S$$

which is compatible with the structural base change isomorphisms.

Given a premotivic adjunction $\varphi^* : \mathscr{T} \to \mathscr{T}'$, the cartesian section $\varphi^*(\mathbb{E})$ is again an absolute ring $\mathscr{T}'$-spectrum. A φ-module over $(\mathbb{E}, \mathscr{T})$ will be a module over $(\varphi^*(\mathbb{E}), \mathscr{T})$.

In both cases, when the context is clear, we will simply say $\mathbb{F}$ is an $\mathbb{E}$-module.

Remark 1.1.9. Obvious examples of modules over an absolute ring spectrum $\mathbb{E}$ are given by $\mathbb{E}$-algebras, defined in 1.1.1. We will see many examples of that kind in Paragraph 2.2.12.

1.2. **Associated bivariant theory.**

1.2.1. We now give the the basic definitions of bivariant theories suitable to our needs taken with some small variants from [FM81].

Let us fix $\mathcal{FS}$ the subcategory of the category of arrows in $\mathscr{S}$ whose objects are the s-morphisms and maps are cartesian squares. Usually, an object $f : X \to S$ of $\mathcal{FS}$ will be denoted by X/S when no confusion can arise. Similarly, a morphism Δ:

$$\begin{array}{ccc} Y & \rightarrow & X \\ \downarrow & \Delta & \downarrow \\ T & \xrightarrow[f]{} & S \end{array}$$

will be indicated by the map $f : T \to S$. Let $\mathscr{A}$ be the category of bigraded abelian groups with morphisms the homogeneous ones of degree $(0, 0)$.

A *bivariant theory without products* [22] will be a contravariant functor

$$\mathbb{E} : \mathcal{FS} \to \mathscr{A}, X/S \mapsto \mathbb{E}_{**}(X/S)$$

which is also a covariant functor in X with respect to proper morphisms of S-schemes, and satisfies the following projection formula: for any cartesian squares

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S, \end{array}$$

and any element $x \in \mathbb{E}_{**}(X/S)$ one has: $f^* p_*(x) = p'_* f^*(x)$, as soon as all the maps exist. The structural map $f^* : \mathbb{E}_{**}(X/S) \to \mathbb{E}_{**}(X'/S')$ will be referred to as the *base change map* associated with f.

[22] Products will be introduced in the second part of this introduction. Apart from the absence of products, this notion corresponds to a bivariant theory as in [FM81] where independent squares are cartesian squares and confined maps are proper morphisms.

Given any absolute spectrum $(\mathscr{T},\mathbb{E})$, any s-morphism $p : X \to S$ and any pair $(n,m) \in \mathbb{Z}^2$, we put:

$$\mathbb{E}^{BM}_{n,m}(X/S) := \mathrm{Hom}_{\mathscr{T}(X)}(\mathbb{1}_X(m)[n], p^!(\mathbb{E}_S)).$$

The functoriality of $p^!$ then allows us to defines a contravariant functor from $\mathcal{F}\mathscr{S}$ to bigraded abelian groups and the base change map is given by the pullback functor f^* for a given morphism $f : T \to S$. The covariance with respect to a proper S-morphism $f : Y \to X$ can be defined using the map:

$$\mathbb{E}_X \xrightarrow{ad(f^*,f_*)} f_*f^*(\mathbb{E}_X) \simeq f_*(\mathbb{E}_Y) \simeq f_!(\mathbb{E}_Y).$$

where $ad(f^*, f_*)$ is the unit of the relevant adjunction, the first isomorphism uses the structural isomorphism of the absolute spectrum $\mathbb{E}$ and the last isomorphism follows from the fact f is proper (see [CD12a, 2.4.50(2)]).

It is now a formal exercise to check the axioms of a bivariant theory without products are fulfilled for the bifunctor $\mathbb{E}^{BM}_{**}$.

Definition 1.2.2. Under the assumptions above, the bifunctor $\mathbb{E}^{BM}_{**}$ will be called the *Borel-Moore homology*[23] associated with the absolute spectrum $\mathbb{E}$.

When $\mathbb{E} = \mathbb{H}^{\mathscr{T}}$ is the canonical absolute ring spectrum associated with $\mathscr{T}$, we will denote the corresponding Borel-Moore homology by $H^{BM}_{n,m}(X/S,\mathscr{T})$.

Recall that one associates to a bivariant theory a cohomology theory; in our case, we have:

$$\mathbb{E}^{n,m}(X) = \mathbb{E}^{BM}_{-n,-m}\left(X \xrightarrow{1_X} X\right) = \mathrm{Hom}_{\mathscr{T}(X)}\left(\mathbb{1}_X, \mathbb{E}_X(m)[n]\right)$$

which is the usual formula for the cohomology represented by the spectrum $\mathbb{E}_X$.

1.2.3. According to the six functors formalism, for any étale s-morphism $f : X \to S$, we have a canonical isomorphism of functors $\mathfrak{p}_f : f^! \simeq f^*$ (cf. for example [CD12b, 2.4.50(3)]). Therefore, we also get a canonical isomorphism:

$$\mathbb{E}^{BM}_{n,m}(X/S) \simeq \mathbb{E}^{-n,-m}(X).$$

As the isomorphism $\mathfrak{p}_f$ is compatible with composition (see Proposition 2.3.9), we obtain that $\mathbb{E}^{BM}_{n,m}(X/S)$ is functorial in X with respect to étale morphisms.

Recall now a classical terminology in motivic homotopy theory. We say a cartesian square:

(1.2.3.a)
$$\begin{array}{ccc} Y' & \xrightarrow{k} & X' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow{i} & X, \end{array}$$

is Nisnevich (resp. cdh) distinguished if i is an open (resp. closed) immersion, u is an étale (resp. proper) morphism and the induced map $(X'-Y') \to (X-Y)$ on the underlying reduced subschemes is an isomorphism.

[23] This terminology extends the classical terminology in motivic homotopy theory, usually applied in the case where S is the spectrum of a field. Note that we will see other bivariant theories associated with $\mathbb{E}$ so we have chosen to use that terminology following the tradition of our field.

Now, the following properties are direct consequences of the Grothendieck six functors formalism.

Proposition 1.2.4. *Let $\mathbb{E}$ be an absolute spectrum. The following properties hold:*

(1) Homotopy invariance*.– For any vector bundle $p : E \to S$, and any s-scheme X/S, the base change map:*
$$p^* : \mathbb{E}^{BM}_{n,m}(X/S) \to \mathbb{E}^{BM}_{n,m}(X \times_S E/E)$$
is an isomorphism.

(2) Étale invariance*.– Given any s-schemes $X/T/S$ such that $u : T \to S$ is étale, there exists a canonical isomorphism:*
$$u^* : \mathbb{E}^{BM}_{**}(X/S) \xrightarrow{\sim} \mathbb{E}^{BM}_{**}(X/T)$$
which is natural with respect to base change in S and the covariance in X/T for proper morphisms.

(3) Localization*.– For and s-scheme X/S and any closed immersion $i : Z \to X$ with complementary open immersion $j : U \to X$, there exists a canonical* localization long exact sequence *of the form:*
$$\mathbb{E}^{BM}_{n,m}(Z/S) \xrightarrow{i_*} \mathbb{E}^{BM}_{n,m}(X/S) \xrightarrow{j^*} \mathbb{E}^{BM}_{n,m}(U/S) \xrightarrow{\partial_i} \mathbb{E}^{BM}_{n-1,m}(Z/S)$$
which is natural with respect to the contravariance in S, the contravariance in X/S for étale morphisms and the covariance in X/S for proper morphisms.

(4) Descent property*.– for any square* (1.2.3.a) *of s-schemes over S which is either Nisnevich or cdh distinguished, there exists a canonical long exact sequence:*
$$\mathbb{E}^{BM}_{n,m}(X/S) \xrightarrow{i^*+u^*} \mathbb{E}^{BM}_{n,m}(Y/S) \oplus \mathbb{E}^{BM}_{n,m}(X'/S) \xrightarrow{v^*-k^*} \mathbb{E}^{BM}_{n,m}(Y'/S) \to \mathbb{E}^{BM}_{n-1,m}(X/S)$$
natural with respect to the contravariance in S, the contravariance in X/S for étale morphisms and the covariance in X/S for proper morphisms.

The proof is again an exercise using the properties of the motivic triangulated category $\mathscr{T}$. More precisely: (1) follows from the homotopy property, (2) from the isomorphism $\mathfrak{p}_f : f^! \simeq f^*$ for an étale s-morphism $f : T \to S$, (3) from the localization property (and for the functoriality, from the uniqueness of the boundary operator at the triangulated level, see [CD12b, 2.3.3]), (4) from the Nisnevich and cdh descent properties of $\mathscr{T}$ (see [CD12b, 3.3.4 and 3.3.10]).

Remark 1.2.5. An important remark for this work is the fact that the Borel-Moore homology associated with an absolute spectrum $\mathbb{E}$, restricted to the subcategory of $\mathcal{F}\mathscr{S}$ whose objects are *closed immersions*, coincides with the cohomology with support and coefficients in $\mathbb{E}$. Indeed, this can be seen from the localization property in the case of a closed immersion $i : Z \to S$. And in fact, using the definition of cohomology with support introduced in [Dég14b, 1.2.5], we get an equality:
$$\mathbb{E}^{BM}_{n,m}(i : Z \to S) = \mathbb{E}^{-n,-m}_Z(X).$$

This explains why the properties used in *op. cit.* are exactly the same than the ones of bivariant theories (a fact the author became aware after writing *op. cit.*).

1.2.6. The Borel-Moore homology associated with an absolute spectrum is functorial: given a morphism $(\varphi, \phi) : (\mathscr{T}, \mathbb{E}) \to (\mathscr{T}', F)$ of absolute spectra, and an s-morphism $p : X \to S$, we define $(\varphi, \phi)_*$ — often simply denoted by ϕ_*:

$$\begin{aligned}\mathbb{E}^{BM}_{n,m}(X/S) = \mathrm{Hom}_{\mathscr{T}(X)}(f_!(\mathbb{1}_X)(m)[n], \mathbb{E}_S) &\xrightarrow{\varphi^*} \mathrm{Hom}_{\mathscr{T}'(X)}(\varphi^* f_!(\mathbb{1}_X)(m)[n], \varphi^*\mathbb{E}_S) \\ &\simeq \mathrm{Hom}_{\mathscr{T}'(X)}(f_!(\mathbb{1}'_X)(m)[n], \varphi^*\mathbb{E}_S) \\ &\xrightarrow{\phi_*} \mathrm{Hom}_{\mathscr{T}'(X)}(f_!(\mathbb{1}'_X)(m)[n], \mathbb{F}_S) = \mathbb{F}^{BM}_{n,m}(X/S)\end{aligned}$$

where the isomorphism comes from the exchange isomorphism

$$\varphi^* f_! \xrightarrow{\sim} f_! \varphi^*$$

associated with the premotivic adjunction (φ^*, φ_*) (cf. [CD12b, 2.4.53]).

It is not difficult (using the compatibility of the various exchange transformations involved) to prove $(\varphi, \phi)_*$ is compatible with the base change maps, the covariant functoriality in X/S with respect to proper maps and the contravariant functoriality in X/S with respect to étale maps. In a word, $(\varphi, \phi)_*$ is a natural transformation of bivariant theories without products.

Note moreover that $(\varphi, \phi)_*$ is compatible with composition of morphisms of absolute spectra and an isomorphism of absolute spectra induces an isomorphism of bivariant theories.

Interesting examples will be given later, after the consideration of products.

Remark 1.2.7. Consider the setting of Remark 1.1.7. Assume in addition that the section $\mathbb{H}^{\mathscr{T}}$ is cartesian. Then one gets a morphism of absolute ring spectra (cf. Definition 1.1.4) $(\mathbb{H}^{\mathscr{T}}, S\mathscr{H}) \to (\mathbb{1}, \mathscr{T})$ as for any scheme S, we have morphisms of monoids:

$$\tau^*(\mathbb{H}^{\mathscr{T}}_S) = \tau^*(\tau_*(\mathbb{1}_S)) \xrightarrow{ad'(\tau^*, \tau_*)} \mathbb{1}_S.$$

Note this morphism is not an isomorphism in the sense of 1.1.4. However, the map induced on Borel-Moore homologies of an s-morphism $f : X \to S$

$$(\mathbb{H}^{\mathscr{T}})^{BM}_{**}(X/S) \to H^{BM}_{**}(X/S, \mathscr{T})$$

is an isomorphism as the functor φ^* commutes with direct images $f_!$ (see [CD12b, 2.4.53]).

1.2.8. Recall now that a *bivariant theory* $\mathbb{E}$ in the sense of Fulton and MacPherson[24] is a bivariant theory without products as introduced in 1.2.1 such that for any s-schemes $Y \to X \to S$, there is given a product:

$$\mathbb{E}_{n,m}(Y/X) \otimes \mathbb{E}_{s,t}(X/S) \to \mathbb{E}_{n+s,m+t}(Y/S), (y, x) \mapsto y.x$$

satisfying the following axioms:

[24] More precisely, when independent squares are cartesian squares, confined map are proper morphisms and the category of values is that of bigraded abelian groups;

- *Associativity.–* given s-morphisms $Z/Y/X/S$, for any triple (z, y, x), we have:
$$(z.y).x = z.(y.x).$$
- *Compatibility with pullbacks.–* Given s-morphisms $Y/X/S$ and a morphism $f : S' \to S$ inducing $g : X' \to X$ after pullback along X/S, for any pair (y, x), we have: $f^*(y.x) = g^*(y).f^*(x)$.
- *Compatibility with pushforwards.–* Given s-morphisms $Z \xrightarrow{f} Y \to X \to S$ such that f is proper, for any pair (z, y), one has: $f_*(z.y) = f_*(z).y$.
- *Projection formula (second).–* Given a cartesian square of s-schemes over S:
$$\begin{array}{ccc} Y' & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f} & X \to S, \end{array}$$
such that f is proper, for any pair (y, x), one has: $g_*(f^*(y).x') = y.f_*(x')$.

Consider now an absolute ring spectrum $(\mathscr{T}, \mathbb{E})$. Then one can define a product of the above form on the associated Borel-Moore homology defined in 1.2.2. Consider indeed s-morphisms $Y \xrightarrow{f} X \xrightarrow{p} S$ and classes
$$y : \mathbb{1}_Y(m)[n] \to f^!(\mathbb{E}_X), x : \mathbb{1}_X(s)[t] \to p^!(\mathbb{E}_S).$$
Let us first recall that one gets a canonical pairing[25]
$$Ex^{!*}_{\otimes} : p^*(M) \otimes p^!(N) \to p^!(M \otimes N) \tag{1.2.8.a}$$
obtained by adjunction from the following map:
$$p_!(p^*(M) \otimes p^!(N)) \xrightarrow{\sim} M \otimes p_!p^!(N) \xrightarrow{1 \otimes ad'(p_!,p^!)} M \otimes N.$$
Then one associates to x:
$$\tilde{x} : \mathbb{E}_X(s)[t] \xrightarrow{1_{\mathbb{E}_X} \otimes x} \mathbb{E}_X \otimes p^!(\mathbb{E}_S) \simeq p^*(\mathbb{E}_S) \otimes p^!(\mathbb{E}_S) \xrightarrow{Ex^{*!}_{\otimes}} p^!(\mathbb{E}_S \otimes \mathbb{E}_S) \xrightarrow{p^!(\mu)} p^!(\mathbb{E}_S)$$
where the map μ is the multiplication map of the ring spectrum $\mathbb{E}_S$. Then, one defines the product as the following composite map:
$$y.x : \mathbb{1}_Y(m+s)[n+t] \xrightarrow{y(s)[t]} f^!(\mathbb{E}_X)(s)(t] \xrightarrow{f^!(\tilde{x})} f^!p^!(\mathbb{E}_S) = (pf)^!(\mathbb{E}_S).$$
This is now a lengthy exercise to prove that the axioms stated previously are satisfied for the product just defined and the bifunctor $\mathbb{E}^{BM}_{**}$. We refer the reader to [Dég14b, proof of 1.2.10] for details[26].

Remark 1.2.9. (1) Products on bivariant theories obviously induce products on the associated cohomology theory. In the case of the bivariant theory of the above definition these induced products are nothing else than the usual cup-products.

[25]This is classical: see also [SGA4.5, IV, 1.2.3].

[26]the corresponding axioms are proved in *loc. cit.* when confined maps are closed immersions (see Remark 1.2.5). In fact the proof does not change when confined maps are only assumed to be proper rather than being closed immersions.

(2) If one considers the case $Y = X$, the product of the preceding paragraph gives an action of the cohomology of X on the Borel-Moore homology of X/S.
(3) We have seen in Remark 1.2.5 that cohomology with support is a particular instance of Borel-Moore homology. In fact, the product introduced in [Dég14b, 1.2.8] for cohomology with supports coincides with the one defined here restricted to Borel-Moore homology of closed immersions. This is obvious as the formulas in each case are exactly the same.

Example 1.2.10. From the examples of 1.1.2, we get respectively the Borel-Moore motivic Λ-homology, the Borel-Moore étale motivic Λ-homology, the *Borel-Moore homotopy invariant K-theory* and the *Borel-Moore algebraic cobordism*:

$$\mathbb{H}^{BM}_{**}(X/S,\Lambda), \mathbb{H}^{BM,\text{ét}}_{**}(X/S,\Lambda), \mathbf{KGL}^{BM}_{**}(X/S), \mathbf{MGL}^{BM}_{**}(X/S).$$

(1) The Borel-Moore motivic homology $\mathbb{H}^{BM}_{**}(X/S,\Lambda)$ could also be called *bivariant higher Chow groups*. In fact, it follows from [CD15, 8.13] that for any s-scheme X/k where k is a field of characteristic exponent p, one has a canonical isomorphism:

$$\varphi : \mathbb{H}^{BM}_{n,m}(X/k,\mathbb{Z}[1/p]) \xrightarrow{\sim} CH_m(X, n-2m)[1/p]$$

where the right hand side is the relevant Bloch's higher Chow group.[27]
(2) It follows from 1.1.6(1) that when $\mathscr{S}$ is the category of $\mathbb{Z}[P^{-1}]$-schemes and $\Lambda = \mathbb{Z}/n\mathbb{Z}$ with n a product of primes in P, the Borel-Moore étale motivic Λ-homology $\mathbb{H}^{\text{ét}}_{**}(X/S,\Lambda)$ coincides with the bivariant étale theory with Λ-coefficients as considered in [FM81, 7.4].
(3) Recall that the absolute ring spectrum **KGL** satisfies Bott periodicity. In particular we get a canonical isomorphism:

$$\mathbf{KGL}^{BM}_{n,m}(X/S) \simeq \mathbf{KGL}^{BM}_{n+2,m+1}(X/S).$$

In particular, the double indexing is superfluous and we sometime consider bivariant K-theory as $\mathbb{Z}$-graded according to the formula:

$$\mathbf{KGL}^{BM}_{n,m}(X/S) \simeq \mathbf{KGL}^{BM}_{n-2m}(X/S).$$

In general, this bivariant K-theory does not coincide with the bivariant K-theory $\mathrm{K}_{\mathrm{alg}}$ of [FM81, 1.1]. Indeed the theory of *loc. cit.* does not satisfies the homotopy property (Prop. 1.2.4) when considering non regular schemes (as algebraic K-theory).

Note however that according to [Jin18], one gets a canonical isomorphism:

$$\mathbf{KGL}^{BM}_{n}(X/S) \simeq G_n(X)$$

for a quasi-projective morphism $f : X \to S$ with S regular, where G_* is Thomason's G-theory, or equivalently Quillen's K'-theory as we work with noetherian schemes

[27] In fact, the assumption that X is equidimensional in *loc. cit.* can be avoided as follows: coming back to the proof of Voevodsky in [VSF00, chap. 5, 4.2.9], one sees that in any case the map φ exists — it is induced by and inclusion of groups of cycles. Therefore, to prove it is an isomorphism, we reduce to the equidimensional case by noetherian induction, as the map φ is compatible with the localization sequence (see the proof of *loc. cit.*).

(see [TT90, 3.13]). The isomorphism of Jin is functorial with respect to proper covariance and étale contravariance.

Remark 1.2.11. In general, there should exist a natural transformation of bivariant theories:

$$K_{\mathrm{alg},n}(X \to S) \to \mathbf{KGL}_n^{BM}(X/S)$$

which extends the known natural transformations on associated cohomologies and which is compatible with the Chern character with values in the motivic bivariant rational theory (see below).

1.2.12. Let $(\varphi, \phi) : (\mathscr{T}, \mathbb{E}) \to (\mathscr{T}', \mathbb{F})$ be a morphism of ring spectra (Definition 1.1.4). Then one checks that the associated natural transformation of bivariant theories defined in Paragraph 1.2.6

$$\phi_* : \mathbb{E}_{n,m}^{BM}(X/S) \to \mathbb{F}_{n,m}^{BM}(X/S)$$

is compatible with the product structures on each Borel-Moore homology (Paragraph 1.2.8).[28] So in fact, ϕ_* is a *Grothendieck transformation* in the sense of [FM81, I. 1.2].

Note that this natural transformation then formally induces a natural transformation on cohomology theories, compatible with cup-products:

$$\phi_* : \mathbb{E}^{n,m}(X) \to \mathbb{F}^{n,m}(X),$$

as usual. This construction gives many interesting examples.

Example 1.2.13. (1) Let ℓ be a prime number. Assume one of the following settings:
- $\mathscr{S}$ is the category of all schemes, $\Lambda = \mathbb{Q}$, $\Lambda_\ell = \mathbb{Q}_\ell$;
- $\mathscr{S}$ is the category of schemes over a prime field F with characteristic exponent p such that $\ell \neq p$, $\Lambda = \mathbb{Z}[1/p]$ and $\Lambda_\ell = \mathbb{Z}_\ell$.

Then, one gets from Example 1.1.6(2) a natural transformation of bivariant theories:

$$\mathbb{H}_{**}^{BM}(X/S, \Lambda) \xrightarrow{\sim} \mathbb{H}_{**}^{BM,\text{ét}}(X/S, \Lambda_\ell)$$

whose associated natural transformation on cohomology is the (higher) cycle class map in étale ℓ-adic cohomology.

