FUNDAMENTAL CLASSES IN MOTIVIC HOMOTOPY THEORY

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Abstract. We develop the theory of fundamental classes in the setting of motivic homotopy theory. Using this we construct, for any motivic spectrum, an associated bivariant theory in the sense of Fulton–MacPherson. We import the tools of Fulton's intersection theory into this setting: (refined) Gysin maps, specialization maps, and formulas for excess intersections, self-intersections, and blow-ups. We also develop a theory of Euler classes of vector bundles in this setting. For the Milnor–Witt spectrum recently constructed by Déglise–Fasel, we get a bivariant theory extending the Chow–Witt groups of Barge–Morel, in the same way the higher Chow groups extend the classical Chow groups. As another application we prove a motivic Gauss–Bonnet formula, computing Euler characteristics in the motivic homotopy category.

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1. Introduction

Given a smooth algebraic variety X over a field k, let $\mathrm{CH}_*(X)$ denote the Chow groups of algebraic cycles on X. Recall that following Fulton [Ful98], the Chow groups admit

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direct images for proper morphisms (covariant functoriality) and Gysin maps for quasi-projective local complete intersection[1] morphisms (contravariant functoriality). Moreover, these operations are compatible in a way that can be summarized by saying that the Chow groups form a *bivariant theory* in the sense of Fulton–MacPherson [FM81].

Using his formalism of motivic categories, Voevodsky constructed a theory called *Borel–Moore motivic homology*, which is bigraded and extends the classical Chow groups:

$$H^{BM}_{2n,n}(X,\mathbb{Z}) \simeq CH_n(X)$$

at least if k is a field of characteristic zero[2]. According to [Dég17], Borel–Moore motivic homology itself is a bivariant theory.

In this paper, we work with the even more general formalism of $\mathbb{A}^1$-homotopy theory. Given any motivic ring spectrum $\mathbb{E} \in S\mathscr{H}(S)$ over a base scheme S, we construct a bivariant theory defined on schemes over S (separated and of finite type, but possibly singular). This recovers Borel–Moore motivic homology in the case of the motivic Eilenberg–MacLane spectrum, but also applies to a vast variety of other interesting new examples. An especially interesting example is the bivariant theory represented by the *Milnor–Witt spectrum* [DF17a, DF17b]. This theory is a natural extension of the *Chow–Witt groups* of Barge–Morel [BM00, Fas08], in the same way that Borel–Moore motivic homology is an extension of the Chow groups. Moreover, it can be viewed as a natural quadratic refinement of Borel–Moore motivic homology: roughly speaking, elements of the Chow–Witt group are "algebraic cycles with coefficients in quadratic forms", and there are forgetful maps from the Chow–Witt groups to the Chow groups which are compatible with the direct image and Gysin maps.

Much of the interest in Chow–Witt groups comes from the fact that they are natural receptacles for *Euler classes* of vector bundles. The Euler class provides an obstruction for a vector bundle to split off a trivial summand of rank one (see [BM00, Mor12, FS09]). We develop a theory of Euler classes in our general bivariant theories, and show that also in this setting they vanish for vector bundles that split off a trivial summand of rank one. Moreover, we also use the Euler class to prove excess intersection formulas in our bivariant theories: the Euler class of the excess bundle measures the failure of the base change formula for a non-transverse intersection. This formula also specializes to self-intersection and blow-up formulas. In Borel–Moore motivic homology, for instance, the Euler class coincides with the top Chern class, and we recover the usual excess intersection formula. In the example of Milnor–Witt bivariant theory, this formula is new, even for just the Chow–Witt groups.

The idea of refining classical formulas to the quadratic setting has been explored recently by many authors [Fas07, Fas08, Fas09, CF14, Hoy15, KW16, Lev17b]. In this direction another application of the theory we develop is a motivic refinement of the classical Gauss-Bonnet formula. Given a smooth proper S-scheme X, there is a so-called categorical Euler

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[1] We will abbreviate use "lci" as an abbreviation.

[2] For fields of positive characteristic p, this isomorphism is known to hold up to inverting p. Moreover, it holds even when X is singular; see [CD14, Cor. 8.13].

characteristic $\chi^{cat}(X/S)$ [Hoy15], which lives in the bivariant theory represented by the motivic sphere spectrum $\mathbb{S}_S$. A simple application of our excess intersection formula then computes this invariant as the degree of the Euler class of the tangent bundle $T_{X/S}$ (see Theorem 4.4.1). This result is a generalization of a recent theorem of Levine [Lev17b, Thm. 1], which applies when S is the spectrum of a field and $p : X \to S$ is smooth projective.