(2) Assume $\mathscr{S}$ is the category of all schemes. Then one gets from 1.1.6(3) a higher bivariant Chern character:

$$ch_n : \mathbf{KGL}_n^{BM}(X/S)_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus_{i \in \mathbb{Z}} \mathbb{H}_{2i+n,i}^{BM}(X/S, \mathbb{Q})$$

which is in fact a Grothendieck transformation in the sense of [FM81, I, 1.2]. From [Dég14b, 5.3.3], it coincides with Gillet's higher Chern character on the associated cohomology theories. Therefore, it extends Fulton and MacPherson Chern character [FM81, II. 1.5], denoted in *loc. cit.* by τ.

Suppose S is a regular scheme and X/S is an s-scheme. Given the result of Jin (Remark 1.2.11) and Riou ([Rio10]), one gets Adams operations ψ^i on Thomason's

[28] This comes again as the exchange transformations involved in the functoriality and in the products are compatible.

G-theory $G_n(X)$ and the above isomorphism identifies $\mathbb{H}^{BM}_{2i+n,i}(X/S,\mathbb{Q})$ with the eigenvector space of $G_n(X)_\mathbb{Q}$ for the eigenvalue r^i of ψ^i, $r \neq 0$ being a fixed integer.

Finally, when $S = \mathrm{Spec}(k)$ is the spectrum of a field, from Example 1.2.10(2) the above higher Chern character can be written as an isomorphism:

$$ch_n : G_n(X)_\mathbb{Q} \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} CH_i(X,n) \otimes \mathbb{Q}$$

for any s-scheme X/k.

More examples will be given in the section dealing with orientations.

Remark 1.2.14. Let us observe finally that the product structure on the Borel-Moore homology associated with an absolute ring spectrum $(\mathbb{E},\mathscr{T})$ can be extended to the setting of modules over ring spectra. Indeed, given a premotivic adjunction $\varphi^* : \mathscr{T} \to \mathscr{T}'$ and a φ-module $\mathbb{F}$ over $(\mathbb{E},\mathscr{T})$ with structural maps ϕ_S as in Definition 1.1.8, we get a product:

$$\mathbb{E}^{BM}_{n,m}(Y/X) \otimes \mathbb{F}^{BM}_{s,t}(X/S) \to \mathbb{F}^{BM}_{n+s,m+t}(Y/S), (y,x) \mapsto y.x,$$

using the construction of Paragraph 1.2.8. Let us be more explicit. First we remark that we have a Grothendieck transformation from the Borel-Moore homology represented by $(\mathbb{E},\mathscr{T})$ to that represented by $(\varphi^*(\mathbb{E}),\mathscr{T}')$, according to 1.2.12. Thus, we can replace $\mathbb{E}$ by $\varphi^*(\mathbb{E})$ to describe the above product. In other words, we can assume $\mathscr{T} = \mathscr{T}'$, $\varphi^* = Id$. Then, given classes:

$$y : \mathbb{1}_Y(m)[n] \to f^!(\mathbb{E}_X), x : \mathbb{1}_X(t)[s] \to p^!(\mathbb{F}_S)$$

one associates to x the following map:

$$\tilde{x} : \mathbb{E}_X(t)[s] \xrightarrow{1_{\mathbb{E}_X}\otimes x} \mathbb{E}_X \otimes p^!(\mathbb{F}_S) \simeq p^*(\mathbb{E}_S) \otimes p^!(\mathbb{F}_S) \xrightarrow{Ex^{*!}_{\otimes}} p^!(\mathbb{E}_S \otimes \mathbb{F}_S) \xrightarrow{p^!(\nu_S)} p^!(\mathbb{F}_S)$$

using the pairing (1.2.8.a) and the structural map ν_S of the module $\mathbb{F}_S$ over $\mathbb{E}_S$. Then one defines the product as the following composite map:

$$y.x : \mathbb{1}_Y(m+t)[n+s] \xrightarrow{y(t)[s]} f^!(\mathbb{E}_X)(t)[s] \simeq f^!(\mathbb{E}_X(t)[s]) \xrightarrow{f^!(\tilde{x})} f^!p^!(\mathbb{E}_S) = (pf)^!(\mathbb{F}_S).$$

Similarly, we also define a right action:

$$\mathbb{F}^{BM}_{n,m}(Y/X) \otimes \mathbb{E}^{BM}_{s,t}(X/S) \to \mathbb{F}^{BM}_{n+s,m+t}(Y/S).$$

It is straightforward to check these two products satisfy the associativity, compatibility with pullbacks and pushforwards, and projection formula like the products in bivariant theories (cf. Paragraph 1.2.8).

1.3. **Proper support.**

1.3.1. Let $(\mathbb{E},\mathscr{T})$ be an absolute spectrum and $p : X \to S$ be an s-morphism. The six functors formalism gives us two other theories which depend on X/S as follows:

$$\mathbb{E}^{n,m}_c(X/S) = \mathrm{Hom}_{\mathscr{T}(S)}\big(\mathbb{1}_S, p_!(\mathbb{E}_X)(m)[n]\big),$$
$$\mathbb{E}_{n,m}(X/S) = \mathrm{Hom}_{\mathscr{T}(S)}\big(\mathbb{1}_S(m)[n], p_!p^!(\mathbb{E}_X)\big).$$

Using the same techniques as in Paragraph 1.2.1, one gets the following functoriality:

- $\mathbb{E}_c^{n,m}(X/S)$ is contravariant in X/S with respect to cartesian squares, contravariant in X with respect to proper S-morphisms and covariant in X with respect to étale S-morphisms;
- $\mathbb{E}_{n,m}(X/S)$ is contravariant in X/S with respect to cartesian squares, covariant in X with respect to all S-morphisms and contravariant in X with respect to finite S-morphisms.

So in each cases, $\mathbb{E}_c^{**}$ and E_{**} are contravariant functors from the category $\mathscr{F}\mathscr{S}$ (see 1.2.1) to the category of bigraded abelian groups. In fact, they are bivariant theories without products where independent squares are the cartesian squares and confined maps are respectively the proper morphisms and the étale morphisms.

Definition 1.3.2. Given the notations above, the functor $\mathbb{E}_c^{**}$ (resp. $\mathbb{E}_{**}$) will be called the *cohomology with compact support* (resp. *homology*) associated with $\mathbb{E}$.

Example 1.3.3. These notions were classically considered when the base S is the spectrum of a field k. Let p be the characteristic exponent of k.

(1) When $\mathbb{E}$ is the absolute Λ-spectrum of étale cohomology as in 1.1.2(1), our formulas for X/k gives the classical étale cohomology with support.
(2) More generally, when $\mathbb{E}$ is the spectrum associated with a mixed Weil theory over k as in 1.1.2(5), one recovers the classical notion of the corresponding cohomology with compact support (eg. Betti, De Rham, rigid). See also Corollary 1.3.5.
(3) When $k = \mathbb{C}$ (or more generally, one has a given embedding of k into $\mathbb{C}$), and $\mathbb{E} = \mathbb{H}_B$ is the spectrum representing Betti cohomology with integral coefficients,[29] one gets an isomorphism:

$$(\mathbb{H}_B)_{n,m}(X/k) = H_n^{sing}(X(\mathbb{C}), \mathbb{Z})$$

which is canonical if $m = 0$ and only depends on the choice of a trivialization of $\mathbb{H}_B^{1,1}(\mathbb{G}_m)$ if $m > 0$.

Indeed, one obtains using Grothendieck-Verdier duality for the triangulated motivic category $D(-, \mathbb{Z})$ that when $p : X \to k$ is the canonical projection, the complex

$$p_! p^!(\mathbb{1}_k)$$

is the dual of the complex $p_* p^*(\mathbb{1}_k)$ which is quasi-isomorphic to $C_*^{sing}(X(\mathbb{C}))$ by definition. So the result follows from the classical definition of singular homology.[30]
(4) When $\mathbb{E}$ is the absolute motivic $\mathbb{Z}[1/p]$-spectrum, it follows from [CD15, 8.7] that for any s-scheme X/k and any integer $n \in \mathbb{Z}$, one gets a canonical isomorphism:

$$\mathbb{H}_{n,0}(X/k, \mathbb{Z}[1/p]) \simeq H_n^{Sus}(X)[1/p]$$

[29] Either as the one obtained through the corresponding Mixed Weil theory or as the canonical spectrum associated with the motivic triangulated category $X \mapsto D(X(\mathbb{C}), \mathbb{Z})$ of [Ayo10, section 1]. For the fact these two versions give the same answer, see [CD12a, 17.1.7].

[30] One could also use Corollary 1.3.5 to conclude here.

where the left hand side is the homology in the above sense associated with the motivic absolute spectrum $\mathbb{H}\Lambda$ and the right hand side is Suslin homology (cf. [SV96]).[31]

Let us collect some properties of these two new types of bivariant theories.

Proposition 1.3.4. *Let $\mathbb{E}$ be an absolute spectrum. The following properties hold:*

(1) Homotopy invariance.*– For any s-scheme X/S and any vector bundle $p : E \to X$, the push-forward map in bivariant homology:*

$$p_* : \mathbb{E}_{**}(E/S) \to \mathbb{E}_{**}(X/S)$$

is an isomorphism.

(2) Proper invariance.*– Given any s-schemes $X/T/S$ such that T/S is proper, there exists a canonical isomorphism:*

$$\mathbb{E}_c^{**}(X/S) \xrightarrow{\sim} \mathbb{E}_c^{**}(X/T)$$

which is natural with respect to the functorialities of compactly supported cohomology (cf. 1.3.1).

(3) Comparison.*– For any s-scheme X/S one has natural transformations:*

$$\mathbb{E}_c^{n,m}(X/S) \to \mathbb{E}^{n,m}(X),$$
$$\mathbb{E}_{n,m}(X/S) \to \mathbb{E}^{BM}_{n,m}(X/S)$$

which are isomorphisms when X/S is proper.

(4) Localisation.*– For any s-scheme X/S and any closed immersion $i : Z \to X$ with complementary open immersion $j : U \to X$, there exists a canonical* localization long exact sequence *of the form:*

$$\mathbb{E}_c^{n,m}(U/S) \xrightarrow{j_*} \mathbb{E}_c^{n,m}(X/S) \xrightarrow{i^*} \mathbb{E}_c^{n,m}(Z/S) \to \mathbb{E}_c^{n+1,m}(U/S)$$

which is natural with respect to the functorialities of compactly supported cohomology (cf. 1.3.1).

Property (1) follows from the homotopy invariance of the category $\mathscr{T}$, which implies that the adjunction map $p_!p^! \to 1$ is an isomorphism. Property (2) and (3) follows from the existence, for $p : X \to S$, of the natural transformation of functors

$$\alpha_p : p_! \to p_*$$

which is an isomorphism when p is proper. Property (4) is a direct translation of the existence of the localization triangle $j_!j^* \to 1 \to i_!i^* \xrightarrow{+1}$.

The following corollary justifies our terminology.

[31] Note that though Suslin homology is defined for s-morphisms X/S, it does not seem that the above identification extends to cases where S is of positive dimension.

Corollary 1.3.5. *Consider an S-scheme X with an open S-immersion $j : X \to \bar{X}$ such that $\bar{X}$ is proper over S. Let X_∞ be the reduced complement of j, $i : X_\infty \to \bar{X}$ the corresponding immersion.*

Then one has canonical long exact sequences:

$$\mathbb{E}^{n-1,m}(\bar{X}) \xrightarrow{i^*} \mathbb{E}^{n-1,m}(X_\infty) \to \mathbb{E}_c^{n,m}(X/S) \to \mathbb{E}^{n,m}(\bar{X}) \xrightarrow{i^*} \mathbb{E}^{n,m}(X_\infty),$$

$$\mathbb{E}_{n,m}(X_\infty/S) \xrightarrow{i_*} \mathbb{E}_{n,m}(\bar{X}/S) \to \mathbb{E}_{n,m}^{BM}(X/S) \to \mathbb{E}_{n-1,m}(X_\infty/S) \xrightarrow{i_*} \mathbb{E}_{n-1,m}(\bar{X}/S).$$

Indeed, the first long exact sequence is obtained from points (3) and (4) of the preceding proposition and the second one from Proposition 1.2.4(4) and point (3) of the previous proposition.

Remark 1.3.6.

1. The first long exact sequence gives us the usual way to get compactly supported cohomology out of a compactification, which can also be interpreted as a canonical isomorphism:
$$\mathbb{E}_c^{n,m}(X/S) \simeq \mathbb{E}^{n,m}(\bar{X}, X_\infty)$$
where the right hand side is the cohomology of the pair $(\bar{X}, X_\infty)$ (as classically considered in algebraic topology). The second long exact sequence is less usual and gives a way to get back Borel-Moore homology from homology. In fact, it gives us an interpretation of Borel-Moore homology as the compactly supported theory associated with homology.
2. Homology and cohomology with compact support also admit *descent long exact sequences* with respect to Nisnevich and cdh distinguished squares as in 1.2.4(4). We leave the formulation to the reader.

1.3.7. Assume finally that $(\mathbb{E}, \mathscr{T})$ has a ring structure.

Then one can define a product, for s-morphisms $Y \xrightarrow{f} X \xrightarrow{p} S$,

$$\mathbb{E}_c^{n,m}(Y/X) \otimes \mathbb{E}_c^{s,t}(X/S) \to \mathbb{E}_c^{n+s,m+t}(Y/S), (y, x) \mapsto y.x,$$

so that the functor $\mathbb{E}_c^{**}$ becomes a bivariant theory in the sense of Fulton and MacPherson (cf. 1.2.8). Indeed, given classes

$$y : \mathbb{1}_X(m)[n] \to f_!(\mathbb{E}_Y), x : \mathbb{1}_S(t)[s] \to p_!(\mathbb{E}_S)$$

we define

$$y' : \mathbb{E}_X(t)[s] \xrightarrow{1_{\mathbb{E}_X} \otimes y} \mathbb{E}_X \otimes f_!(\mathbb{E}_Y) \xrightarrow{PF} f_!(f^*\mathbb{E}_X \otimes \mathbb{E}_Y) \simeq f_!(\mathbb{E}_Y \otimes \mathbb{E}_Y) \xrightarrow{\mu} f_!(\mathbb{E}_Y)$$

(where PF stands for projection formula) and then

$$y.x : \mathbb{1}_S(m+t)[n+s] \xrightarrow{x} p_!(\mathbb{E}_S)(t)[s] \simeq p_!(\mathbb{E}_S(t)[s]) \xrightarrow{p_!(y')} p_!f_!(\mathbb{E}_Y) = (pf)_!(\mathbb{E}_Y).$$

Again the formulas required for the product of a bivariant theory (cf. 1.2.8) follow from the six functors formalism.

Such a product does not exist on the bivariant theory without products $\mathbb{E}_{**}$ defined above. Instead one can define an exterior product:

$$\mathbb{E}_{**}(X/S) \otimes \mathbb{E}_{**}(Y/S) \to \mathbb{E}_{**}(X \times_S Y/S).$$

using the following pairing:

$$\begin{aligned} p_!p^!(\mathbb{E}_S) \otimes q_!q^!(\mathbb{E}_S) &\xrightarrow{PF} p_!\big(p^!(\mathbb{E}_S) \otimes p^*q_!q^!(\mathbb{E}_S)\big) \xrightarrow{BC} p_!\big(p^!(\mathbb{E}_S) \otimes q'_!p'^*q^!(\mathbb{E}_S)\big) \\ &\xrightarrow{PF} p_!q'_!\big(q'^*p^!(\mathbb{E}_S) \otimes p'^*q^!(\mathbb{E}_S)\big) \simeq a_!\big(q'^*p^!(\mathbb{E}_S) \otimes p'^*q^!(\mathbb{E}_S)\big) \\ &\xrightarrow{Ex^{*!}} a_!\big(q'^*p^!(\mathbb{E}_S) \otimes q'^!p^*(\mathbb{E}_S)\big) \xrightarrow{(1.2.8.a)} a!a^!(\mathbb{E}_S \otimes \mathbb{E}_S) \xrightarrow{\mu} a!a^!(\mathbb{E}_S) \end{aligned}$$

(where BC stands for base change formula) for a cartesian square of s-morphisms

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p'} & Y \\ {\scriptstyle q'}\downarrow & \searrow^{a} & \downarrow{\scriptstyle q} \\ X & \xrightarrow{p} & S. \end{array}$$

One can check this product is associative, compatible with pushforwards and base changes (we will not use these properties).

In a more original way, one can define the following *cap-product*, pairing of bivariant theories:

(1.3.7.a) $$\mathbb{E}_c^{**}(X/S) \otimes \mathbb{E}_{**}^{BM}(X/S) \to \mathbb{E}_{**}(X/S), (a, b) \mapsto a \cap b$$

which, using the preceding notations, is induced by the following pairing of functors:

$$\begin{aligned} p_*p^!(\mathbb{E}_S) \otimes p_!p^*(\mathbb{E}_S) &\xrightarrow{PF} p_!\big(p^*p_*p^*(\mathbb{E}_S) \otimes p^*(\mathbb{E}_S)\big) \xrightarrow{ad'(p^*,p_*)} p_!\big(p^*(\mathbb{E}_S) \otimes p^*(\mathbb{E}_S)\big) \\ &\xrightarrow{(1.2.8.a)} p_!p^!(\mathbb{E}_S \otimes \mathbb{E}_S) \xrightarrow{\mu} p_!p^!(\mathbb{E}_S). \end{aligned}$$

Remark 1.3.8. Our definition of cap-product is an extension of the classical definition of cap-product, between cohomology and homology, as defined for classical spectra ([Ada64, III, §9]). In fact, when S is the spectrum of a field k and X/k is projective, it also coincides with the cap-product appearing in Bloch-Ogus axioms [BO74].

2. Fundamental classes

2.1. **Abstract fundamental classes.** Let us recall the following basic definitions from [FM81].

Definition 2.1.1. Let $\mathbb{E}$ be an absolute ring spectrum and $f : X \to S$ be an s-morphism.

An *orientation* for the morphism f with coefficients in $\mathbb{E}_{**}^{BM}$ will be the choice of an element $\eta_f \in \mathbb{E}_{**}^{BM}(X/S)$ in the bivariant theory associated with $\mathbb{E}$ (Def. 1.2.2).

Given a locally constant function $d : X \to \mathbb{Z}$, with values $d(i)$ on the connected components X_i of X, for $i \in I$, we will say that η_f has dimension d if it belongs to the group:

$$\bigoplus_{i \in I} \mathbb{E}_{2d(i),d(i)}^{BM}(X_i/S).$$

Accordingly, we introduce the following notation for any $\mathscr{T}$-spectrum $\mathbb{F}$ over X:

$$\mathbb{F}(d)[2d] = \bigoplus_{i \in I} \mathbb{F}|_{X_i}(d(i))[2d(i)].$$

Then one defines an *orientation of degree d* as a map:

$$\eta_f : \mathbb{E}_X(d)[2d] \to f^!\mathbb{E}_S.$$

Remark 2.1.2. The word orientation in the context of the previous definition has been chosen by Fulton and MacPherson in [FM81]. We will also use the terminology *fundamental class* for such an orientation when it is part of a coherent system of orientations: see Definition 2.1.9. In our main example, the choice of an orientation in the sense of $\mathbb{A}^1$-homotopy theory (see Definition 2.2.2) will indeed canonically determine such a coherent system.

Example 2.1.3. Consider any absolute ring spectrum $\mathbb{E}$. Then any étale s-morphism $f : X \to S$ admits a canonical orientation $\bar{\eta}_f$ of degree 0. Take:

(2.1.3.a) $$\bar{\eta}_f : \mathbb{1}_X \xrightarrow{\eta_X} \mathbb{E}_X \xrightarrow{\tau_f^{-1}} f^*(\mathbb{E}_S) \xrightarrow{\mathfrak{p}_f^{-1}} f^!(\mathbb{E}_S)$$

where η_X is the unit of the ring spectrum $\mathbb{E}_S$, τ_f is the base change isomorphism (Def. 1.1.1) and $\mathfrak{p}_f$ is the purity isomorphism of the six functors formalism (see 1.2.3).

Remark 2.1.4. As the previous example is a basic piece of our main result, Theorem 2.5.3, we recall the definition of the isomorphism $\mathfrak{p}_f$ of the above example. We consider the pullback square:

$$\begin{array}{ccccc} X & \xrightarrow{\delta} & X \times_S X & \xrightarrow{f''} & X \\ & \searrow^{=} & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow{f} & S \end{array}$$

where δ is the diagonal immersion, which is both open and closed according to our assumptions on f (étale and separated). Then we define $\mathfrak{p}_f$ as follows:

$$f^* \simeq \delta^! f''^! f^* \xrightarrow{Ex^{!*}} \delta^! f'^* f^! \xrightarrow{(1)} \delta^* f'^* f^! \simeq f^!.$$

To get the isomorphism (1), we come back to the construction of exceptional functors following Deligne (see [CD12b, 2.2]). Indeed, as δ is an open immersion, we get a canonical identification $\delta_! \simeq \delta_\sharp$ of functors so that we get a canonical isomorphism $\delta^! \simeq \delta^*$ of their respective right adjoints as required.

2.1.5. Consider an absolute ring spectrum $\mathbb{E}$, and an orientation η_f of an s-morphism $f : X \to S$.

Given an s-scheme Y/X and using the product of the Borel-Moore $\mathbb{E}$-homology, one can associate to η_f a map:

$$\delta(Y/X, \eta_f) : \mathbb{E}^{BM}_{**}(Y/X) \to \mathbb{E}^{BM}_{**}(Y/S), y \mapsto y.\eta_f.$$

Note that going back to the definition of this product (Par. 1.2.8), this map can be described up to shift and twist as the composition on the left with the following morphism of $\mathscr{T}(X)$:

(2.1.5.a) $$\tilde{\eta}_f : \mathbb{E}_X(*)[*] \xrightarrow{1_{\mathbb{E}_X} \otimes \eta_f} \mathbb{E}_X \otimes f^!(\mathbb{E}_S) \simeq f^*(\mathbb{E}_S) \otimes f^!(\mathbb{E}_S) \xrightarrow{Ex^{*!}_{\otimes}} f^!(\mathbb{E}_S \otimes \mathbb{E}_S) \xrightarrow{\mu} f^!(\mathbb{E}_S).$$

Definition 2.1.6. Consider the above assumptions. One says that the orientation η_f is

- *strong* if for any s-scheme Y/X, the map $\delta(Y/X, \eta_f)$ is an isomorphism.
- *universally strong* if the morphism $\tilde{\eta}_f$ is an isomorphism in $\mathscr{T}(X)$.

As remarked in [FM81], a strong orientation of X/S is unique up to multiplication by an invertible element in $\mathbb{E}^{0,0}(X)$. The notion of universally strong is new, as it makes sense only in our context. Obviously, universally strong implies strong according to Paragraph 2.1.5.

Remark 2.1.7. Consider the notations of the above definition.

(1) The property of being universally strong for an orientation η_f as above implies that for any smooth morphism $p : T \to S$, the orientation $p^*(\eta_f)$ of $f \times_S T$ is strong — this motivates the name. We will see more implications of this property in Section 4.

(2) The data of the orientation $\bar{\eta}_f$ is equivalent to the data of the map $\tilde{\eta}_f$ as the map $\bar{\eta}_f$ is equal to the following composite:

$$\mathbb{1}_X(*)[*] \xrightarrow{\eta_X} \mathbb{E}_X(*)[*] \xrightarrow{\tilde{\eta}_f} f^!\mathbb{E}_S$$

where η_X is the unit of the ring spectrum $\mathbb{E}_X$.

Example 2.1.8. Consider the notations of the previous definition and assume that f is étale as in Example 2.1.3.

It follows from point (2) of the preceding remark that the map $\tilde{\eta}_f$ associated with the orientation $\bar{\eta}_f$ of the latter example is equal to the following composite morphism:

$$\tilde{\eta}_f : \mathbb{E}_X \xrightarrow{\tau_f^{-1}} f^*(\mathbb{E}_S) \xrightarrow{\mathfrak{p}_f^{-1}} f^!(\mathbb{E}_S),$$

using the notations of the example. Thus, $\tilde{\eta}_f$ is an isomorphism: the canonical orientation η_f of an étale morphism f is universally strong.

Definition 2.1.9. Given a class $\mathscr{C}$ of morphisms of schemes closed under composition, a *system of fundamental classes* for $\mathscr{C}$ with coefficients in $\mathbb{E}$ will be the datum for any $f \in \mathscr{C}$ of an orientation $\eta_f^{\mathscr{C}}$ such that for any composable maps $Y \xrightarrow{g} X \xrightarrow{f} S$ in $\mathscr{C}$ one has the relation:

$$\eta_g^{\mathscr{C}}.\eta_f^{\mathscr{C}} = \eta_{f \circ g}^{\mathscr{C}}$$

using the product of the bivariant theory $\mathbb{E}_{**}^{BM}$. This relation will be referred to as the *associativity formula.*

Recall the aim of this paper is to construct a system of fundamental classes for a class of morphisms as large as achievable under the minimal possible choices.

2.2. Global orientations.

2.2.1. Recall from our convention that we assume from now on that any motivic triangulated category $\mathscr{T}$ is equipped with a premotivic adjunction:

$$\tau^* : S\mathscr{H} \to \mathscr{T}.$$

Consider an absolute ring $\mathscr{T}$-spectrum $\mathbb{E}$. Let us fix a scheme S in $\mathscr{S}$. Then $\tau_*(\mathbb{E}_S)$ is a motivic ring spectrum and for any smooth scheme X/S, for any pair $(n,m) \in \mathbb{Z}^2$, one gets an isomorphism:

$$\begin{aligned}\mathrm{Hom}_{S\mathscr{H}(S)}(\Sigma^\infty X_+, \tau_*(\mathbb{E}_S)(m)[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{T}(S)}(\tau^*(\Sigma^\infty X_+), \mathbb{E}_S(m)[n]) \\ \simeq \mathrm{Hom}_{\mathscr{T}(S)}(f_\sharp(\mathbb{1}_X), \mathbb{E}_S(m)[n]) \simeq \mathbb{E}^{n,m}(X)\end{aligned}$$

In other words, the ring spectrum $\tau_*(\mathbb{E}_S)$ in $S\mathscr{H}(S)$ represents the cohomology $\mathbb{E}^{**}$ restricted to smooth S-schemes and the above isomorphism is also compatible with cup-products.[32]

Therefore, one can apply all the definitions and results of orientation theory of motivic homotopy theory for which we refer to [Dég14b]. In the remainder of this section, we recall these results, applied more specifically to our situation.

As usual $\tilde{\mathbb{E}}^{**}$ denotes the reduced cohomology with coefficients in $\mathbb{E}$. As, by definition, $M_S(\mathbb{P}^1_S) = \mathbb{1}_S \oplus \mathbb{1}_S(1)[2]$ and because $\mathbb{1}_S(1)$ is $\otimes$-invertible, we get a canonical isomorphism:

$$\tilde{\mathbb{E}}^{2,1}(\mathbb{P}^1_S) \xrightarrow{\psi} \mathbb{E}^{0,0}(S)$$

where $\mathbb{P}^1_S$ is pointed by ∞. Therefore, the unit η_S of the ring spectrum $\mathbb{E}$ induces a canonical cohomology class $\sigma^{\mathbb{E}}_S = \psi^{-1}(\eta_S) \in \tilde{\mathbb{E}}^{2,1}(\mathbb{P}^1_S)$ — classically called the *stability class.*

As in [Dég14b, Def. 2.1.2], we let $\mathbb{P}^\infty_S$ be the colimit, in the category of Nisnevich sheaves of sets over the category of smooth S-schemes, of the inclusions $\mathbb{P}^n_S \to \mathbb{P}^{n+1}_S$ by means of the first coordinates.

Definition 2.2.2. Consider the above notations. An *orientation* of the absolute ring $\mathscr{T}$-spectrum $\mathbb{E}$ will be the datum, for any scheme S in $\mathscr{S}$, of a class $c_S \in \tilde{\mathbb{E}}^{2,1}(\mathbb{P}^\infty_S)$ such that:

- the restriction of c_S to $\mathbb{P}^1_S$ equals the stability class $\sigma^{\mathbb{E}}_S$ defined above;
- for any morphism $f : T \to S$, one has: $f^*(c_S) = c_T$.

For short, we will say that $(\mathbb{E}, c)$ is an absolute oriented ring $\mathscr{T}$-spectrum (or simply spectrum).

A morphism of absolute oriented ring spectra $(\mathscr{T}, \mathbb{E}, c) \to (\mathscr{T}', \mathbb{F}, d)$ will be a morphism of absolute ring spectra (Def. 1.1.4) (φ, ψ) such that for any scheme S, the map induced on cohomology

$$\psi_* : \tilde{\mathbb{E}}^{2,1}(\mathbb{P}^\infty_S) \to \tilde{\mathbb{F}}^{2,1}(\mathbb{P}^\infty_S)$$

sends c_S to d_S.

Remark 2.2.3. We will show later (Example 2.3.7) that an orientation of the ring spectrum $\mathbb{E}_S$ does correspond to a family of orientations of the associated Borel-Moore homology in the sense of Definition 2.1.1.

Example 2.2.4. Each of the absolute ring spectra of Example 1.1.2 admits a canonical orientation; see [Dég14b, 2.1.4].

[32] Beware however that $\tau_*(\mathbb{E}_S)$ for various schemes S only gives a section of $S\mathscr{H}$, not necessarily a *cartesian* one.

2.2.5. Consider the previous assumptions and notations. Recall one can build out of the orientation c a complete theory of characteristic classes. The first building block is the first Chern class which follows rightly from the class c, seen as a morphism. Indeed, from [Dég14b, 2.1.8]:

$$\begin{aligned} c_1 : \mathrm{Pic}(S) \rightarrow \mathrm{Hom}_{\mathscr{H}(S)}(S_+, \mathbb{P}^\infty_S) &\xrightarrow{\Sigma^\infty} \mathrm{Hom}_{S\mathscr{H}(S)}(\Sigma^\infty S_+, \Sigma^\infty \mathbb{P}^\infty_S) \\ &\xrightarrow{\tau^*} \mathrm{Hom}_{\mathscr{T}(S)}(\mathbb{1}_S, M_s(\mathbb{P}^\infty_S)) \xrightarrow{(c_S)_*} \mathrm{Hom}_{\mathscr{T}(S)}(\mathbb{1}_S, \mathbb{E}_S(1)[2]) = \mathbb{E}^{2,1}(S), \end{aligned}$$

— $\mathscr{H}(S)$ is Morel and Voevodsky's pointed unstable homotopy category.

One then deduces from [Dég14b, 2.1.13] that the cohomology theory $\mathbb{E}^{**}$ satisfies the classical projective bundle formula, which will be freely used in the rest of the text. Further, one gets higher Chern classes (see [Dég14b, Def. 2.1.16]) satisfying the following properties:

Proposition 2.2.6. *Considering the above notations and a given base scheme X in $\mathscr{S}$, the following assertions hold:*

(1) *For any vector bundle E over X, there exist* Chern classes $c_i(E) \in \mathbb{E}^{2i,i}(X)$ *uniquely defined by the formula:*

$$\sum_{i=0}^{n} p^*(c_i(E)).\big(-c_1(\lambda)\big)^{n-i} = 0, \tag{2.2.6.a}$$

$c_0(E) = 1$ *and* $c_i(E) = 0$ *for* $i \notin [0, n]$. *As usual, we define the* total Chern class *in the polynomial ring* $\mathbb{E}^{**}(X)[t]$:

$$c_t(E) = \sum_i c_i(E).t^i.$$

(2) *Chern classes are nilpotent, compatible with pullbacks in X, invariant under isomorphisms of vector bundles and satisfy the Whitney sum formula: for any vector bundles E, F over X,*

$$c_t(E \oplus F) = c_t(E).c_t(F).$$

(3) *There exists a (commutative) formal group law $F_X(x, y)$ with coefficients in the ring $\mathbb{E}^{**}(X)$ such that for any line bundles L_1, L_2 over X, the following relation holds:*

$$c_1(L_1 \otimes L_2) = F_X\big(c_1(L_1), c_1(L_2)\big) \in \mathbb{E}^{2,1}(X),$$

— which is well defined as the cohomology class $c_1(L_i)$ is nilpotent. Moreover, for any morphism $f : Y \to X$, one gets the relation: $f^*(F_X(x, y)) = F_Y(x, y)$, in other words, the morphism of rings $f^* : \mathbb{E}^{**}(X) \to \mathbb{E}^{**}(Y)$ induces a morphism of formal group laws.

This is the content of [Dég14b, 2.1.17, 2.1.22].

Remark 2.2.7. Consider a morphism of oriented ring spectra:

$$(\varphi, \psi) : (\mathscr{T}, \mathbb{E}, c) \to (\mathscr{T}', \mathbb{F}, d)$$

as in Definition 2.2.2, and $\psi_*:\mathbb{E}^{**}(X)\to\mathbb{F}^{**}(X)$ the map induced in cohomology. Let us denote by $c_n(E)$ (resp. $d_n(E)$) the n-th Chern class in $\mathbb{E}^{**}(X)$ (resp. $\mathbb{F}^{**}(X)$) associated with a vector bundle E/X using the previous proposition. It follows from the construction of Chern classes and the fact ψ_* respects the orientation that we get the relation:

$$\psi_*(c_n(E))=d_n(E).$$

Before going down the path of characteristic classes, let us recall that, according to Morel, the existence of an orientation on an absolute ring spectrum implies the associated cohomology is graded commutative. Actually, this property holds for the associated bivariant theory in the following terms.

Proposition 2.2.8. *Let $(\mathbb{E},c)$ be an absolute oriented ring $\mathscr{T}$-spectrum. Then for any cartesian square of s-morphisms*

$$\begin{array}{ccc} Y & \rightarrow & X \\ \downarrow & & \downarrow p \\ T & \xrightarrow[f]{} & S, \end{array}$$

and any pair $(x,t)\in\mathbb{E}^{BM}_{n,i}(X/S)\times\mathbb{E}^{BM}_{m,j}(T/S)$, the following relation holds in $\mathbb{E}^{BM}_{n+m,i+j}(Y/S)$:

$$p^*(t).x=(-1)^{nm}f^*(x).t.$$

Proof. For cohomology with support, this was proved in [Dég14b, 2.1.15]. The proof here is essentially the same. Recall that $M(\mathbb{G}_{m,S})=\mathbb{1}_S\oplus\mathbb{1}_S(1)[1]$. The map permuting the factors of $\mathbb{G}_m\times\mathbb{G}_m$ therefore induces an endomorphism of $\mathbb{1}_S(1)[1]$ which after desuspension and untwisting gives an element[33]

$$\epsilon\in\operatorname{End}_{\mathscr{T}(S)}(\mathbb{1}_S)=H_{0,0}(S,\mathscr{T}).$$

Formally, we get (under the assumptions of the proposition) the following relation:

$$f^*(t).x=(-1)^{nm-ij}\epsilon^{ij}.p^*(x).t \tag{2.2.8.a}$$

where the multiplication by $(-1)^{nm-ij}\epsilon^{ij}$ is seen via the action of $\mathbb{E}_{0,0}(S)$ on $\mathbb{E}^{BM}_{**}(X,S)$ — see Remark 1.2.9(2). Therefore, we are done as $\epsilon=-1$ in $\mathbb{E}_{0,0}(S)$. Indeed, according to relation (2.2.8.a) applied with $Y=X=T=S$, $(n,i)=(m,j)=(2,1)$, we have $c^2=-\epsilon.c^2\in\mathbb{E}^{4,2}(\mathbb{P}^2)$ and the projective bundle theorem for $\mathbb{E}^{**}$ thus concludes. □

2.2.9. Suppose given a motivic triangulated category $\mathscr{T}$. Recall that given a vector bundle $p:E\to X$ with zero section s, one defines the Thom space attached with E/X as:

$$M\mathrm{Th}(E/X):=p_\sharp s_*(\mathbb{1}_X).$$

This is also the image under the right adjoint $\tau^*:S\mathscr{H}(X)\to\mathscr{T}(X)$ of the classical Thom space $E/E-X$. Given an absolute $\mathscr{T}$-spectrum $\mathbb{E}$, we define the $\mathbb{E}$-cohomology of the Thom space of E as:

$$\mathbb{E}^{n,m}(\mathrm{Th}(E))=\operatorname{Hom}_{\mathscr{T}(X)}(M\mathrm{Th}(E),\mathbb{E}_X(m)[n]).$$

[33] In fact, this element is the image of Morel's element $\epsilon\in\pi_0(S^0_S)$ by the functor $\tau^*:S\mathscr{H}(S)\to\mathscr{T}(S)$, justifying our notation.

Note that by adjunction, one immediately gets an isomorphism:

(2.2.9.a) $$\mathbb{E}^{n,m}(\mathrm{Th}(E)) \xrightarrow{\alpha_s^*} \mathbb{E}_X^{n,m}(E) = \mathbb{E}^{BM}_{-n,-m}(X \xrightarrow{s} E)$$

where the map $\alpha_s : s_! \to s_*$ is the canonical isomorphism obtained from the six functors formalism, as s is proper. Finally, one gets the classical short exact sequence:

(2.2.9.b) $$0 \to \mathbb{E}^{n,m}(\mathrm{Th}(E)) \xrightarrow{\partial} \mathbb{E}^{n,m}(\mathbb{P}(E \oplus 1)) \xrightarrow{\nu^*} \mathbb{E}^{n,m}(\mathbb{P}(E)) \to 0$$

where $\nu : \mathbb{P}(E) \to \mathbb{P}(E \oplus 1)$ is the canonical immersion of the projective bundle associated with E/X into its projective completion — cf. the construction of [Dég14b, 2.2.1].

Proposition 2.2.10. *Let* $(\mathbb{E}, c)$ *be an absolute oriented ring* $\mathscr{T}$*-spectrum and* E/X *be a vector bundle of rank* r*.*

One defines the Thom class *of* E *in* $\mathbb{E}^{2r,r}(\mathbb{P}(E \oplus 1))$ *as:*

(2.2.10.a) $$\mathfrak{t}(E) = \sum_{i=0}^{r} p^*(c_i(E)).\big(-c_1(\lambda)\big)^{r-i}.$$

Then $\mathfrak{t}(E)$ *induces a unique class* $\bar{\mathfrak{t}}(E) \in \mathbb{E}^{2r,r}(\mathrm{Th}(E))$*, called the* refined Thom class*, such that* $\partial(\bar{\mathfrak{t}}(E)) = \mathfrak{t}(E)$*.*

Moreover, $\mathbb{E}^{**}(\mathrm{Th}(E))$ *is a free graded* $\mathbb{E}^{**}(X)$*-module of rank* 1 *with base* $\bar{\mathfrak{t}}(E)$*. In other words, the sequence* (2.2.9.b) *is split and we get a canonical isomorphism:*

(2.2.10.b) $$\tau_E : \mathbb{E}^{**}(X) \to \mathbb{E}^{**}(Th(E)), x \mapsto x.\,\bar{\mathfrak{t}}(E).$$

For the proof, see [Dég14b, 2.2.1, 2.2.2]. The preceding isomorphism is traditionally called the *Thom isomorphism* associated with the vector bundle E/X. It follows from Remark 2.2.7 that morphisms of oriented absolute ring spectra respect Thom classes as well as refined Thom classes.

Remark 2.2.11. Note that the proposition makes sense even when the rank of E/X is not constant. Indeed, in any case, the rank is locally constant on X, *i.e.* constant over each connected component X_i of X and we just take direct sums of the Thom classes restricted to each connected component, in the canonical decomposition:

$$\mathbb{E}^{**}(\mathrm{Th}(E)) = \bigoplus_i \mathbb{E}^{**}\left(\mathrm{Th}\left(E|_{X_i}\right)\right).$$

2.2.12. The natural functor $\tau^* : S\mathscr{H}(S) \to \mathscr{T}(S)$ is monoidal. In particular, we get a canonical absolute ring $\mathscr{T}$-spectrum $\tau^*(\mathbf{MGL})$, the avatar of algebraic cobordism in $\mathscr{T}$. Note that by definition, it satisfies the following formula for any base scheme S:

$$\tau^*(\mathbf{MGL}_S) = \mathrm{hocolim}_{n \geq 0}\, M\mathrm{Th}_S(\gamma_n)(-n)[-2n]$$

where γ_n is the tautological vector bundle on the infinite Grassmannian of n-planes over S. Note that, by adjunction, a structure of a $\tau^*(\mathbf{MGL}_S)$-module (resp. $\tau^*(\mathbf{MGL}_S)$-algebra) over a $\mathscr{T}$-spectrum $\mathbb{E}_S$ is the same thing as a structure of $\mathbf{MGL}_S$-module (resp. $\mathbf{MGL}_S$-algebra) over $\tau_*(\mathbb{E}_S)$. Thus, according to [Vez01, 4.3] (see also [Dég14b, 2.2.6]), there is a bijection between the following sets:

(1) the orientations c on $\mathbb{E}$ as defined in 2.2.2;
(2) the structures of an **MGL**-algebra on $\mathbb{E}$ as defined in 1.1.4.

In particular, the class c induces a unique morphism of absolute ring spectra:

$$(\tau, \phi^c) : \mathbf{MGL} \to \mathbb{E}.$$

This morphism induces a natural transformation of cohomology theories, compatible with cup-products (see 1.2.12):

$$\phi^c_* : \mathbf{MGL}^{n,m}(X) \to \mathbb{E}^{n,m}(X)$$

which by definition satisfies the property that $\tau^c_*(c^{\mathbf{MGL}}) = c$ in $\tilde{\mathbb{E}}^{2,1}(\mathbb{P}^\infty_S)$. In other words, (τ, ϕ^c) is a morphism of oriented ring spectra.

This fact suggest the following definition.

Definition 2.2.13. A *weak orientation* of an absolute $\mathscr{T}$-spectrum $\mathbb{E}$ is a structure of a τ-module over the absolute ring spectrum **MGL** — in short, an **MGL**-module structure.

Remark 2.2.14. (1) Typically, an **MGL**-module $\mathbb{E}$ will not possess Chern classes or Thom classes but will possess a structural action of the ones which naturally exists for **MGL**. As we will see below, this is enough to obtain Gysin morphisms and duality results for the cohomology represented by $\mathbb{E}$.
(2) When $\mathbb{E}$ is an absolute ring spectrum, the difference between a weak orientation and an orientation is the one between an **MGL**-module structure and an **MGL**-algebra structure.

Remark 2.2.15. In this example, we consider one of the following assumptions on a given ring of coefficients Λ:

- the category of schemes $\mathscr{S}$ can be arbitrary and $\Lambda = \mathbb{Q}$;
- the category $\mathscr{S}$ is a subcategory of the category of k-schemes for a field k of characteristic exponent p and $\Lambda = \mathbb{Z}[1/p]$.

We denote by $\mathbb{H}\Lambda$ the motivic (Eilenberg-MacLane) absolute spectrum with coefficients in Λ (Example 1.1.2). Then given an absolute oriented ring $\mathscr{T}$-spectrum $(\mathbb{E}, c)$ which is Λ-linear and whose associated formal group law is additive, there exists a unique morphism of absolute ring spectra

$$(\tau, \tilde{\phi}^c) : \mathbb{H}\Lambda \to \mathbb{E}$$

such that $\tilde{\phi}^c(c^{\mathbb{H}\Lambda}) = x$.

Indeed, in the first case, this follows from [Dég14b, Th. 14.2.6], and in the second case from Hoyois-Hopkins-Morel Theorem (see [Hoy15a]). See [Dég14b, 5.3.1, 5.3.9] for more details.

2.2.16. Consider again the setting of Paragraph 2.2.9. Recall from [Dég14b, 2.4.18] that the association $E \mapsto M\mathrm{Th}(E)$ can be uniquely extended to a monoidal functor:

$$M\mathrm{Th} : \underline{K}(X) \to \mathrm{Pic}(\mathscr{T}, \otimes)$$

where $\underline{K}(X)$ is the Picard groupoid of *virtual vector bundles* over X ([Del87, 4.12]) and $\mathrm{Pic}(\mathscr{T}, \otimes)$ that of $\otimes$-invertible objects of $\mathscr{T}(X)$, morphisms being isomorphisms. Actually, this extension follows from the fact that for any short exact sequence of vector bundles over X:

$$(\sigma) \qquad 0 \to E' \to E \to E'' \to 0$$

there exists a canonical isomorphism:

$$(2.2.16.a) \qquad \epsilon_\sigma : M\mathrm{Th}(E') \otimes M\mathrm{Th}(E'') \to M\mathrm{Th}(E)$$

and the isomorphisms of this form satisfy the coherence conditions of [Del87, 4.3].

Then the definition of the refined Thom class can be extended to Thom spaces of virtual vector bundles using the following lemma (see [Dég14b, 2.4.7]).

Lemma 2.2.17. *Consider as above an exact sequence* (σ) *of vector bundles over a scheme* X*. Then the following relation holds in* $\mathbb{E}^{**}(\mathrm{Th}(E))$*:*

$$\bar{\mathfrak{t}}(E/X) = \bar{\mathfrak{t}}(E'/X).\,\bar{\mathfrak{t}}(E/E')$$

using the product $\mathbb{E}^{**}_X(E') \otimes \mathbb{E}^{**}_{E'}(E) \to \mathbb{E}^{**}_X(E)$ *of the corresponding bivariant theories.*

2.2.18. Consider an arbitrary vector bundle E/X. Then we obviously get a perfect pairing of $\mathbb{E}^{**}(X)$-modules:

$$\mathbb{E}^{**}(\mathrm{Th}(E)) \otimes \mathbb{E}^{**}(\mathrm{Th}(-E)) \to \mathbb{E}^{**}(X), (a, b) \mapsto a \otimes_X b.$$

We let $\bar{\mathfrak{t}}(-E)$ be the unique element of $\mathbb{E}^{**}(\mathrm{Th}(-E))$ such that $\bar{\mathfrak{t}}(E) \otimes_X \bar{\mathfrak{t}}(-E) = 1$ so that $\bar{\mathfrak{t}}(-E)$ is a basis of the $E^{**}(X)$-module $\mathbb{E}^{**}(\mathrm{Th}(-E))$.

Let now v be a virtual vector bundle over X. Then we deduce from the preceding lemma that for any X-vector bundles E and E' such that $v = [E] - [E']$, the class

$$\bar{\mathfrak{t}}(v) = \bar{\mathfrak{t}}(E) \otimes_X \bar{\mathfrak{t}}(-E')$$

is independent of the choice of E and E'.

Definition 2.2.19. Consider the notations above. We define the *Thom class* of the virtual vector bundle v over X as the element $\bar{\mathfrak{t}}(v) \in \mathbb{E}^{**}(\mathrm{Th}(v))$ defined by the preceding relation.

Recall $\bar{\mathfrak{t}}(v)$ is a basis of $\mathbb{E}^{**}(\mathrm{Th}(v))$ as an $\mathbb{E}^{**}(X)$-module. In other words, the map:

$$(2.2.19.a) \qquad \tau_v : \mathbb{E}^{**}(X) \to \mathbb{E}^{**}(Th(v)), x \mapsto x.\,\bar{\mathfrak{t}}(v).$$

is an isomorphism, again called the *Thom isomorphism* associated with the virtual vector bundle v. Besides, the preceding lemma shows we have the relation:

$$(2.2.19.b) \qquad \bar{\mathfrak{t}}(v + v') = \bar{\mathfrak{t}}(v) \otimes_X \bar{\mathfrak{t}}(v')$$

in $\mathbb{E}^{**}(\mathrm{Th}(v + v'))$.

2.2.20. Consider again the setting of Paragraph 2.2.9. According to the projection formulas of the motivic triangulated category $\mathscr{T}$, one gets an isomorphism of functors in M, object of $\mathscr{T}(X)$,

$$(2.2.20.a) \qquad p_\sharp s_*(M) = p_\sharp s_*(\mathbb{1}_S \otimes s^* p^* M) \simeq p_\sharp s_*(\mathbb{1}_S) \otimes M = M\mathrm{Th}_S(E) \otimes M.$$

Therefore, $p_\sharp s_*$ is an equivalence of categories with quasi-inverse:

$$M \mapsto s^! p^*(M) = M\mathrm{Th}_S(-E) \otimes M. \tag{2.2.20.b}$$

The following proposition is a reinforcement of Proposition 2.2.10.

Proposition 2.2.21. *Let $(\mathbb{E}, c)$ be an absolute oriented $\mathscr{T}$-spectrum.*

Then for a vector bundle E/X with zero section s, the refined Thom class $\bar{\mathfrak{t}}(E)$, seen as an element $\bar{\eta}_s$ of $\mathbb{E}^{BM}_{**}(X \xrightarrow{s} E)$ through the identification (2.2.9.a), is a universally strong orientation of s, with degree equal to the rank r of E/X.

Moreover, these orientations form a system of fundamental classes with respect to the class of morphisms made by the zero sections of vector bundles (over schemes in $\mathscr{S}$).

Proof. Let us consider the following map:

$$\mathfrak{p}'_E : M\mathrm{Th}(E) \otimes \mathbb{E}_X \xrightarrow{\bar{\mathfrak{t}}(E)\otimes 1} \mathbb{E}_X \otimes \mathbb{E}_X(r)[2r] \xrightarrow{\mu} \mathbb{E}_X(r)[2r]$$

where μ is the multiplication map of the ring spectrum $\mathbb{E}_X$. It follows formally from this construction that the map

$$\begin{aligned} \mathbb{E}^{n,m}(\mathrm{Th}(-E)) &= \mathrm{Hom}_{\mathscr{T}(X)}\left(M\mathrm{Th}(-E), \mathbb{E}(m)[n]\right) \simeq \mathrm{Hom}_{\mathscr{T}(X)}\left(\mathbb{1}_X, M\mathrm{Th}(E) \otimes \mathbb{E}(m)[n]\right) \\ &\xrightarrow{(\mathfrak{p}'_E)_*} \mathrm{Hom}_{\mathscr{T}(X)}(\mathbb{1}_X, \mathbb{E}(m+r)[n+2r]) = \mathbb{E}^{n+2r,m+r}(X) \end{aligned}$$

induced by $\mathfrak{p}'_E$ after applying the functor $\mathrm{Hom}_{\mathscr{T}(X)}(\mathbb{1}_X, -(*)[*])$ is equal to the inverse of the Thom isomorphism (2.2.19.a). Because Thom classes are stable under pullbacks, we further deduce that for any smooth morphism $f : Y \to X$, the map induced after applying the functor $\mathrm{Hom}_{\mathscr{T}(X)}(M_X(Y), -(m)[n])$ is equal to the inverse of the Thom isomorphism associated with the virtual vector bundle $(-f^{-1}(E))$ over Y. As the objects $M_X(Y)(-m)$ for Y/X smooth and $m \in \mathbb{Z}$ form a family of generators for the triangulated category $\mathscr{T}(X)$ (according to our conventions on motivic triangulated categories), we deduce that $\mathfrak{p}'_E$ is an isomorphism.

Now, one can check going back to definitions that the following isomorphism:

$$\mathbb{E}_X = s^! p^*(M\mathrm{Th}(E) \otimes \mathbb{E}_X) \xrightarrow{s^! p^*(\mathfrak{p}'_E)} s^! p^*(\mathbb{E}_X(r)[2r]) \simeq s^!(\mathbb{E}_X)(r)[2r]$$

is equal to the map $\tilde{\eta}_s(r)[2r]$ associated to $\bar{\eta}_s$ as in (2.1.5.a). This implies the first claim.

Then the second claim is exactly Lemma 2.2.17. □

Remark 2.2.22. We will remember from the above proof that, in the condition of the proposition, given any virtual vector bundle v over X with virtual rank r, the following map:

$$\mathfrak{p}'_v : M\mathrm{Th}(v) \otimes \mathbb{E}_X \xrightarrow{\bar{\mathfrak{t}}(v)\otimes 1} \mathbb{E}_X \otimes \mathbb{E}_X(r)[2r] \xrightarrow{\mu} \mathbb{E}_X(r)[2r] \tag{2.2.22.a}$$

is an isomorphism — this follows from the case where $v = [E]$ explicitly treated in the proof, relation (2.2.19.b) and the fact $\bar{\mathfrak{t}}(0) = 1$. This map obviously represents the Thom isomorphism (2.2.19.b) so we will also call it the *Thom isomorphism* when no confusion can arise.

2.3. The smooth case.

2.3.1. Let $\mathscr{T}$ be a triangulated motivic category with, according to our conventions, a premotivic adjunction $\tau^* : S\mathscr{H} \to \mathscr{T}$.

As usual we call closed S-pair any pair of S-schemes (X, Z) such that X/S is smooth and Z is a closed immersion. One defines the motive of X with support in Z as:

$$M_S(X/X - Z) := p_\sharp i_*(\mathbb{1}_Z)$$

where p is the structural morphism of X/S, i the immersion of Z in X. Alternatively, one can equivalently put:

$$M_S(X/X - Z) := \tau^*(\Sigma^\infty X/X - Z)$$

where $X/X - Z$ is the quotient computed in the category pointed Nisnevich sheaves of sets over the category of smooth S-schemes, seen as an object of the pointed $\mathbb{A}^1$-homotopy category over S.

A particular example that we have already seen is given for a vector bundle E/X, with X/S smooth. We then put:

$$M\mathrm{Th}_S(E) := M_S(E/E - X)$$

extending the definition of Paragraph 2.2.9 — for which we had $X = S$.

These objects satisfy a classical formalism which has been summarized in [Dég08, 2.1]. In particular, they are covariant in the closed S-pair (X, Z) — recall morphisms of closed pairs are given by commutative squares which are topologically cartesian[34]; one says such a morphism is *cartesian* if the corresponding square is cartesian.

Given a closed S-pair (X, Z), we define the associated deformation space as:

$$D_Z X := B_{Z\times\{0\}}(\mathbb{A}^1_X) - B_Z X.$$

Let us put $D = D_Z X$ for the rest of the discussion. This deformation space contains as a closed subscheme the scheme $\mathbb{A}^1_Z \simeq B_Z(\mathbb{A}^1_Z)$. It is flat over $\mathbb{A}^1$ and the fiber of the closed pair $(D, \mathbb{A}^1_Z)$ over 1 (resp. 0) is (X, Z) (resp. $(N_Z X, Z)$). Therefore one gets cartesian morphisms of closed S-pairs:

(2.3.1.a) $$(X, Z) \xrightarrow{d_1} (D, \mathbb{A}^1_Z) \xleftarrow{d_0} (N_Z X, Z).$$

Theorem 2.3.2 (Morel-Voevodsky)**.** *Consider a closed S-pair (X, Z) such that Z/S is smooth.*

Then the induced maps

$$M_S(X/X - Z) \xrightarrow{d_{1*}} M_S(D/D - \mathbb{A}^1_Z) \xleftarrow{d_{0*}} M\,\mathrm{Th}_S(N_Z X)$$

are isomorphisms in $\mathscr{T}(S)$.

The proof is well known — see for example [CD12b, Th. 2.4.35].

[34] *i.e.* cartesian on the underlying topological spaces.

Definition 2.3.3. Under the assumptions of the previous theorem, we define the *purity isomorphism* associated with (X, Z) as the composite isomorphism:

$$\mathfrak{p}_{(X,Z)} : M_S(X/X-Z) \xrightarrow{d_{1*}} M_S(D/D-\mathbb{A}^1_Z) \xrightarrow{d_{0*}^{-1}} M\,\mathrm{Th}_S(N_Z X).$$

2.3.4. We can derive a functorial version of the preceding purity isomorphism. Consider for simplicity the case of a closed immersion $i : S \to X$ which admits a smooth retraction $p : X \to S$. We deduce from the preceding construction the following isomorphism functorial in a given object $\mathbb{E}$ of $\mathscr{T}(S)$:

$$\mathfrak{p}_{p,i} : p_\sharp i_*(\mathbb{E}) = p_\sharp i_*(\mathbb{1} \otimes s^*p^*\mathbb{E}) \xrightarrow{\ \sim\ } M_S(X/X-S) \otimes \mathbb{E} \xrightarrow{\mathfrak{p}_{(X,Z)}} M\mathrm{Th}_S(N_S X) \otimes \mathbb{E},$$

where p is the structural map of X/S, and the first isomorphism is given by the projection formulas associated with i_* and $p_\sharp$. As the $\mathscr{T}$-spectrum $M\mathrm{Th}_S(N_S X)$ is $\otimes$-invertible, we deduce that the functor $p_\sharp s_*$ is an equivalence of categories. Then by adjunction, we get a dual isomorphism:

$$\mathfrak{p}'_{p,i} : i^! p^*(\mathbb{E}) \to M\mathrm{Th}_S(-N_S X) \otimes \mathbb{E} \tag{2.3.4.a}$$

using the notation of Paragraph 2.2.16. Note by the way this map can be written as the following composite of isomorphisms:

$$\begin{aligned}\mathfrak{p}'_{p,i} : i^! p^*(\mathbb{E}) \xrightarrow{\sim} \underline{\mathrm{Hom}}(M_S(X/X-S), \mathbb{E}) &\xrightarrow{\left(\mathfrak{p}^{-1}_{(X,Z)}\right)^*} \underline{\mathrm{Hom}}(M\mathrm{Th}_S(N_S X), \mathbb{E}) \\ &\simeq M\mathrm{Th}_S(-N_S X) \otimes \mathbb{E}\end{aligned} \tag{2.3.4.b}$$

Consider now an absolute oriented spectrum $(\mathscr{T}, \mathbb{E}, c)$. Let n be the function on S which measures the local codimension of S in X. It is locally constant as i is a regular closed immersion — as it admits a smooth retraction. Then we deduce from the previous purity isomorphism and from the Thom isomorphism (2.2.22.a) the following one:

$$\tilde{\eta}_i : \mathbb{E}_S(n)[2n] \xrightarrow{(\mathfrak{p}'_{-N_S X})^{-1}} M\mathrm{Th}(-N_S X) \otimes \mathbb{E}_S \xrightarrow{(\mathfrak{p}'_{p,i})^{-1}} i^! p^*(\mathbb{E}_S) \simeq i^!(\mathbb{E}_X).$$

As explained in Remark 2.1.7(2), we can associate to this isomorphism the following orientation:

$$\bar{\eta}_i : \mathbb{1}_S(n)[2n] \to \mathbb{E}_S(n)[2n] \xrightarrow{\tilde{\eta}_i} i^!(\mathbb{E}_X)$$

which is therefore universally strong.

Proposition 2.3.5. *Consider the above notations and assumptions. The universally strong orientations $\bar{\eta}_i$ constructed above form a system of fundamental classes (Definition 2.1.9) with coefficients in $\mathbb{E}$ for closed immersions which admit a smooth retraction.*

Indeed, all what remains to be proved is the associativity formula. This follows from the use of the double deformation space and the associativity formula for Thom classes (Proposition 2.2.21). The reader can consult [Dég14b, 2.4.9] for details.

Remark 2.3.6. (1) Note the same result could have been derived replacing closed immersions which admits a smooth retraction by closed immersions between smooth schemes over some fixed base. We will derive this case later from our more general results.

(2) We will need the following normalisation property of the fundamental classes constructed above. In the assumptions of the proposition, we consider the deformation diagram (2.3.1.a):

$$\begin{array}{ccccc} S & \longrightarrow & \mathbb{A}^1_S & \longleftarrow & S \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{d_1} & D & \xleftarrow{d_0} & N_S X. \end{array}$$

It induces pullback morphisms on Borel-Moore homology:

$$\mathbb{E}^{BM}_{**}(i : S \to X) \xleftarrow{d_1^*} \mathbb{E}^{BM}_{**}(\nu : \mathbb{A}^1_S \to D) \xrightarrow{d_0^*} \mathbb{E}^{BM}_{**}(s : S \to N_S X) \simeq \mathbb{E}^{**}(\mathrm{Th}(N_S X))$$

which are isomorphisms according to Theorem 2.3.2.

It rightly follows from the above construction that one has the relation:

$$d_0^*(d_1^*)^{-1}(\bar{\eta}_i) = \bar{\mathfrak{t}}(N_S X).$$

This relation implies that the system of fundamental classes of the preceding proposition, restricted to zero sections of vector bundles, coincides with that of Proposition 2.2.21: indeed, when X/S is a vector bundle, one obtains that D is isomorphic to $\mathbb{A}^1_X$, $N_S X$ is isomorphic to X and the maps d_0 and d_1 corresponds respectively to the zero and unit sections of $\mathbb{A}^1_X$ through these isomorphisms.

Example 2.3.7. Fixing a base scheme S, one can interpret the global orientation $c = c_S \in \mathbb{E}^{2,1}(\mathbb{P}^\infty_S)$ as a sequence of classes $c_n \in \mathbb{E}^{2,1}(\mathbb{P}^n_S)$, $n > 0$. Let us consider the immersion: $\mathbb{P}^{n-1}_S \xrightarrow{\nu^{n-1}_S} \mathbb{P}^n_S$ of the hyperplane at infinity (say $\mathbb{P}^{n-1}_S \times \{\infty\}$). Then the canonical exact sequence:

$$0 \to \mathbb{E}^{2,1}_{\mathbb{P}^{n-1}_S}(\mathbb{P}^n_S) \to \mathbb{E}^{2,1}(\mathbb{P}^n_S) \xrightarrow{j^*} \mathbb{E}^{2,1}(\mathbb{P}^n_S - \mathbb{P}^{n-1}_S) \simeq \mathbb{E}^{2,1}(S) \to 0$$

is split exact and the class c_n uniquely lifts to a class $\bar{c}_n$ in

$$\mathbb{E}^{2,1}_{\mathbb{P}^{n-1}_S}(\mathbb{P}^n_S) \simeq \mathbb{E}^{BM}_{-2,-1}(\mathbb{P}^{n-1}_S \xrightarrow{\nu^{n-1}_S} \mathbb{P}^n_S);$$

So the family $(c_n)_{n>0}$ corresponds to a family of orientations for the closed immersions ν^{n-1}_S.

According to the previous remark, we get the equality:

$$\bar{\eta}_{\nu^{n-1}_S} = c_1^{\mathbb{P}^{n-1}}(\mathcal{O}_{\mathbb{P}^n}(-1)) = \bar{c}_n,$$

using the notations of Definition 2.4.2 in the the middle term. In fact one can interpret an orientation of the ring spectrum $\mathbb{E}_S$ as a family of orientations of the immersions ν^∞_n satisfying suitable conditions.

2.3.8. Let us recall how the homotopy purity theorem of Morel and Voevodsky (stated above as Theorem 2.3.2) is used according to the method of Ayoub to define the relative purity isomorphism of the six functors formalism.

Consider now an arbitrary smooth s-morphism $f : X \to S$. We look at the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\delta} & X \times_S X & \xrightarrow{f_2} & X \\ & \searrow^{=} & {\scriptstyle f_1}\downarrow & \Delta & \downarrow{\scriptstyle f} \\ & & X & \xrightarrow[f]{} & S \end{array}$$

where the square Δ is cartesian and δ is the diagonal immersion. Let T_f be the tangent bundle of X/S, that is the normal bundle of the immersion δ. Interpreting the construction of Ayoub (see also [CD12b, 2.4.39]), we then introduce the following natural transformation:

$$\mathfrak{p}_f : f^* \simeq \delta^! f_2^! f^* \xrightarrow{Ex^{!*}(\Delta)} \delta^! f_1^* f^! \xrightarrow{\mathfrak{p}'_{f_1,\delta}} M\mathrm{Th}_X(-T_f) \otimes f^!. \tag{2.3.8.a}$$

By adjunction, one gets a natural transformation:

$$\mathfrak{p}'_f : f_\sharp \to f_!\big(M\mathrm{Th}_X(T_f) \otimes -\big). \tag{2.3.8.b}$$

Then one deduces from the axioms of motivic triangulated categories that $\mathfrak{p}_f$ and $\mathfrak{p}'_f$ are isomorphisms, simply called the *purity isomorphisms associated with* f.[35] Besides, we will use the following functoriality result satisfied by these purity isomorphisms.

Proposition 2.3.9 (Ayoub)**.** *Consider smooth s-morphisms* $Y \xrightarrow{g} X \xrightarrow{f} S$ *with respective tangent bundles* T_g *and* T_f*. Then the following diagram of natural transformations is commutative:*

$$\begin{array}{ccc} M\mathrm{Th}_Y(T_g) \otimes g^*\big(M\mathrm{Th}_X(T_f) \otimes f^*\big) & \xrightarrow{\mathfrak{p}_g . \mathfrak{p}_f} & g^! \circ f^! \\ \sim\downarrow & & \| \\ M\mathrm{Th}_Y(T_g) \otimes M\mathrm{Th}_X(g^{-1}T_f) \otimes g^* f^* & & \| \\ {\scriptstyle \epsilon_\sigma}\downarrow & & \| \\ M\mathrm{Th}_Y(T_{fg}) \otimes g^* f^* & & \| \\ \| & & \| \\ M\mathrm{Th}_Y(T_{fg}) \otimes (fg)^* & \xrightarrow{\mathfrak{p}_{fg}} & (fg)^! \end{array}$$

where the first isomorphism comes from the fact that g^* *is monoidal and Thom spaces are compatible with base change while the isomorphism* ϵ_σ *stands for* (2.2.16.a) *associated with the exact sequence of vector bundles:*

$$0 \to T_g \to T_{fg} \to g^{-1}(T_f) \to 0. \tag{σ}$$

[35] This is one of the main results of [Ayo07a], though it was proved there only in the quasi-projective case. The extension to the general case was first made in [CD12b, 2.4.26].

For the proof, we refer the reader to [Ayo07a, 1.7.3] in the quasi-projective case – actually Ayoub proves the assertion for the right adjoints but this is obviously equivalent to our statement. Then the general case is reduced to the quasi-projective one using the localization property of motivic triangulated categories.

2.3.10. It is now easy to deduce from the preceding results canonical new orientations for our bivariant theories using the method of Paragraph 2.3.4.

Let us fix again an absolute oriented ring $\mathscr{T}$-spectrum $(\mathbb{E}, c)$. Given a smooth morphism $f : X \to S$ of relative dimension d (seen as a locally constant function on X), we define the following isomorphism:

$$\tilde{\eta}_f : \mathbb{E}_X(d)[2d] \xrightarrow{(\mathfrak{p}'_{T_f})^{-1}} M\mathrm{Th}(T_f) \otimes \mathbb{E}_X \simeq M\mathrm{Th}(T_f) \otimes f^*(\mathbb{E}_S) \xrightarrow{(\mathfrak{p}_f)^{-1}} f^!(\mathbb{E}_S)$$

where $\mathfrak{p}'_{T_f}$ is the Thom isomorphism (2.2.22.a) associated with the tangent bundle T_f of f and $\mathfrak{p}_f$ is Ayoub's purity isomorphism (2.3.8.a).

Following Remark 2.1.7(2), we then define the following orientation of f:

$$\bar{\eta}_f : \mathbb{1}_S(d)[2d] \to \mathbb{E}_S(d)[2d] \xrightarrow{\tilde{\eta}_f} f^!(\mathbb{E}_X).$$

Combining this construction together with the preceding proposition, we have obtained:

Proposition 2.3.11. *The universally strong orientations $\bar{\eta}_f$ constructed above form a system of fundamental classes (Definition 2.1.9) with coefficients in $\mathbb{E}$ for smooth s-morphisms.*

Remark 2.3.12. (1) As the relative dimension d of a smooth s-morphism $f : Y \to X$ is equal to the rank of its tangent bundle, the fundamental class $\bar{\eta}_f$ has degree d in the sense of Definition 2.1.9.

(2) It obviously follows from the constructions of Paragraphs 2.1.4 and 2.3.8 that the orientations constructed here for arbitrary smooth s-morphisms extend the definition given in Example 2.1.8 for étale s-morphisms.

(3) For future reference, we will recall the following characterisation of the orientation $\bar{\eta}_f \in \mathbb{E}^{BM}_{**}(X/S)$ constructed above. We have:

$$\eta_f = \mathfrak{p}'^*_f(\bar{\mathfrak{t}}(-T_f))$$

where the map $\mathfrak{p}'^*_f$ is induced by the isomorphism (2.3.8.b) as follows:

(2.3.12.a)
$$\mathbb{E}^{**}(\mathrm{Th}(-T_f)) \simeq \mathrm{Hom}(f_\sharp(M\mathrm{Th}(-T_f)), \mathbb{E}_S) \xrightarrow{\mathfrak{p}'^*_f} \mathrm{Hom}(f_!(\mathbb{1}_X), \mathbb{E}_S) = \mathbb{E}^{BM}_{**}(X/S)$$

— here Hom are understood with their natural $\mathbb{Z}^2$-graduation.

Let us finally note the following lemma for later use.

Lemma 2.3.13. *Consider the notations of the previous proposition together with a cartesian square:*

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ {\scriptstyle g}\downarrow & \Theta & \downarrow{\scriptstyle f} \\ T & \xrightarrow[p]{} & S \end{array}$$

such that f is a smooth s-morphism. Then the following relation holds in $\mathbb{E}^{BM}_{**}(Y/T)$: $p^*(\bar{\eta}_f) = \bar{\eta}_g$.

Proof. We reduce to prove the analogous fact for the isomorphism constructed in 2.3.10. Coming back to definitions (see in particular (2.3.8.a) and (2.3.4.b)), one reduces to prove the following diagram of natural transformations is commutative:

$$\begin{array}{ccccccc}
q^*f^* & \xrightarrow{\sim} & q^*\underline{\mathrm{Hom}}(M(X^2/X^2-\Delta_X), f^!) & \longrightarrow & q^*(M\mathrm{Th}(-T_f)\otimes f^!) & \longrightarrow & q^*f^!((d)) \\
\| & & \downarrow{\sim} & & \downarrow{\sim} & & \| \\
\| & {\scriptstyle (1)} & \underline{\mathrm{Hom}}(q^*M(X^2/X^2-\Delta_X), q^*f^!) & {\scriptstyle (2)} & q^*\,M\mathrm{Th}(-T_f)\otimes q^*f^! & {\scriptstyle (3)} & q^*f^!((d)) \\
\| & & \downarrow{\scriptstyle \underline{\mathrm{Hom}}(\phi,\psi)} & & \downarrow{\scriptstyle \phi'\otimes\psi} & & \downarrow{\scriptstyle \psi} \\
g^*q^* & \xrightarrow{\sim} & \underline{\mathrm{Hom}}(M(Y^2/Y^2-\Delta_Y), g^!p^*) & \longrightarrow & M\mathrm{Th}(-T_g)\otimes g^!p^* & \longrightarrow & g^!p^*((d))
\end{array}$$

where:

- d means the relative dimension of f which we can assume to be constant, and which is equal to the relative dimension of g, and we have denoted $-((d))$ the twist $-(d)[2d]$.
- ψ stands for the reciprocal isomorphism of the exchange transformation: $Ex^{!*}: g^!p^* \to q^*f!$ associated with the square Θ — which is an isomorphism as f is smooth;
- we have put $X^2 = X \times_S X$ and $Y^2 = Y \times_T Y = X \times_S X \times_S T$; ϕ is the isomorphism associated with the cartesian squares:

$$\begin{array}{ccccc}
Y & \xrightarrow{\delta_Y} & Y^2 & \xrightarrow{g_1} & Y \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q\times q} & & \downarrow{\scriptstyle q} \\
X & \xrightarrow[\delta_X]{} & X^2 & \xrightarrow[f_1]{} & X
\end{array}$$

  where f_1 (resp. g_1) stands for the projection on the first factor.
- ϕ' is the isomorphism induced by the identification $q^{-1}(T_f) \simeq T_g$ — which also expresses that the square Θ is transversal.

Then, diagram (1) is commutative as it is made of exchange transformations, diagram (2) is commutative has the deformation diagram (2.3.1.a) is functorial with respect to cartesian morphisms of closed pairs — applied to the cartesian square (*) — and diagram (3) is commutative as the Thom class (2.2.10.a) is stable under pullbacks. □

2.4. **The regular closed immersion case according to Navarro.**

2.4.1. Let us now recall the construction of Navarro of a system of fundamental classes for regular closed immersions which extends the one constructed in Proposition 2.3.5. The construction can be safely transported to our generalized context as we assume the existence of a premotivic adjunction $\tau^* : \mathscr{SH} \to \mathscr{T}$.

Given a closed pair (X, Z), $U = X - Z$ the open complement, we recall that the relative Picard group $\mathrm{Pic}(X, Z)$ is the group of isomorphisms classes of pairs (L, u) where L is a line bundle on X and $u : L|_U \to \mathbb{A}^1_U$ a trivialization of L over U.

As remarked by Navarro ([Nav16, Rem. 3.8]), one deduces from a classical result of Morel and Voevodsky that there is a natural bijection:

$$\epsilon_{X,Z} : \mathrm{Pic}(X,Z) \xrightarrow{\sim} [X/U, \mathbb{P}^\infty_X]$$

where the right hand-side stands for the unstable $\mathbb{A}^1$-homotopy classes of pointed maps over X. Thus, one obtains Chern classes with support as in [Nav16, 1.4].

Definition 2.4.2. Let $(\mathbb{E}, c)$ be an absolute oriented ring $\mathscr{T}$-spectrum.

Then, given any closed pair (X, Z) with open complement U, one defines the first Chern class map with support and coefficients in $\mathbb{E}$ as the following composite:

$$\begin{aligned} c_1^Z : \mathrm{Pic}(X,Z) \xrightarrow{\epsilon_{X,Z}} [X/U, \mathbb{P}^\infty_X] \to \mathrm{Hom}_{S\mathscr{H}(X)}(\Sigma^\infty X/U, \Sigma^\infty \mathbb{P}^\infty_S) \\ \xrightarrow{\tau^*} \mathrm{Hom}_{\mathscr{T}(X)}\big(M(X/X-Z), M(\mathbb{P}^\infty_X)\big) \\ \xrightarrow{(c_X)_*} \mathrm{Hom}_{\mathscr{T}(X)}(M(X/X-Z), \mathbb{E}_X(1)[2]) = \mathbb{E}^{2,1}_Z(X). \end{aligned}$$

These Chern classes satisfy the following properties (see [Nav16, 1.39, 1.40]):

Proposition 2.4.3. *Consider the notations of the above definition.*

1. *Given any cartesian morphism $(Y,T) \to (X,Z)$ of closed pairs and any element $(L,u) \in \mathrm{Pic}(X,Z)$, one has: $f^*c_1^Z(L,u) = c_1^T\big(f^{-1}(L), f^{-1}(u)\big)$.*
2. *For any $(L,u) \in \mathrm{Pic}(X,Z)$, the cohomology class $c_1^Z(L)$ is nilpotent in the ring $\mathbb{E}^{**}_Z(X)$.*
3. *Let F_X be the formal group law associated with the orientation c_X of $\mathbb{E}_X$. Then, given any classes (L_1,u_1), (L_2,u_2) in $\mathrm{Pic}(X,Z)$, one has the following relation in $\mathbb{E}^{2,1}_Z(X)$:*
$$c_1^Z(L_1 \otimes L_2, u_1 \otimes u_2) = F_X\big(c_1^Z(L_1,u_1), c_1^Z(L_2,u_2)\big),$$
using the $\mathbb{E}^{**}(X)$-module structure on $\mathbb{E}^{**}_Z(X)$ and point (2).

The second tool needed in the method of Navarro is the following version of the blow-up formula for oriented theories (see [Nav16, 2.6]).

Proposition 2.4.4. *Consider an absolute oriented ring $\mathscr{T}$-spectrum $(\mathbb{E}, c)$. Let (X, Z) be a closed pair of codimension n, B be the blow up of X in Z and consider the following cartesian square:*

$$\begin{array}{ccc} P & \xrightarrow{k} & B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X. \end{array}$$

Then the following sequence is split exact:

$$0 \to \mathbb{E}^{**}_Z(X) \xrightarrow{p^*} \mathbb{E}^{**}_P(B) \xrightarrow{\overline{k^*}} \mathbb{E}^{**}(P)/\mathbb{E}^{**}(X) \to 0.$$

Letting $\mathcal{O}_P(-1)$ (resp. $\mathcal{O}_B(-1)$) be the canonical line bundle over P (resp. B), and putting $c = c_1(\mathcal{O}(-1))$ (resp. $b = c_1^P(\mathcal{O}_B(-1))$), a section is given by the following $\mathbb{E}^{**}(X)$-linear morphism:

$$s : \mathbb{E}^{**}(P)/\mathbb{E}^{**}(X) \simeq \big(\oplus_{i=1}^{n-1} \mathbb{E}^{**}(X).c^i\big) \to \mathbb{E}^{**}_P(B), c^i \mapsto b^i.$$

2.4.5. Consider again the assumption and notations of the preceding proposition. We can now explain the construction of Navarro [Nav16, 2.7].

One defines a canonical class in $\mathbb{E}_P^{2n,n}(B)$ (where n is seen as a locally constant function on P) as follows:

$$\bar{\eta}'_i := -\left(\sum_{i=0}^{n-1} q^*(c_i(N_Z X)).(-c)^{n-i}\right).b$$

where we have used the $\mathbb{E}^{**}(P)$-module structure on $\mathbb{E}_P^{**}(B)$. Then, according to the projective bundle theorem for $P = \mathbb{P}(N_Z X)$, one deduces that $k^*(\bar{\eta}'_i) = c_n(N_Z X)$ so that this class is zero in the quotient $\mathbb{E}^{**}(P)/\mathbb{E}^{**}(X)$. Therefore, according to the preceding proposition, there exists a unique class $\bar{\eta}_i \in \mathbb{E}_Z^{2n,n}(X)$ such that:

$$p^* \bar{\eta}_i = \bar{\eta}'_i\,.$$

This will be called the *orientation* of i associated with the orientation c of the ring spectrum $\mathbb{E}$.

According to Navarro's work, we get the following result.

Proposition 2.4.6. *The orientations $\bar{\eta}_i$ constructed above form a system of fundamental classes with coefficients in $\mathbb{E}$ for regular closed immersions.*

Besides, this system coincides with that of Proposition 2.3.5 when restricted to the closed immersions which admit a smooth retraction.

To prove the first assertion, we need to prove the associativity formula; this is [Nav16, Th. 2.14]. The second assertion follows from the compatibility of $\bar{\eta}_i$ with respect to base change along transversal squares ([Nav16, 2.12]) and from the fact the orientation defined above coincides with the refined Thom class when i is the zero section of a vector bundle ([Nav16, 2.19]).

Example 2.4.7. Consider the notations of the previous proposition.

(1) Let $i : D \to X$ be the immersion of a regular divisor in a scheme X. We let $\mathcal{O}(-D)$ be the line bundle on X corresponding to the inverse of the ideal sheaf of Z in X (following the convention of [EGA4, 21.2.8.1]). The sheaf $\mathcal{O}(-D)$ has support in D and admits a canonical trivialization s over $X - D$. As blowing-up a divisor does not do anything, it follows from the previous construction that we have the relation:

$$\bar{\eta}_i = c_1^D\big(\mathcal{O}(-D), s\big).$$

(2) Let $i : Z \to X$ be a regular closed immersion of codimension n. Recall we have a base change map:

$$i^* : \mathbb{E}_Z^{**}(X) = \mathbb{E}_{**}^{BM}(Z/X) \to \mathbb{E}_{**}^{BM}(Z/Z) = \mathbb{E}^{**}(Z),$$

— equivalently, the map forgetting the support.

Then, from the construction of Paragraph 2.4.5, we deduce the relation:

$$i^*(\bar{\eta}_i) = c_n(N_Z X).$$

2.5. **The global complete intersection case.** We are finally ready for the main theorem of this work. We have so far constructed several systems of fundamental classes and we now show how to glue them. The main result in order to do so is the following lemma.

Lemma 2.5.1. *Consider an absolute oriented ring spectrum* $(\mathbb{E}, c)$*. Let* $f : X \to S$ *be a smooth s-morphism and* $s : S \to X$ *a section of* f*.*

Then using the notations of Propositions 2.3.5 and 2.3.11, the following relation holds in $\mathbb{E}^{00}(X)$*:*

$$\bar{\eta}_s \,.\, \bar{\eta}_f = 1.$$

Proof. Let $V = N_S(X)$ be the normal bundle of S in X. By construction of the deformation space, we get a commutative diagram made of cartesian squares:

$$\begin{array}{ccccc}
S & \xrightarrow{s} & X & \xrightarrow{f} & S \\
{\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle s_1} \\
\mathbb{A}^1_S & \longrightarrow & D_S(X) & \xrightarrow{\tilde{f}} & \mathbb{A}^1_S \\
{\scriptstyle s_0}\uparrow & & \uparrow{\scriptstyle d_0} & & \uparrow{\scriptstyle s_0} \\
S & \xrightarrow{\sigma} & V & \xrightarrow{p} & S
\end{array}$$

where the two left columns are made by the deformation diagram (2.3.1.a) associated with the closed pair (X, Z), the morphism p is the canonical projection of V/S and $\tilde{f}$ is given by the composite map $D_S(X) \to D_S(S) \simeq \mathbb{A}^1_S$. An easy check shows that $\tilde{f}$ is smooth.

According to $\mathbb{A}^1$-homotopy invariance, the pullbacks $s_0^*, s_1^* : \mathbb{E}^{**}(\mathbb{A}^1_S) \to \mathbb{E}^{**}(S)$ are the same isomorphism. The orientations of the form η_f for f smooth are stable under pullbacks (Lemma 2.3.13) so that applying Remark 2.3.6, we are reduced to prove the relation

$$\bar{\eta}_\sigma \,.\, \bar{\eta}_p = 1.$$

In other words, we can assume $X = V$ is a vector bundle over S, $s = \sigma$ is its zero section and $f = p$ its canonical projection. We have seen in Remark 2.3.6 that η_σ coincides with the refined Thom class of V/S, via the canonical isomorphism (2.2.9.a). Similarly, from Remark 2.3.12(3), the orientation $\bar{\eta}_f$ is induced by the Thom class of the S-vector bundle $T_p = p^{-1}E$ via the isomorphism (2.3.12.a). As we obviously have: $\bar{\mathfrak{t}}(V) \cdot \bar{\mathfrak{t}}(-V) = 1$, we are reduced to prove the following lemma:

Lemma 2.5.2. *Given a vector bundle* V/S *with zero section* σ *and canonical projection* p*, the following diagram is commutative:*

$$\begin{array}{ccc}
\mathbb{E}^{BM}_{**}(S/V) \otimes \mathbb{E}^{BM}_{**}(V/S) & \xrightarrow{\bar{\mu}} & \\
{\scriptstyle \alpha_\sigma^* \otimes \mathfrak{p}'^*_p}\downarrow & & \mathbb{E}^{**}(S) \\
\mathbb{E}^{**}(M\mathrm{Th}(V)) \otimes \mathbb{E}^{**}(M\mathrm{Th}(-V)) & \xrightarrow{\mu} &
\end{array}$$

where μ *is the product on cohomology,* $\bar{\mu}$ *is the product of bivariant theory, while* α_σ^* *(resp.* $\mathfrak{p}'^*_p$*) is induced by the isomorphism* $\alpha_\sigma : \sigma_! \to \sigma_*$ *as* σ *is proper (resp. the purity isomorphism* (2.3.8.a)*).*

In the proof of this lemma, we will restrict ourselves to classes of degree $(0,0)$. It will healthily simplify notations and the proof for other degrees is the same. Let us thus consider the following maps, corresponding to Borel-Moore homology classes:

$$y : \sigma_!(\mathbb{1}_S) \to \mathbb{E}_V, v : p_!(\mathbb{1}_V) \to \mathbb{E}_S.$$

The commutativity of the preceding diagram then amounts to prove the following relation:

$$\bar{\mu}(y \otimes v) = \mu\big(\alpha_\sigma^*(y) \otimes \mathfrak{p}_p'^*(v)\big),$$

so we put: $\tilde{y} = \alpha_\sigma^*(y)$ and $\tilde{v} = \mathfrak{p}_p'^*(v)$. Then the preceding relation can be translated into the commutativity of the following diagram of isomorphisms:

$$\begin{array}{ccccccc}
 & & p_!\sigma_!(\mathbb{1}_S) & \xrightarrow{p_!(y)} & p_!(\mathbb{E}_V) \xrightarrow{\sim} \mathbb{E}_S \otimes p_!(\mathbb{1}_S) \xrightarrow{1\otimes v} & \mathbb{E}_S \otimes \mathbb{E}_S & \\
 & \overset{\epsilon_{p,\sigma}}{\nearrow} & \uparrow \mathfrak{p}'_p & & & \| & \searrow \mu_{\mathbb{E}} \\
 & & p_\sharp\big(\sigma_!(\mathbb{1}_S) \otimes M\mathrm{Th}(-T_p)\big) & & & \| & \\
 & & \downarrow Ex_\sharp^\otimes & & & \| & \\
\mathbb{1}_S & (1) & p_\sharp\sigma_!(\mathbb{1}_S) \otimes M\mathrm{Th}(-V) & & (2) & \| & E_S \\
 & & \downarrow \alpha_\sigma & & & \| & \\
 & & p_\sharp\sigma_*(\mathbb{1}_S) \otimes M\mathrm{Th}(-V) & & & \| & \nearrow \mu_{\mathbb{E}} \\
 & \underset{can}{\searrow} & \| & & & & \\
 & & M\mathrm{Th}(V) \otimes M\mathrm{Th}(-V) & \xrightarrow{\tilde{y}\otimes\tilde{v}} & & \mathbb{E}_S \otimes \mathbb{E}_S, &
\end{array}$$

where:

- the morphism labelled $\epsilon_{p,\sigma}$ stands for the inverse of the canonical isomorphism $Id = (p \circ \sigma)_! \to p_!\sigma_!$ coming from the fact $f \mapsto f_!$ corresponds to a 2-functor;
- the map labelled $Ex_\sharp^\otimes$ is the exchange isomorphism corresponding to the projection formula for $f_\sharp$ (cf. [CD12b, §1.1.24]) — using the identification $M\mathrm{Th}(-T_p) = p^* M\mathrm{Th}(-V)$;
- the map labelled *can* follows from the definition of the Thom space of the virtual bundle $(-V)$.

The commutativity of part (2) follows directly from the definition of $\tilde{y}$ and $\tilde{v}$. Thus we only need to show the commutativity of part (1) of the above diagram.

After taking tensor product with $M\mathrm{Th}(V)$, the diagram (1) can be simplified as follows:

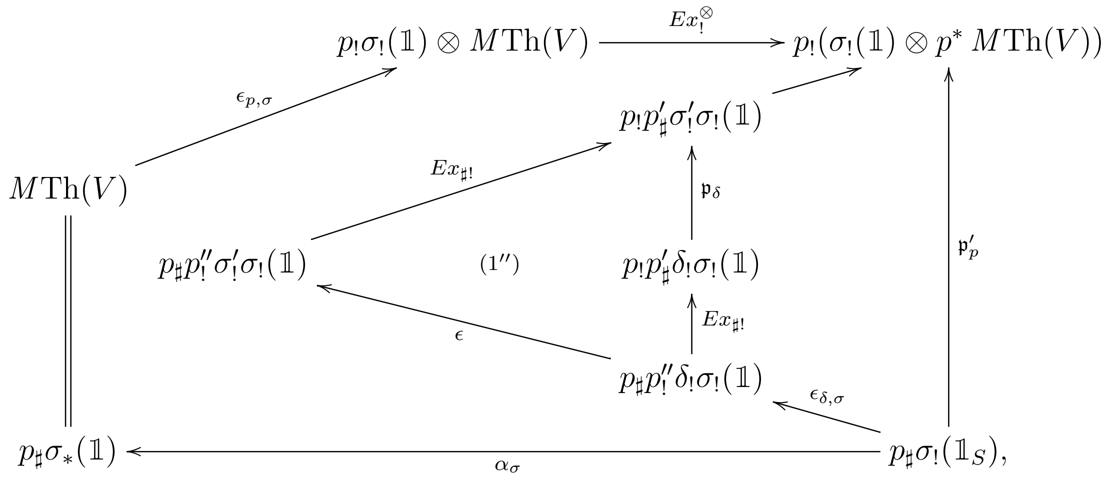
$$\begin{array}{ccccc}
 & & p_!\sigma_!(\mathbb{1}_S) \otimes M\mathrm{Th}(V) & \xrightarrow{Ex_!^\otimes} & p_!(\sigma_!(\mathbb{1}) \otimes p^* M\mathrm{Th}(V)) \\
 & \overset{\epsilon_{p,\sigma}}{\nearrow} & & & \uparrow \mathfrak{p}'_p \\
M\mathrm{Th}(V) & & (1') & & p_\sharp\sigma_!(\mathbb{1}_S) \\
 & & & & \downarrow \alpha_\sigma \\
 & & = = = = = = & & p_\sharp\sigma_*(\mathbb{1}_S),
\end{array}$$

where the arrow $Ex_!^\otimes$ stands for the exchange isomorphism of the projection formula for $p_!$ (see [CD12b, 2.2.12]).

Let us first summarize the geometric situation in the following commutative diagram of schemes:

$$\begin{array}{ccccccc} S & \xrightarrow{\sigma} & V & \underset{\sigma'}{\overset{\delta}{\longrightarrow}} & W & \xrightarrow{p'} & V \\ & & {\scriptstyle p}\downarrow & \Theta & \downarrow{\scriptstyle p''} & \Delta & \downarrow{\scriptstyle p} \\ & & S & \xrightarrow[\sigma]{} & V & \xrightarrow[p]{} & S, \end{array}$$

where each square is cartesian and δ denotes the obvious diagonal embedding. The map $p'' : W = V \times_S V \to V$ is the projection on the second factor and in particular, we get: $\sigma'(v) = (v, p(v))$. Note finally that σ is an equalizer of (δ, σ').
Then coming back to the definition of the purity isomorphism $\mathfrak{p}'_p$ (cf. Paragraph 2.3.8), diagram $(1')$ can be divided as follows:



where $\mathfrak{p}_\delta$ is (induced by) the purity isomorphism associated with the closed immersion δ (Definition 2.3.3), the map labelled $Ex_{\sharp!}$ stands for the obvious exchange isomorphism associated with the cartesian square Δ, and ϵ is the isomorphism coming from the fact $f \mapsto f_!$ corresponds to a 2-functor and the relation $\delta \circ \sigma = \sigma' \circ \sigma$.

In this diagram, the commutativity of the right hand side follows by definition of $\mathfrak{p}'_p$ and the commutativity of the left hand side follows from the definition of the involved exchange isomorphisms and the fact $\alpha_f : f_! \to f_*$ corresponds to a morphism of 2-functors. The geometry is hidden in the commutativity of the diagram labelled $(1'')$.

We can divide again $(1'')$ as follows:

$$\begin{array}{ccccc} p_\sharp p''_!\sigma'_!\sigma_!(\mathbb{1}) & \xrightarrow{Ex_{\sharp!}} & p_!p'_\sharp\sigma'_!\sigma_!(\mathbb{1}) & = & p_!p'_\sharp\sigma'_!\sigma_!(\mathbb{1}) \\ {\scriptstyle\epsilon}\uparrow & & \uparrow{\scriptstyle\epsilon} & (1''') & \uparrow{\scriptstyle\mathfrak{p}_\delta} \\ p_\sharp p''_!\delta_!\sigma_!(\mathbb{1}) & \xrightarrow{Ex_{\sharp!}} & p_!p'_\sharp\delta_!\sigma_!(\mathbb{1}) & = & p_!p'_\sharp\delta_!\sigma_!(\mathbb{1}) \end{array}$$

The commutativity of the left hand square follows from the naturality of the exchange isomorphism, it remains part ($1'''$). We can certainly erase the functor $p_!$ in each edge of this diagram. Then the right-most vertical map can be expressed as follows:

$$M(W/W-\delta(V))\otimes\sigma_!(\mathbb{1})\xrightarrow{\mathfrak{p}_\delta} M\mathrm{Th}(N_\delta/V)\otimes\sigma_!(\mathbb{1})$$

and the commutativity of diagram ($1'''$) means that this map is the identity. Using once again the projection formula for σ, this map can be expressed as follows:

$$\sigma_!\sigma^*\big(M(W/W-\delta(V))\big)\xrightarrow{\sigma_!\sigma^*(\mathfrak{p}_\delta)}\sigma_!\sigma^*\,M\mathrm{Th}(N_\delta/V).$$

So we are reduced to show that $\sigma^*(\mathfrak{p}_\delta)$ is the identity map through the obvious identifications:

$$\sigma^*M(W/W-\delta(V))=M(V/V-S)=\sigma^*\,M\mathrm{Th}(N_\delta/V).$$

This is an easy geometric fact: let us consider the deformation diagram for the closed immersion $\delta:V\to W$:

$$\begin{array}{ccccc} V & \to & \mathbb{A}^1_V & \leftarrow & V \\ \downarrow & & \downarrow & & \downarrow \\ W & \to & D_\delta & \leftarrow & N_\delta. \end{array}$$

Note that the closed immersion δ and σ' are transversal — *i.e.* the square Θ is transversal. In other words, $\sigma^*(N_\delta)=N_\sigma=V$ as a vector bundle over S. In particular, the pullback of the preceding diagram of V-schemes along the immersion $\sigma:S\to V$ is the following one:

$$\begin{array}{ccccc} S & \to & \mathbb{A}^1_S & \leftarrow & S \\ \downarrow & & \downarrow & & \downarrow \\ V & \to & \mathbb{A}^1_V & \leftarrow & V. \end{array}$$

where we have used the identifications $\sigma^*(D_\delta)=D_\sigma=\mathbb{A}^1_V$; the last identification is justified by the fact σ is the zero section of a vector bundle. In this diagram, the vertical maps are the unit and zero sections of the affine lines involved. Therefore, by homotopy invariance, we get that $\sigma^*(\mathfrak{p}_\delta)$ is identified to the identity map as required. □

We are now ready to state our main theorem.

Theorem 2.5.3. *Let $(\mathbb{E},c)$ be an absolute oriented ring $\mathscr{T}$-spectrum.*

There exists a unique system of fundamental classes $\bar\eta_f\in\mathbb{E}^{BM}_{**}(X/S)$ for gci morphisms f with coefficients in $\mathbb{E}$ such that, in addition to the associativity property, one has:

1. *If f is a smooth morphism, $\bar\eta_f$ coincides with the fundamental class defined in Proposition 2.3.11.*
2. *If $i:Z\to X$ is a regular closed immersion, $\bar\eta_i$ coincides with the fundamental class defined in 2.4.6.*

If d is the relative dimension of f, seen as a Zariski local function on X, the class η_f has dimension d (Definition 2.1.1).

Proof. Because any gci morphism $p : X \to S$ admits a factorization

$$X \xrightarrow{i} P \xrightarrow{f} S$$

where f is smooth and i is a regular closed immersion, we have to prove that the class $\bar{\eta}_i \cdot \bar{\eta}_f$ is independent of the factorization.

To prove this, we are reduced by usual arguments (see for example [Dég08, proof of 5.11]) and the help of Lemma 2.3.13 to show the associativity property:

$$\bar{\eta}_g \cdot \bar{\eta}_f = \bar{\eta}_{fg}$$

in the following three cases:

(a) f and g are smooth morphisms;
(b) f and g are regular closed immersions;
(c) g is a smooth morphism and f is a section of g.

Case (a) follows from Proposition 2.3.11, case (b) from Proposition 2.4.6 and case (c) from the preceding lemma. Then the associativity formula in the general case follows using standard arguments (see for example [Dég08, proof of 5.14]) from (a), (b) and (c).

The last assertion follows as the degree of $\bar{\eta}_i$ (resp. $\bar{\eta}_p$) is the opposite of the codimension n of i (resp. the dimension r of p) and we have: $d = r - n$. □

Remark 2.5.4. When f is a smooth morphism, it follows from Proposition 2.3.11 that $\bar{\eta}_f$ is universally strong (Definition 2.1.6). When $f = s$ is the section of a smooth s-morphism, property (2) in the above theorem and the last assertion of Proposition 2.4.6 shows that the class $\bar{\eta}_s$ constructed in the above theorem coincides with that of Proposition 2.3.5. Therefore $\bar{\eta}_s$ is universally strong according to *loc. cit.* This remark will be amplified in Section 4.

Definition 2.5.5. Given the assumptions of the previous proposition, for any gci morphism $f : X \to S$, we call $\bar{\eta}_f$ the *fundamental class* of f associated with the orientation c of the absolute ring spectrum $\mathbb{E}$.

In case $f = i : Z \to X$ is a regular closed immersion, we will also use the notation:

$$\bar{\eta}_X(Z) := \bar{\eta}_i$$

seen as an element of $\mathbb{E}^{2c,c}_Z(X)$ where c is the codimension of i — as a locally constant function on Z.

This fundamental class only depends upon the choice of the (global) orientation c of $\mathbb{E}$. If there is a possible confusion about the chosen orientation, we write $\bar{\eta}^c_f$ instead of $\bar{\eta}_f$.

We often find another notion of fundamental class in the literature that we introduce now for completeness.

Definition 2.5.6. Let $(\mathbb{E}, c)$ be an absolute oriented ring spectrum and $f : Y \to X$ be a proper gci morphism. We define the fundamental class of f in $\mathbb{E}$-cohomology, denoted by η_f, as the image of $\bar{\eta}_f$ by the map:

$$\mathbb{E}^{BM}_{**}(Y/X) \xrightarrow{f_!} \mathbb{E}^{BM}_{**}(X/X) = \mathbb{E}^{**}(X).$$

We will give more details on these classes in section 3.3.

2.6. **The quasi-projective lci case.** We end this section by presenting an alternative method to build fundamental classes, when restricting to quasi-projective lci morphism. This is based on the following uniqueness result.

Theorem 2.6.1. *Let $(\mathbb{E}, c)$ be an absolute oriented $\mathscr{T}$-spectrum.*

Then a family of orientations $\bar{\eta}_f \in \mathbb{E}^{BM}_{**}(X/S)$ attached to quasi-projective lci morphisms $f : X \to S$ is uniquely characterized by the following properties:

(1) *If $j : U \to X$ is an open immersion, $\bar{\eta}_j$ is equal to the following composite:*
$$\mathbb{1}_X \xrightarrow{\eta} \mathbb{E}_X \simeq j^*(\mathbb{E}_U) = j^!(\mathbb{E}_U)$$
where η is the unit of the ring spectrum $\mathbb{E}_X$.

(2) *If s is the zero section of a line bundle L/S, one has: $\bar{\eta}_s = c_1^Z(L)$ (notation of Definition 2.4.2).*

(3) *If p is the projection map of $\mathbb{P}^n_S/S$, $\bar{\eta}_p$ is given by the construction* (2.3.8.a).

(4) *Given any composable pair of morphism (f, g), one has the relation $\bar{\eta}_f \, . \, \bar{\eta}_g = \bar{\eta}_{g\circ f}$ whenever f and g are immersions, or f is the projection of $\mathbb{P}^n_S/S$ and g is an immersion.*

(5) *Let $i : Z \to X$ be a regular closed immersion and $f : X' \to X$ a morphism transversal to i. Put $k = f^{-1}(i)$. Then the following relation holds: $f^* \bar{\eta}_i = \bar{\eta}_k$.*

(6) *Consider the blow-up square of a closed immersion i of codimension n,*
$$\begin{array}{ccc} E & \xrightarrow{k} & B \\ \downarrow & & \downarrow{\scriptstyle p} \\ Z & \xrightarrow{i} & X \end{array}$$
the following relation holds: $p^* \bar{\eta}_i = c_{n-1}(N_Z X) . \bar{\eta}_k$.

Proof. Let us prove the uniqueness of $\bar{\eta}_f$ for a quasi-projective lci morphism f. The map f admits a factorization $f = pji$ where p is the projection of $\mathbb{P}^n_S$ for a suitable integer $n \geq 0$, j is an open immersion and i a closed immersion. According to property (4), we reduce the case of f to that of p, j or i. The case of j follows from (1), and that of p follows from (3).

So we are reduced to the case of a regular closed immersion $i : Z \to X$. According to property (6) and its notations, we are reduced to the case of the regular closed immersion k. In other words, we can assume i has codimension 1.

Then Z corresponds to an effective Cartier divisor in X, and therefore to a line bundle L/X with a canonical section $s : X \to L$ such that the following diagram is cartesian:
$$\begin{array}{ccc} Z & \xrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{s_0} & L \end{array}$$
where s_0 is the zero-section of the line bundle L/X. According to relation (5), we get: $s^* \bar{\eta}_{s_0} = \bar{\eta}_i$. This uniquely characterize $\bar{\eta}_i$ because $\bar{\eta}_{s_0}$ is prescribed by relation (2). □

2.6.2. In fact, it is possible to show the existence of fundamental classes in the quasi-projective lci case by using the constructions in the preceding proof and the techniques of [Dég08] (see more specifically [Dég08, Sec. 5]). This gives an alternate method where the construction of Ayoub 2.3.8 is avoided.

The interest of this method is that, instead of using the axiomatic of triangulated motivic categories, one can directly work with a given bivariant theory satisfying suitable axioms: in fact, the properties stated in Proposition 1.2.4. Then one can recover the construction of a system of fundamental classes for quasi-projective lci morphisms and proves the properties that we will see in the forthcoming section.

3. Intersection and generalized Riemann-Roch formulas

3.1. **Base change formula.** In all this section, we will fix once and for all an absolute oriented ring $\mathscr{T}$-spectrum $(\mathbb{E}, c)$. We first state the following extension of the classical excess intersection formula.

Proposition 3.1.1 (Excess of intersection formula)**.** *Consider a cartesian square*

$$\begin{array}{ccc} Y & \xrightarrow{g} & T \\ {\scriptstyle q}\downarrow & \Delta & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & S \end{array}$$

of schemes such that f, g are gci. Let $\tau_f \in K_0(X)$ (resp. $\tau_g \in K_0(Y)$) be the virtual tangent bundle of f (resp. g). We put $\xi = p^*(\tau_f) - \tau_g$ as an element of $K_0(Y)$,[36] *and let $e(\xi)$ be the top Chern class of ξ in $\mathbb{E}^{**}(Y)$. Then the following formula holds in $\mathbb{E}^{BM}_{**}(Y/T)$:*

$$p^*(\bar{\eta}_f) = e(\xi) \cdot \bar{\eta}_g \,.$$

In fact, we can consider a factorization of f into a regular closed immersion i and a smooth morphism p. Because of property (3) of Theorem 2.5.3, we are reduced to the case $f = i$, regular closed immersion, or $f = p$, smooth morphism. The first case is [Nav16, Cor. 2.12] while the second case was proved in Lemma 2.3.13.

Example 3.1.2. Of course, an interesting case is obtained when the square Δ is transversal, *i.e.* $\xi = 0$: it shows, as expected, that fundamental classes are stable by pullback along transversal morphisms.[37]

3.1.3. Fundamental classes in the case of closed immersions give an incarnation of intersection theory. Let us consider the enlightening case of divisors. In fact one can extend slightly the notion of fundamental classes from effective Cartier divisors to that of *pseudo-divisors* as defined by Fulton [Ful98, 2.2.1]

[36] The element ξ is called the *excess intersection bundle* associated with the square Δ;

[37] To be clear: a morphism of schemes $p : T \to S$ is transversal to a gci morphism $f : X \to S$ if $g = f \times_S T$ is gci and $p^*(\tau_f) = \tau_g$ as elements of $K_0(X \times_S T)$.

Definition 3.1.4. Let $D = (\mathcal{L}, Z, s)$ be a pseudo-divisor on a scheme X. We define the fundamental class of D in X with coefficients in $(\mathbb{E}, c)$ as:

$$\bar{\eta}_X(D) := c_1^Z(\mathcal{L}, s) \in \mathbb{E}_Z^{2,1}(X).$$

The following properties of these extended fundamental classes immediately follows from Proposition 2.4.3.

Proposition 3.1.5. *Let X be a scheme.*

(1) *For any pseudo-divisor D on X with support Z, the class $\bar{\eta}(D)$ is nilpotent in the ring $\mathbb{E}_Z^{**}(X)$.*

(2) *Let $(D_1, ..., D_r)$ be pseudo-divisors on X with support in a subscheme $Z \subset X$ and $(n_1, ..., n_r) \in \mathbb{Z}^r$ an r-uple/ One has the following relations in the ring $\mathbb{E}_Z^{**}(X)$:*

$$\bar{\eta}(n_1.D_1 + \ldots + n_r.D_r) = [n_1]_F . \bar{\eta}(D_1) +_F \ldots +_F [n_r]_F . \bar{\eta}(D_r)$$

where $+_F$ (resp. $[n]_F$ for an integer $n \in \mathbb{Z}$) means the addition (resp. n-th self addition) for the formal group law with coefficients in $\mathbb{E}^{**}(X)$ associated with the orientation c (Proposition 2.2.6).

(3) *Let $f : Y \to X$ be any morphism of schemes. Then for any pseudo-divisor D with support Z, $T = f^{-1}(Z)$, on has in $\mathbb{E}_T^{**}(Y)$:*

$$f^*(\bar{\eta}_X(D)) = \bar{\eta}_Y(f^*(D))$$

where f^* on the right hand side is the pullback of pseudo-divisors as defined in [Ful98, 2.2.2].

In particular, it is worth to derive the following corollary which describes more precisely the pullback operation on fundamental classes associated with divisors.

Corollary 3.1.6. *Let X be a normal scheme.*

(1) *For any Cartier divisor D on X, one has the relation*

$$\bar{\eta}_X(D) = [n_1]_F . \bar{\eta}_X(D_1) +_F \ldots +_F [n_r]_F . \bar{\eta}_X(D_r)$$

in $\mathbb{E}_Z^{2,1}(X)$,where Z in the support of D, $(D_i)_i$ the family of irreducible components of Z and n_i is the multiplicity of D at D_i.[38]

(2) *Let $f : Y \to X$ be a dominant morphism of normal schemes. Then the pullback divisor $E = f^{-1}(D)$ is defined, as a Cartier divisor, and if one denotes $(E_j)_{j=1,...,r}$ the family of irreducible components of the support T of E, and m_j the intersection multiplicity of E_j in the pullback of D along f (*i.e.* the multiplicity of E_j in the Cartier divisor E), one has the relation in $\mathbb{E}_T^{2,1}(Y)$:*

$$f^*(\bar{\eta}_X(D)) = [m_1]_F . \bar{\eta}_Y(E_1) +_F \ldots +_F [m_r]_F . \bar{\eta}_Y(E_r).$$

[38] In case D is effective, this is the geometric multiplicity of D, seen as a regular closed subscheme of X, at the generic point of D_i.

3.2. **Riemann-Roch formulas.**

3.2.1. We now show that we can derive from our theory many generalized Riemann-Roch formulas in the sense of Fulton and MacPherson's bivariant theories ([FM81, I.1.4]). This is based on the construction of Todd classes. Let us fix a morphism of absolute ring spectra

$$(\varphi, \phi) : (\mathscr{T}, \mathbb{E}) \to (\mathscr{T}', \mathbb{F})$$

as in Definition 1.1.4.

Suppose c (resp. d) is an orientation of the ring spectrum $\mathbb{E}$ (resp. $\mathbb{F}$). Given a base scheme S, we obtain following Paragraph 1.2.12 a morphism of graded rings:

$$\phi_*^{\mathbb{P}^\infty_S} : \mathbb{E}^{**}(\mathbb{P}^\infty_S) \to \mathbb{F}^{**}(\mathbb{P}^\infty_S)$$

— induced by the Grothendieck transformation of *loc. cit.* According to the projective bundle theorem satisfied by the oriented ring spectra $(\mathbb{E}_S, c_S)$ and $(\mathbb{F}_S, d_S)$, this corresponds to a morphism of rings:

$$\mathbb{E}^{**}(S)[[u]] \to \mathbb{F}^{**}(S)[[t]]$$

and we denote by $\Psi_\phi(t)$ the image of u by this map. In other words, the formal power series $\Psi_\phi(t)$ is characterized by the relation:

$$\phi_*^{\mathbb{P}^\infty_S}(c) = \Psi_\phi(d). \tag{3.2.1.a}$$

Note that the restriction of $\phi_*^{\mathbb{P}^\infty_S}(c)$ to $\mathbb{P}^0_S$ (resp. $\mathbb{P}^1_S$) is 0 (resp. 1) because c is an orientation and φ is a morphism of ring spectra. Thus we can write $\Psi_\phi(t)$ as:

$$\Psi_\phi(t) = t + \sum_{i>1} \alpha_i^S.t^i$$

where $\alpha_i^S \in \mathbb{F}^{2-2i,1-i}(S)$. In particular, the power series $\Psi_\phi(t)/t$ is invertible.

We next consider the commutative monoid $\mathcal{M}(S)$ generated by the isomorphism classes of vector bundles over S modulo the relations $[E] = [E'] + [E'']$ coming from exact sequences

$$0 \to E' \to E \to E'' \to 0.$$

Then $\mathcal{M}$ is a presheaf of monoids on the category $\mathscr{S}$ whose associated presheaf of abelian groups is the functor K_0.

Note that $\mathbb{F}^{00}(S)$, equipped with the cup-product, is a commutative monoid. We will denote by $\mathbb{F}^{00\times}(S)$ the group made by its invertible elements.

Proposition 3.2.2. *There exists a unique natural transformation of presheaves of monoids over the category* $\mathscr{S}$

$$\mathrm{Td}_\phi : \mathcal{M} \to \mathbb{F}^{00}$$

such that for any line bundle L over a scheme S,

$$\mathrm{Td}_\phi(L) = \frac{t}{\Psi_\phi(t)}.d_1(L). \tag{3.2.2.a}$$

Moreover, it induces a natural transformation of presheaves of abelian groups:

$$\mathrm{Td}_\phi : K_0 \to \mathbb{F}^{00\times}.$$

The proof is straightforward using the splitting principles (see [Dég14b, 4.1.2]).

Remark 3.2.3. According to the construction of the first Chern classes for the oriented ring spectra $(\mathbb{E}, c)$ and $(\mathbb{F}, d)$ together with Relations (3.2.1.a) and (3.2.2.a), we get for any line bundle L/S the following identity in $\mathbb{F}^{2,1}(S)$:

(3.2.3.a) $$\varphi_S\big(c_1(L)\big) = \mathrm{Td}_\varphi(-L) \cup d_1(L).$$

Definition 3.2.4. Consider the context and notations of the previous proposition.

Given any virtual vector bundle v over a scheme S, the element $\mathrm{Td}_\phi(v) \in \mathbb{F}^{00}(S)$ is called the *Todd class* of v over S associated with the morphism of ring spectra (φ, ϕ).

The main property of Todd classes is the following formula.

Lemma 3.2.5. *Consider the above notations and assumptions.*

Then for any smooth S-scheme X and any virtual vector bundle v over X, the following relation holds in $\mathbb{F}^{**}(\mathrm{Th}(v))$:

$$\phi_*\left(\bar{\mathfrak{t}}^{\mathbb{E}}(v)\right) = \mathrm{Td}_\phi(-v).\,\bar{\mathfrak{t}}^{\mathbb{F}}(v)$$

where $\bar{\mathfrak{t}}(v)$ denotes the Thom class associated with v (see Definition 2.2.19).

Proof. Recall that for any virtual bundles v and v' over X (see Paragraph 2.2.18), the tensor product in $\mathscr{T}(S)$ gives a pairing

$$\otimes_X : \mathbb{E}^{**}(\mathrm{Th}(v)) \otimes_X \mathbb{E}^{**}(\mathrm{Th}(v')) \to \mathbb{E}^{**}(\mathrm{Th}(v+v'))$$

and similarly for $\mathbb{F}^{**}$. It follows from definitions that the natural transformation of cohomology theories $\phi_* : \mathbb{E}^{**} \to \mathbb{F}^{**}$ is compatible with this product.

Moreover, we have the relations:

$$\bar{\mathfrak{t}}(v+v') = \bar{\mathfrak{t}}(v) \otimes_X \bar{\mathfrak{t}}(v'),$$
$$\mathrm{Td}_\phi(v+v') = \mathrm{Td}_\phi(v) + \mathrm{Td}_\phi(v').$$

Therefore, by definition of the Thom class of a virtual bundle (see 2.2.19), it is sufficient to check the relation of the proposition when $v = [E]$ is the class of a vector bundle over X. Besides, using again the preceding relations and the splitting principle, one reduces to the case of a line bundle L. But then, in the cohomology of the projective completion $\bar{L}$ of L/X, we have the following relation

$$\mathfrak{t}(L) = c_1(\xi_L)$$

where ξ_L is the universal quotient bundle. Thus the desired relation follows from relation (3.2.3.a) and the fact $\mathrm{Td}_\phi(\xi_L) = \mathrm{Td}_\phi(L)$. □

We can now derive the generalized Riemann-Roch formula.

Theorem 3.2.6. *Let $(\mathscr{T}, \mathbb{E}, c)$ and $(\mathscr{T}', \mathbb{F}, d)$ be absolute oriented ring spectra together with a morphism of ring spectra:*

$$(\varphi, \phi) : (\mathscr{T}, \mathbb{E}) \to (\mathscr{T}', \mathbb{F}).$$

Using the notations of the Definitions 2.5.5 and 3.2.4, for any gci morphism $f : X \to S$ with virtual tangent bundle τ_f, one has the following relation:

$$\phi_*(\bar{\eta}_f^{\mathbb{E}}) = \mathrm{Td}_\phi(\tau_f).\, \bar{\eta}_f^{\mathbb{F}}\,.$$

Proof. As f is gci and because of the associativity property of our system of fundamental classes, we are reduced to the cases where $f = i$ is a regular closed immersion and f is a smooth morphism.

In the first case, we can use the deformation diagram (2.3.1.a) and the fact fundamental classes are stable by transversal base change to reduce to the case of the zero section $s = f$ of a vector bundle E/X. Then we recall that the fundamental class η_s coincides with the Thom class associated with E so that the preceding lemma concludes.

In the second case, we come back to the construction of Paragraph 2.3.10, and more precisely Remark 2.3.12(3). It is clear that the map ϕ_* is compatible with the isomorphisms (2.3.12.a), compute either in $\mathscr{T}$ or in $\mathscr{T}'$, as they are all build using exchange transformations. So we are reduced again to the case where the fundamental class is $\bar{\mathfrak{t}}(-T_f)$, which follows from the preceding lemma. □

Example 3.2.7. Let us fix a gci morphism $f : X \to S$

(1) Given an absolute ring spectrum $\mathbb{E}$, we have seen in Paragraph 2.2.12 that the data of an orientation c on $\mathbb{E}$ is equivalent to that of a morphism of ring spectra

$$\phi : \mathbf{MGL} \to \mathbb{E}$$

such that $\phi_*(c^{\mathbf{MGL}}) = c$, where $c^{\mathbf{MGL}}$ is the canonical orientation of $\mathbf{MGL}$. In that case, the previous theorem gives us the relation:

$$\phi_*(\bar{\eta}_f^{\mathbf{MGL}}) = \bar{\eta}_f^{\mathbb{E}}\,.$$

In other words, the fundamental classes $\bar{\eta}_f$ are all induced by the one defined in algebraic cobordism.

(2) Next we can apply the previous formula to the morphisms of absolute ring spectra of Example 1.1.6(2). Let us fix a prime ℓ and consider the two following cases:
- $\mathscr{S}$ is the category of all schemes, $\Lambda = \mathbb{Q}$, $\Lambda_\ell = \mathbb{Q}_\ell$;
- $\mathscr{S}$ is the category of k-schemes for a field k if characteristic $p \neq \ell$, $\Lambda = \mathbb{Z}$, $\Lambda_\ell = \mathbb{Z}_\ell$;

Then according to *loc. cit.*, we get a morphism of absolute ring spectra:

$$\rho_\ell : \mathbb{H}\Lambda \to \mathbb{H}_{\text{ét}}\Lambda_\ell,$$

corresponding to the higher étale cycle class. As the formal group laws associated with the canonical orientations on each spectra are additive, the morphism of formal group law associated with the induced morphism $\rho_\ell : \mathbb{H}^{**}(-,\Lambda) \to \mathbb{H}^{**}_{\text{ét}}(-,\Lambda)$ is the identity. Therefore, the Todd class associated with ϕ is constant equal to 1 and we get:

$$\rho_\ell(\bar{\eta}_f) = \bar{\eta}_f^{\text{ét}}\,.$$

(3) Consider the Chern character

$$\mathrm{ch}: \mathbf{KGL} \longrightarrow \bigoplus_{i\in\mathbb{Z}} \mathbb{H}\mathbb{Q}(i)[2i],$$

of Example 1.1.6(3). As explained in [Dég14b, 5.3.3], the formal group law associated with the canonical orientation of **KGL** is multiplicative: $F_{\mathbf{KGL}}(x,y) = x + y - \beta.xy$ where β is the Bott element in algebraic K-theory and the formal group law on rational motivic cohomology is the additive formal group law. As β is sent to 1 by the Chern character, the morphism of formal group law associated with ϕ is necessarily the exponential one, $t \mapsto 1 - \exp(-t)$. Therefore, we get the Todd class associated with ch_t defined on a line bundle L/X as:

$$\mathrm{Td}(L) = \frac{c_1(L)}{1 - \exp(-c_1(L))}.$$

Recall this formula makes sense as $c_1(L)$ is nilpotent in the motivic cohomology ring $\mathbf{H}^{**}_{\mathrm{B}}(X)$ (recall the notation of Example 1.1.2(2)). So in fact the transformation Td is the usual Todd class in motivic cohomology, and we have obtained the following generalized Riemann-Roch formula:

$$\mathrm{ch}(\bar{\eta}_f^{\mathbf{KGL}}) = \mathrm{Td}(\tau_f).\,\bar{\eta}_f^{\mathbb{H}\mathbb{Q}}.$$

This generalizes the original formula of Fulton and MacPherson (cf. [FM81, II, 1.4]).

For more examples, we refer the reader to [Dég14b, §5].

3.3. Gysin morphisms.

3.3.1. Recall that a weakly oriented (Definition 2.2.13) absolute $\mathscr{T}$-spectrum $\mathbb{E}$ is a τ-module over the ring spectrum **MGL** where $\tau^* : S\mathscr{H} \to \mathscr{T}$ is the premotivic adjunction fixed according to our convention.

Recall from Remark 1.2.14 that we get in particular an action:

$$\mathbf{MGL}^{BM}_{n,m}(Y/X) \otimes \mathbb{E}^{BM}_{s,t}(X/S) \to \mathbb{E}^{BM}_{n+s,m+t}(Y/S).$$

Then Theorem 2.5.3 induces the following constructions.

Definition 3.3.2. Consider the above notation. Then for any gci morphism $f : Y \to X$, with relative dimension d and fundamental class $\bar{\eta}_f \in \mathbf{MGL}^{BM}_{2d,d}(Y/X)$, we define the following *Gysin morphisms*:

- if f is a morphism of s-schemes over a base S, one gets a pullback
$$f^* : \mathbb{E}^{BM}_{**}(X/S) \to \mathbb{E}^{BM}_{**}(Y/S), x \mapsto \bar{\eta}_f\,.x$$
homogeneous of degree $(2d, d)$;
- if f is proper, one gets a pushforward
$$f_* : \mathbb{E}^{**}(Y) = \mathbb{E}^{BM}_{**}(Y/Y) \xrightarrow{.\bar{\eta}_f} \mathbb{E}^{BM}_{**}(Y/X) \xrightarrow{f_!} \mathbb{E}^{BM}_{**}(X/X) = \mathbb{E}^{**}(X)$$
homogeneous of degree $(-2d, -d)$.

According to this definition, and the fact we have in fact constructed in Theorem 2.5.3 a system of fundamental classes, which includes in particular the compatibility with composition (Definition 2.1.9), we immediately get that these Gysin morphisms are compatible with composition.

Remark 3.3.3. (1) When we consider the stronger case of an absolute oriented ring spectrum $\mathbb{E}$ in the sense of Definition 2.2.2, in the preceding definition, one can consider $\bar{\eta}_f$ as the fundamental class associated with f in $\mathbb{E}^{BM}_{2d,d}(X/S)$ and only use the product of the bivariant theory $\mathbb{E}^{BM}_{**}$. In fact, the two definitions coincide because of the Grothendieck-Riemann-Roch formula below (Proposition 3.3.11) and the fact the morphism $\phi : \mathbf{MGL} \to \mathbb{E}$ of ring spectra corresponding to the chosen orientation c sends the canonical orientation of $\mathbf{MGL}$ to c.

(2) Gysin morphisms, in the case of the Borel-Moore homology associated with an absolute oriented ring $\mathscr{T}$-spectrum $(\mathbb{E}, c)$, extends the one already obtained with respect to étale morphism in Paragraph 1.2.3. This rightly follows from the construction of the fundamental class in the case of étale morphisms.

Example 3.3.4. We can use the construction of the preceding definition in the case of all the ring spectra of Example 1.1.2 (according to Example 2.2.4). This gives back the Gysin morphisms on representable cohomologies as constructed in [Nav16], and notably covariant functoriality of Spitzweck integral motivic cohomology (cf. Ex. 1.1.2(4)) with respect to any gci proper morphism of schemes.

The important new case we get out of our theory is given when Λ is any ring (resp. $\Lambda = \mathbb{Z}_\ell, \mathbb{Q}_\ell$) and $\mathbb{H}_{\text{ét}}\Lambda$ is the étale motivic absolute Λ-spectrum (resp. ℓ-completed étale motivic absolute spectrum, integral or rational) as in Example 1.1.2(3). The Gysin morphisms obtained here, for the corresponding cohomology and any gci proper morphism of schemes, cannot be deduced from Navarro's result (as explained in the end of Remark 1.1.7).

When Λ is a torsion ring, this gives covariant functoriality for the classical étale cohomology with Λ-coefficients, for any gci proper morphism. This was known for flat proper morphisms by [SGA4, XVII, 2.13] and for proper morphisms between regular schemes by [Dég14b, 6.2.1].

Remark 3.3.5. The only class of morphisms containing both flat morphisms and local complete morphisms is the class of morphisms of finite Tor dimension.[39] This seems to be the largest class of morphisms for which fundamental classes an exceptional functoriality can exist. Indeed, we have the example of Quillen's higher algebraic G-theory: it is contravariant with respect to such morphisms as follows from [Qui10, §7, 2.5]. Unfortunately, our method is powerless to treat this generality.

Example 3.3.6. Another set of examples is obtained in the case of Borel-Moore homology. So applying Example 1.2.10, we get contravariant functoriality with respect to gci morphisms of S-schemes, of the following theories:

[39] Recall: a morphism of schemes $f : Y \to X$ is of finite Tor dimension if $\mathcal{O}_Y$ is a module of finite Tor dimension over $f^{-1}(\mathcal{O}_X)$.

- Bloch's higher Chow groups, when S is the spectrum of a field. This was previously known only for morphisms of smooth schemes according to constructions of Bloch and Levine.
- for Borel-Moore étale homology, both in the case S is the spectrum of a field (classically considered) and in the case S is an arbitrary scheme.
- Note also that according to Example 1.2.10(3), we get contravariance of Thomason's G-theory, or equivalently, Quillen K'-theory with respect to any gci morphism of s-schemes over a regular base. This contravariance coincides with the classical one but we will not check that here.[40]

The properties of fundamental classes obtained in the beginning of this section immediately translate to properties of Gysin morphisms.

Proposition 3.3.7. *Let $\mathbb{E}$ be a weakly oriented absolute $\mathscr{T}$-spectrum (Definition 2.2.13). Consider a cartesian square of S-schemes:*

$$\begin{array}{ccc} Y' & \xrightarrow{g} & X' \\ {\scriptstyle q}\downarrow & \Delta & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[f]{} & X. \end{array}$$

such that f is gci and let $\xi \in K_0(X')$ be the excess intersection bundle (see 3.1.1), $e = \mathrm{rk}(\xi)$. Then the following formulas hold:

- *If p is proper, for any $x' \in \mathbb{E}^{BM}_{**}(X'/S)$, one has: $f^*p_*(x') = q_*(c_e(\xi).f^*(x'))$ in $\mathbb{E}^{BM}_{**}(Y/S)$.*
- *If f is proper, for any $y \in \mathbb{E}^{**}(Y)$, one has: $p^*f_*(y) = g_*(c_e(\xi).q^*(y))$ in $\mathbb{E}^{**}(X')$.*

Assume moreover that $\mathbb{E}$ is an absolute oriented ring $\mathscr{T}$-spectrum. Then, if $f : Y \to X$ is gci and proper, for any pair $(x, y) \in \mathbb{E}^{**}(X) \times \mathbb{E}^{**}(Y)$, one gets the classical projection formula:

$$f_*(f^*(x).y) = x.f_*(y).$$

Given Definition 3.3.2, the first two assertions are mere consequences of 3.1.1 as follows from the properties of bivariant theories together with the commutativity property of the product on **MGL** (see Proposition 2.2.8). The last assertion easily follows from the second projection formula of the axioms of bivariant theories (as recalled in Paragraph 1.2.8).

Remark 3.3.8. Other projection formulas can be obtained, for products with respect to bivariant theories and for modules over absolute oriented ring spectra. In each case, the formulation is straightforward, as well as their proof so we left them to the reader.

3.3.9. Consider an absolute oriented ring $\mathscr{T}$-spectrum $(\mathbb{E}, c)$ and a proper gci morphism $f : X \to S$. Applying the above definition, we get a Gysin morphism $f_* : \mathbb{E}^{**}(X) \to \mathbb{E}^{**}(S)$.

[40] In the quasi projective, one can directly use the analogue of Theorem 2.6.1 for Gysin morphisms. The general case requires to identify Borel-Moore homology with coefficients in **KGL** with a suitable bivariant version of G-theory.

The fundamental class in cohomology defined in 2.5.6 is simply: $\eta_f = f_*(1)$ where 1 is the unit of the ring $\mathbb{E}^{**}(X)$.

This is the classical definition, and we can derive from the properties of f_* several properties of fundamental classes. As an illustration, we note that the preceding projection formula (and the graded commutativity of cup-product) immediately gives the following abstract degree formula:

$$f_*f^*(x) = \eta_f\,.x.$$

Moreover, one can compute η_f in many cases (see [Dég14b, 2.4.6, 3.2.12, 5.2.7]). Let us give an interesting example when f is finite.

Proposition 3.3.10. *Let $f : X \to S$ be a finite lci morphism such that there exists a factorization:*

$$X \xrightarrow{i} \mathbb{P}^1_S \xrightarrow{p} S$$

where p is the projection of the projective line over S and i is a closed immersion. Let d be the Euler characteristic of the perfect complex $\mathrm{R}f_*(\mathcal{O}_X)$ over S, seen as a locally constant function on S, and L be the line bundle on $\mathbb{P}^1_S$ corresponding to the immersion i.

Then the invertible sheaf $L(d)$ can be written $L(d) = p^*(L_0)$ where L_0 is a line bundle on S and the following formula holds in $\mathbb{E}^{00}(S)$:

$$\eta_f = d + (d-1).a_{11}.c_1(L_0) + a_{12}.c_1(L_0)^2 + a_{13}.c_1(L_0)^3 + \dots$$

where a_{ij} are the coefficients of the formal group law associated with the orientation c of $\mathbb{E}$ over S.

In particular, if the formal group law of $\mathbb{E}$ is additive, or if L_0/S is trivial, we get the usual degree formula:

$$f_*f^*(x) = d.x.$$

Proof. Let $\lambda = \mathcal{O}(-1)$ be the canonical line bundle on $\mathbb{P}^1_S$. According to our assumptions on f, we get an isomorphism:

$$L = p^{-1}(L_0)(-d) = \lambda^{\otimes, d} \otimes p^{-1}(L_0)$$

where L_0/S is the line bundle expected in the first assertion of the previous statement. In particular, if we denote by $F(x,y)$ the formal group law associated with $(\mathbb{E}, c)$ over S, and put $x = c_1(\lambda)$, $y = p^*c_1(L_0)$, one gets:

$$i_*(1) = c_1(L) = [d]_F.x +_F y = (d.x) +_F y = (d.x) + y + (d.x).\sum_{i>0} a_{1i}.y^i,$$

using the fact $x^i = 0$ if $i > 1$.

Because p is the projection of a projective line one obtains the following explicit formulas (see [Dég08, 5.31]):

$$p_*(x) = 1, p_*(1) = -a_{11}.$$

Therefore, as $y = p^*(y_0)$ where $y_0 = c_1(L_0)$, one obtains:

$$\eta_f = p_*(i_*(1)) = y_0.\underbrace{p_*(1)}_{=-a_{11}} + d.\sum_{i\geq 0} a_{1i}y_0^i$$

□

Similarly one gets the following Grothendieck-Riemann-Roch formulas from the generalized Riemann-Roch formula of Theorem 3.2.6.

Proposition 3.3.11. *Consider the assumptions of Theorem 3.2.6.*
Then for any gci morphism $f : Y \to X$ of s-schemes over S with tangent bundle τ_f, the following diagrams are commutative:

$$\begin{array}{ccc} \mathbb{E}^{BM}_{**}(X/S) & \xrightarrow{f^*} & \mathbb{E}^{BM}_{**}(Y/S) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \mathbb{F}^{BM}_{**}(X/S) & \xrightarrow{\mathrm{Td}_\phi(\tau_f).f^*} & \mathbb{F}^{BM}_{**}(Y/S) \end{array} \qquad \begin{array}{ccc} \mathbb{E}^{**}(Y) & \xrightarrow{f_*} & \mathbb{E}^{**}(X) \\ \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} \\ \mathbb{F}^{**}(Y) & \xrightarrow{f_*[\mathrm{Td}_\phi(\tau_f).]} & \mathbb{F}^{**}(X) \end{array}$$

where in the square on the right hand side we assume in addition f is proper.

Again this follows easily from Theorem 3.2.6 and Definition 3.3.2 given the properties of bivariant theories together with Proposition 2.2.8 for the commutativity of the product on **MGL**.

Example 3.3.12. Our main examples are given by the morphisms of ring spectra of Example 1.1.6, as already exploited in Example 3.2.7.

(1) Assume we are in one of the following cases:
- $\mathscr{S}$ is the category of all schemes, $\Lambda = \mathbb{Q}$, $\Lambda_\ell = \mathbb{Q}_\ell$;
- $\mathscr{S}$ is the category of k-schemes for a field k if characteristic $p \neq \ell$, $\Lambda = \mathbb{Z}$, $\Lambda_\ell = \mathbb{Z}_\ell$;

Then we obtain that the natural transformations induced by the ℓ-adic realization functor gives natural transformations on cohomologies and Borel-Moore homologies that are compatible with Gysin morphisms.

Note in particular that in the second case, we get that the higher cycle class, from higher Chow groups to Borel-Moore étale homology of any k-scheme is compatible with Gysin functoriality (here, pullbacks).

(2) The Chern character as in 3.2.7(3) gives the usual Grothendieck-Riemann-Roch formula from homotopy invariant K-theory to motivic cohomology of [Nav16]. But we also get a Riemann-Roch formula for bivariant theories relative to any base scheme.

Let us be more specific in the case where the base scheme is a field k. Then the Chern character of Example 1.1.6, applied to bivariant theories with respect to the s-morphism $X \to \mathrm{Spec}(k)$, gives an isomorphism:

$$\mathrm{ch} : G_n(X) \to \bigoplus_{i \in \mathbb{Z}} CH_i(Y, n)_{\mathbb{Q}}$$

in view of point (1) and (3) of Example 1.2.10. Considering the Todd class functor Td as defined in Example 3.2.7(3), with coefficients in rational motivic cohomology, we get for any gci morphism $f : Y \to X$ of separated k-schemes of finite type the following commutative diagram:

$$\begin{array}{ccc} G_n(X) & \xrightarrow{\quad f^* \quad} & G_n(Y) \\ {\scriptstyle \mathrm{ch}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{ch}_Y} \\ \bigoplus_{i\in\mathbb{Z}} CH_i(X,n)_{\mathbb{Q}} & \xrightarrow{\mathrm{Td}_\phi(\tau_f).f^*} & \bigoplus_{i\in\mathbb{Z}} CH_i(Y,n)_{\mathbb{Q}}. \end{array}$$

3.3.13. In the case of an absolute oriented ring $\mathscr{T}$-spectrum $(\mathbb{E}, c)$, the Gysin morphisms can also be obtained very easily using the six functors formalism. Indeed, consider a commutative diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle q}\searrow \quad \swarrow{\scriptstyle p} & \\ & S & \end{array}$$

where f is gci of relative dimension d and p, q are s-morphisms. By adjunction, one obtains from the map (2.1.5.a) associated with the fundamental class of f with coefficients in $\mathbb{E}$ (Definition 2.5.5) the following map:

$$\tilde{\eta}'_f : f_!(\mathbb{E}_Y)(d)[2d] \to \mathbb{E}_X$$

When f is proper, we deduce the following *trace map*, well known in the case of étale coefficients:

$$\mathrm{tr}_f : f_*f^*(\mathbb{E}_X)(d)[2d] \simeq f_*(\mathbb{E}_Y) \simeq f_!(\mathbb{E}_Y) \xrightarrow{\tilde{\eta}'_f} \mathbb{E}_X.$$

It is clear that this trace map corresponds to the Gysin morphism associated with f in cohomology.

Let us go back to the case where f is an arbitrary gci morphism fitting into the above commutative diagram. Then, by applying the functor $p_!$ to $\tilde{\eta}'_f$, we get a canonical map:

$$q_!(\mathbb{E}_Y)(d)[2d] = p_!f_!(\mathbb{E}_Y)(d)[2d] \xrightarrow{p_!(\tilde{\eta}'_f)} p_!(\mathbb{E}_X)$$

which induces a *covariant functoriality* on cohomology with compact supports:

$$f_* : \mathbb{E}_c^{**}(Y/S) \to \mathbb{E}_c^{**}(X/S),$$

morphisms of degree $(2d, d)$. This functoriality extends the one we had already seen with respect to étale morphisms in Paragraph 1.3.1.

Finally, if we assume again that f is proper we get the following construction that was found by Adeel Khan (see also [EHK+17]). From the fundamental class of f, we get by adjunction a map:

$$\bar{\eta}'_f : f_!(\mathbb{1}_Y)(d)[2d] \to \mathbb{E}_X.$$

We deduce the following composite map:

$$f_!f^*p^!(\mathbb{E}_S)(d)[2d] \simeq f_!\big(\mathbb{1}_S \otimes f^*p^!(\mathbb{E}_S)\big)(d)[2d] \xrightarrow[(1)]{\sim} f_!\big(\mathbb{1}_S\big) \otimes p^!(\mathbb{E}_S)(d)[2d]$$

$$\xrightarrow{\bar{\eta}'_f \otimes p^!(\mathbb{E}_S)} \mathbb{E}_X \otimes p^!(\mathbb{E}_S) \simeq p^*(\mathbb{E}_S) \otimes p^!(\mathbb{E}_S) \xrightarrow{Ex^{!*}_{\otimes}} p^!(\mathbb{E}_S \otimes \mathbb{E}_S) \xrightarrow{\mu} p^!(\mathbb{E}_S),$$

where (1) is given by the projection formula, $Ex^{!*}_{\otimes}$ by the pairing (1.2.8.a) and μ is the product of the ring spectrum $\mathbb{E}_S$. Using the adjunctions $(f_!, f^!)$ and (f^*, f_*) and applying the functor $p_!$, we get:

$$p_!p^!(\mathbb{E}_S)(d)[2d] \to p_!f_*f^!p^!(\mathbb{E}_S) \simeq p_!f_!f^!p^!(\mathbb{E}_S) = q_!q^!(\mathbb{E}_S),$$

where we have used the fact f is proper. This immediately gives the expect contravariant functoriality:

$$f^* : \mathbb{E}_{**}(X/S) \to \mathbb{E}_{**}(Y/S)$$

which is a morphism of degree $(-2d, -d)$. This functoriality extends the one already mentioned in Paragraph 1.3.1 in the case where f is a finite morphism.

Remark 3.3.14. Therefore one has obtained exceptional functorialities for all the four theories associated with an absolute oriented ring spectrum $(\mathbb{E}, c)$.

Besides, it is clear that the excess intersection formula (Prop. 3.3.7) and the Riemann-Roch formula (Prop. 3.3.11) extends to formulas involving cohomology with compact support and homology. We leave the formulation to the reader not to overburden this paper.

Example 3.3.15. Again, one deduces notable examples from 1.3.3. This gives covariance with respect to gci morphisms of k-schemes of all the classical cohomology with compact supports which corresponds to a Mixed Weil theory.

We also obtain the contravariance with respect to proper gci morphisms of complex schemes for the integral Betti homology. Surprisingly, this result seems new.

4. Absolute purity and duality

4.1. **Purity for closed pairs.**

4.1.1. We will say that a closed pair (X, Z) is *regular* if the corresponding immersion $Z \to X$ is regular. Then, in the deformation diagram (2.3.1.a)

(4.1.1.a)
$$\begin{array}{ccccc} Z & \longrightarrow & \mathbb{A}^1_Z & \longleftarrow & Z \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{d_1} & D_ZX & \xleftarrow{d_0} & N_ZX, \end{array}$$

the closed immersion ν is also regular.

The next definition is an obvious extension of [Dég14b, 1.3.2].

Definition 4.1.2. Let $\mathbb{E}$ be an absolute $\mathscr{T}$-spectrum and (X, Z) be a regular closed pair.

(1) We say that (X, Z) is $\mathbb{E}$-*pure* if the morphisms

$$\mathbb{E}^{**}_Z(X) = \mathbb{E}^{**}(X, Z) \xleftarrow{d_1^*} \mathbb{E}^{**}(D_ZX, \mathbb{A}^1_Z) \xrightarrow{d_0^*} \mathbb{E}^{**}(N_ZX, Z) = \mathbb{E}^{**}(\mathrm{Th}(N_ZX))$$

induced by the deformation diagram (2.3.1.a) are isomorphisms.

(2) We say that (X, Z) is *universally* $\mathbb{E}$-*pure* if for all smooth morphism $Y \to X$, the closed pair $(Y, Y \times_X Z)$ is $\mathbb{E}$-pure.

Example 4.1.3. It follows directly from Morel-Voevodsky's purity theorem (see Th. 2.3.2) that any closed pair (X, Z) of smooth schemes over some base S is universally $\mathbb{E}$-pure.

We can link this definition with Fulton-MacPherson's theory of *strong orientations* (Def. 2.1.6) as follows.

Proposition 4.1.4. *Let* $(\mathbb{E}, c)$ *be an absolute oriented ring* $\mathscr{T}$*-spectrum and* (X, Z) *be a regular closed pair. Consider the notations of diagram* (4.1.1.a)*. Then the following conditions are equivalent:*

(i) *The closed pair* (X, Z) *is* $\mathbb{E}$*-pure.*
(ii) *The orientations* $\bar{\eta}_i$ *and* $\bar{\eta}_\nu$*, associated with the orientation* c *in Definition 2.5.5, are strong.*

Proof. The proof can be summarized in the commutativity of the following diagram:

$$\begin{array}{ccccc}
\mathbb{E}^{**}(Z) & \xleftarrow[\sim]{s_1^*} & \mathbb{E}^{**}(\mathbb{A}^1_Z) & \xrightarrow[\sim]{s_0^*} & \mathbb{E}^{**}(Z) \\
{\scriptstyle .\bar{\eta}_i}\downarrow & & {\scriptstyle .\bar{\eta}_\nu}\downarrow & & {\scriptstyle \sim}\downarrow{\scriptstyle .\bar{\mathfrak{t}}(N_Z X)} \\
\mathbb{E}^{**}_Z(X) & \xleftarrow{d_1^*} & \mathbb{E}^{**}_{\mathbb{A}^1_Z}(D_Z X) & \xrightarrow{d_0^*} & \mathbb{E}^{**}(\mathrm{Th}(N_Z X))
\end{array}$$

The diagram is commutative according to the stability of fundamental classes with respect to transversal pullbacks (Example 3.1.2) and the fact $\bar{\eta}_s = \bar{\mathfrak{t}}(N_Z X)$ (Remark 2.3.6). In this diagram, all arrows indicated with a symbol $\sim$ are obviously isomorphisms. Condition (i) (resp. (ii)) says that the maps d_0^* and d_1^* (resp. $(.\,\bar{\eta}_i)$ and $(.\,\bar{\eta}_\nu)$) are isomorphisms. Thus the equivalence stated in this proposition obviously follows . □

4.1.5. To formulate stronger purity results, we now fix a full sub-category $\mathscr{S}_0$ of $\mathscr{S}$ stable under the following operations:

- For any scheme S in $\mathscr{S}_0$, any smooth S-scheme belongs to $\mathscr{S}_0$.
- For any regular closed immersion $Z \to S$ in $\mathscr{S}_0$, the schemes $N_Z X$ and $B_Z X$ belong to $\mathscr{S}_0$.

The main examples we have in mind are the category $\mathscr{R}$eg of regular schemes (in $\mathscr{S}$) and the category $\mathscr{S}m_S$ of smooth S-schemes for a scheme S in $\mathscr{S}$.

Following again [Dég14b, 1.3.2], we introduce the following useful definition.

Definition 4.1.6. Let $\mathbb{E}$ be an absolute $\mathscr{T}$-spectrum.

(1) We say that $\mathbb{E}$ is $\mathscr{S}_0$*-pure* if for any regular closed pair (X, Z) such that X and Z belongs to $\mathscr{S}_0$, (X, Z) is $\mathbb{E}$-pure.
(2) We say that $\mathscr{T}$ is $\mathscr{S}_0$*-pure* if the unit cartesian section $\mathbb{1}$ of the fibred category $\mathscr{T}$ is $\mathscr{S}_0$*-pure.*

Finally, we will simply say *absolutely pure* for $\mathscr{R}$eg-pure.

Remark 4.1.7. The last definition already appears in [CD16, A.2.9] — in *loc. cit.* one says $\mathscr{T}$ satisfies the absolute purity property.

Example 4.1.8. (1) From Example 4.1.3, all absolute $\mathscr{T}$-spectra $\mathbb{E}$, as well as all motivic triangulated categories $\mathscr{T}$, are $\mathscr{S}m_S$-pure for any scheme S (even a singular one).

(2) Let k be a perfect field whose spectrum is in $\mathscr{S}$. Assume that $\mathbb{E}$-cohomology with support is compatible with projective limit in the following sense: for any essentially affine projective system of closed k-pairs $(X_\alpha, Z_\alpha)_{\alpha\in A}$ whose projective limit (X, Z) is still in $\mathscr{S}$, the canonical map:

$$\varinjlim_{\alpha\in A^{op}} \left(\mathbb{E}^{n,m}_{Z_\alpha}(X_\alpha)\right) \to \mathbb{E}^{n,m}_Z(X)$$

is an isomorphism. This happens in particular if $\mathscr{T}$ is continuous in the sense of [CD12b, 4.3.2].

Then one can deduce from Popescu's theorem that any regular k-pair (X, Z) in $\mathscr{R}$eg is $\mathbb{E}$-pure. In other words, $\mathbb{E}$ is $(\mathscr{R}\mathrm{eg}/k)$-pure (see [Dég14b, 1.3.4(2)]). This fact concerns in particular the spectra of points (5) and (6) in Example 1.1.2.

Moreover, one deduces that any continuous motivic triangulated category $\mathscr{T}$ is $(\mathscr{R}\mathrm{eg}/k)$-pure. This includes modules over a mixed Weil theory, $\mathrm{DM}_{\mathrm{cdh}}(-, \mathbb{Z}[1/p])$ where p is the characteristic of k (see [CD15, ex. 5.11] for the continuity statement).

Example 4.1.9. The following absolute spectra (Example 1.1.6) are absolutely pure:
(1) The homotopy invariant K-theory spectrum **KGL** (see [CD12a, 13.6.3]);
(2) given any $\mathbb{Q}$-algebra Λ, the motivic ring spectrum $\mathbb{H}\Lambda$ (see [CD16, 5.6.2 and 5.2.2]);
(3) the rationalization $\mathbf{MGL}\otimes\mathbb{Q}$ of algebraic cobordism (this follows from the preceding example and [NSOsr09, 10.5]);
(4) Given any ring Λ, the étale motivic ring spectrum $\mathbb{H}_{\text{ét}}\Lambda$ (see [CD16, 5.6.2]).[41]

Example 4.1.10. The ring spectra of the previous examples all corresponds to the following (non-exhaustive) list of absolutely pure motivic triangulated categories:
(1) the category **KGL**- mod of **KGL**-modules (see [CD12a, §13.3]);
(2) $\mathrm{DM}_{\text{Б}}$, $\mathrm{DM}_h(-, \Lambda)$ where Λ is a $\mathbb{Q}$-algebra (see [CD12a, CD15]);
(3) $\mathrm{D}^b_c((-)_{\text{ét}}, \Lambda)$ where $\Lambda = \mathbb{Z}/\ell^n$, $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$ and $\mathscr{S}$ is the category of $\mathbb{Z}[1/\ell]$-schemes;
(4) the category of $(\mathbf{MGL}\otimes\mathbb{Q})$-modules (see [CD12a, §13.3]);
(5) $\mathrm{DM}_h(-, \Lambda)$ where Λ is any ring (see [CD16, §5]);
(6) for any prime ℓ, the ℓ-completed category $\mathrm{DM}_h(-, \Lambda)$ where $\Lambda = \mathbb{Z}_\ell, \mathbb{Q}_\ell$ (see [CD16, §7.2]).

4.2. **Dualities.**

4.2.1. Recall that given an oriented ring $\mathscr{T}$-spectrum $(\mathbb{E}, c)$, we have associated to a gci morphism $f : X \to S$ with relative dimension d the fundamental class $\bar{\eta}_f$ (Definition 2.5.5) and equivalently — equation (2.1.5.a) — a morphism:

$$\tilde{\eta}_f : \mathbb{E}_S(d)[2d] \to f^!(\mathbb{E}_X).$$

[41] Recall the case where Λ is a torsion ring, or $\Lambda = \mathbb{Z}_\ell, \mathbb{Q}_\ell$, follows directly from Thomason's purity theorem ([Tho84]).

Recall we say the orientation $\bar{\eta}_f$ is universally strong when $\tilde{\eta}_f$ is an isomorphism (Definition 2.1.6).

Proposition 4.2.2. *Consider the preceding notations and a subcategory $\mathscr{S}_0 \subset \mathscr{S}$ as in Definition 4.1.6. Then the following conditions are equivalent:*

(i) *$\mathbb{E}$ is $\mathscr{S}_0$-pure;*
(ii) *for any regular closed immersion i in $\mathscr{S}_0$, $\bar{\eta}_i$ is a strong orientation;*
(ii') *for any gci morphism f in $\mathscr{S}_0$, $\bar{\eta}_f$ is a strong orientation.*

This is obvious given definitions and Proposition 4.1.4.

As a corollary, we get the following various formulations of duality statements.

Corollary 4.2.3. *Let (E, c) be an absolute oriented ring $\mathscr{T}$-spectrum which is $\mathscr{S}_0$-pure, following the notations of the previous proposition. Let $f : X \to S$ be a gci s-morphism of relative dimension d and Y/X be an arbitrary s-scheme. Then the following maps are isomorphisms:*

$$\delta_f : \mathbb{E}^{n,i}(X) \to \mathbb{E}^{BM}_{2d-n,d-i}(X/S), x \mapsto x.\bar{\eta}_f, \tag{4.2.3.a}$$

$$\delta_f : \mathbb{E}^{BM}_{n,i}(Y/X) \to \mathbb{E}^{BM}_{2d+n,d+i}(Y/S), y \mapsto y.\bar{\eta}_f, \tag{4.2.3.b}$$

$$\delta^c_f : \mathbb{E}^{n,i}_c(X/S) \to \mathbb{E}_{2d-n,d-i}(X/S), x \mapsto x \cap \bar{\eta}_f \tag{4.2.3.c}$$

where the first two maps are defined using the product of Borel-Moore homology and the last one using the cap-product (1.3.7.a)*.*

Recall that these duality isomorphisms occur in particular whenever X/S is a smooth s-scheme (Example 4.1.8(1)).

Example 4.2.4. (1) As a notable particular case of isomorphism (4.2.3.b), we get the formulation of duality with support due to Bloch and Ogus ([BO74]). Assume $Z = Y \to X$ is a closed immersion, $S = \mathrm{Spec}(k)$ and X is smooth over k. Then the duality isomorphism (4.2.3.b) has the form:

$$\mathbb{E}^{n,i}_Z(X) \simeq \mathbb{E}^{BM}_{2d-n,d-i}(Z/k).$$

As an example, we get the following identification, when p is the characteristic exponent of k:

$$H^n_Z(X, \mathbb{Z}[1/p]) \simeq CH_{d-i}(Z, 2i-n)[1/p]$$

where the left hand side is Voevodsky motivic cohomology of X with support in Z and the right hand side is Bloch's higher Chow group (see Example 1.2.10(1)).

(2) In the extension of the preceding case, assume X and S are regular schemes and $Z = Y \subset X$ is a closed subscheme. Then the duality isomorphism in the case of the absolutely pure spectrum **KGL** gives the classical duality with support isomorphism (see [Sou85]):

$$K^Z_{2i-n}(X) \simeq \mathbf{KGL}^{n,i}_Z(X) \simeq \mathbf{KGL}^{BM}_{2d-n,d-i}(Z/S) \simeq K'_{2i-n}(Z).$$

(3) Let X/S be a smooth proper scheme. Then, the four theories defined in this paper coincide through isomorphisms pictured as follows:

$$\begin{array}{ccc} \mathbb{E}^{n,i}(X) & \xrightarrow[\sim]{\delta_f} & \mathbb{E}^{BM}_{2d-n,d-i}(X/S) \\ \sim\downarrow & & \downarrow\sim \\ \mathbb{E}^{n,i}_c(X/S) & \xrightarrow[\sim]{\delta^c_f} & \mathbb{E}_{2d-n,d-i}(X/S) \end{array}$$

(4) An interesting application of the duality isomorphisms obtained above is the following identification, for a regular s-scheme X over a field k, of dimension d:

$$H^{sing}_n(X)[1/p] \simeq \mathbb{H}_{n,0}(X/k, \mathbb{Z}[1/p]) \xrightarrow{(\delta^c_{X/k})^{-1}} H^{2d-n,d-n}_c(X, \mathbb{Z}[1/p])$$

where the left hand side is Suslin homology and the right hand side is motivic cohomology with compact support. This isomorphism was only known for smooth k-schemes when k is a perfect field and under the resolution of singularities assumption (see [VSF00, chap. 5, Th. 4.3.7]).

Remark 4.2.5. An immediate corollary of the duality isomorphisms (4.2.3.a) and (4.2.3.c) is the existence of certain Gysin morphisms for the four theories. More precisely, under the assumptions of the previous corollary, given a scheme S in $\mathscr{S}_0$, we obtain Gysin maps for all S-morphisms $f : Y \to X$ in $\mathscr{S}_0$ such that in addition Y/S and X/S are gci.

In case f is gci, it follows from the definitions that the Gysin morphisms obtained as in Definition 3.3.2 and Paragraph 3.3.13 coincides with the Gysin morphisms obtained respectively from the isomorphisms (4.2.3.a) and (4.2.3.c).

On the other hand, the morphism f can also simply be a *local* complete intersection morphism, so the Gysin morphisms obtained in this way are slightly more general.

We end-up this paper with the following Riemann-Roch-like statement, involving the previous duality isomorphisms and directly following from the general Riemann-Roch Theorem 3.2.6.

Theorem 4.2.6. *Consider a subcategory $\mathscr{S}_0 \subset \mathscr{S}$ as in Definition 4.1.6, $(\mathbb{E}, c)$ and $(\mathbb{F}, d)$ absolute oriented $\mathscr{S}_0$-pure ring spectra. We adopt the notations of the previous Corollary. Consider in addition a morphism of ring spectra:*

$$(\varphi, \phi) : (\mathscr{T}, \mathbb{E}) \to (\mathscr{T}', \mathbb{F})$$

and $\mathrm{Td}_\phi : K_0 \to \mathbb{F}^{00\times}$ the associated Todd class transformation (Definition 3.2.4).

Then, given an arbitrary gci morphism $f : X \to S$ with virtual tangent bundle τ_f and relative dimension d, for any s-scheme Y/X, the following diagrams are commutative:

$$\begin{array}{ccc} \mathbb{E}^{n,i}(Y/X) & \xrightarrow[\sim]{\delta_f} & \mathbb{E}^{BM}_{2d-n,d-i}(Y/S) \\ \phi_*\downarrow & & \downarrow\phi_* \\ \mathbb{E}^{n,i}(Y/X) & \xrightarrow[\sim]{\delta_f(-.\,\mathrm{Td}_\phi(\tau_f))} & \mathbb{E}^{BM}_{2d-n,d-i}(Y/S), \end{array} \qquad \begin{array}{ccc} \mathbb{E}^{n,i}_c(X) & \xrightarrow[\sim]{\delta^c_f} & \mathbb{E}^{BM}_{2d-n,d-i}(X/S) \\ \phi_*\downarrow & & \downarrow\phi_* \\ \mathbb{E}^{n,i}_c(X) & \xrightarrow[\sim]{\mathrm{Td}_\phi(\tau_f).\delta^c_f} & \mathbb{E}^{BM}_{2d-n,d-i}(X/S). \end{array}$$

The proof is obvious from the formulas in Corollary 4.2.3 and Theorem 3.2.6.

Remark 4.2.7. (1) This theorem has to be compared with [FM81, I.7.2.2].
(2) As indicated to us by Henri Gillet, one immediately deduces from this theorem the Grothendieck-Riemann-Roch formulas for the Gysin morphisms obtained using duality as in Remark 4.2.5.

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