In a different direction, the theory developed here has been applied recently to the problem of recognition of infinite loop spaces in motivic homotopy theory [EHK$^+$17, EHK$^+$18]. Let S be the spectrum of a perfect field of characteristic different from 2. Given a grouplike motivic space in the unstable $\mathbb{A}^1$-homotopy category $\mathscr{H}_\bullet(S)$, it is proven in *loc. cit.* that the obstruction for it to admit an infinite $\mathbb{P}^1$-delooping in $S\mathscr{H}(S)$ is precisely that it admit a (homotopy coherent) system of Gysin maps for finite flat lci morphisms.

Fundamental classes. Given any motivic ring spectrum $\mathbb{E} \in S\mathscr{H}(S)$, we define the *bivariant theory* with coefficients in $\mathbb{E}$, graded by integers $n \in \mathbb{Z}$ and virtual vector bundles e on X, by the following formula:

$$\mathbb{E}_n(X/S, e) := \mathrm{Hom}_{S\mathscr{H}(S)}(\mathrm{Th}_X(e)[n], p^!(\mathbb{E}_S)),$$

for any morphism $p : X \to S$ that is separated of finite type. For a proper morphism $f : X \to Y$ of S-schemes there are direct image homomorphisms

$$f_* : \mathbb{E}_n(X/S, f^*(e)) \to \mathbb{E}_n(Y/S, e).$$

In terms of the six operations, this functoriality comes from the canonical natural transformation $f_*f^! \to \mathrm{Id}$ which is the co-unit of an adjunction $(f_*, f^!)$.

If $f : Y \to X$ is a *quasi-projective lci* morphism, then our main construction defines a class η_f, called the *fundamental class* of f:

$$\eta_f \in \mathbb{E}_0(Y/X, \langle L_f \rangle),$$

where $\langle L_f \rangle$ is the virtual tangent bundle of f. This class gives rise to Gysin homomorphisms of the form

$$f^! : \mathbb{E}_n(X/S, e) \to \mathbb{E}_n(Y/S, f^*(e) + \langle L_f \rangle).$$

The main part of the paper is concerned with the construction of these classes and verification of their expected properties. Note that, for oriented theories like Borel–Moore motivic homology, the orientation provides an isomorphism that replaces the virtual twist $\langle L_f \rangle$ by a shift by the relative virtual dimension $\chi(L_f)$, so that the Gysin homomorphism takes a more familiar shape. In general, we need to take virtual twists into account.

In fact, the fundamental class η_f and Gysin homomorphism $f^!$ also admit a simple interpretation in the language of the six operations. Namely, they come from a natural transformation of functors

$$\mathfrak{p}_f : f^*(-) \otimes \mathrm{Th}_X(L_f) \to f^!$$

that we call the *purity transformation* associated to f (4.2.1.b). When f is *smooth*, this transformation is invertible and coincides with the purity isomorphism (see Example 2.2.3). In general, the purity transformation measures the failure of a given motivic spectrum $\mathbb{E} \in S\mathscr{H}(Y)$ to satisfy the property of *absolute purity* with respect to f (see Definition 4.2.5).

Contents. In Section 2 we construct the bivariant theory and cohomology theory associated to a motivic ring spectrum, and study their basic properties. Following Fulton–MacPherson [FM81], we also introduce the abstract notion of *orientations* of morphisms in this setting; fundamental classes will be examples of orientations. We then show how any choice of orientation gives rise to a *purity* transformation.

The heart of the paper is Section 3, where we construct fundamental classes and verify their basic properties. In the case where f is smooth, the fundamental class comes from the purity theorem (see Example 2.2.3). For the case of a regular closed immersion we use the technique of deformation to the normal cone, re-interpreting constructions of Baum–Fulton–MacPherson [BFM75] and Verdier [Ver76b] in the setting of $\mathbb{A}^1$-homotopy theory. A similar construction was developed independently by Levine [Lev17a]. We then explain how to glue these to obtain a fundamental class for any quasi-projective lci morphism. Throughout this section, we restrict our attention to the bivariant theory represented by the motivic sphere spectrum.

Finally in Section 4, we return to the setting of the bivariant theory represented by any motivic ring spectrum. We show how the fundamental class gives rise to Gysin homomorphisms. We prove the excess intersection formula in this setting (Proposition 4.1.8). We also discuss the purity transformation, the absolute purity property, and duality isomorphisms (identifying bivariant groups with certain cohomology groups). We then import some further constructions from Fulton's intersection theory, including refined Gysin maps and specialization maps. Finally, we conclude with a proof of the motivic Gauss-Bonnet formula mentioned above.

Notations and conventions. All schemes in this paper are assumed to be quasi-compact and quasi-separated. We will use the term *s-morphism* as an abbreviation for "separated morphism of finite type"; similarly an *s-scheme over S* is an S-scheme whose structural morphism is an s-morphism.[3]

We follow [BGI71, Exps. VII–VIII] for our conventions on regular closed immersions and lci morphisms. If X and Z are regular schemes, or are both smooth over some base S, then any closed immersion $Z \to X$ is regular. Given a regular closed immersion $Z \to X$, we write $N_i = N_Z X$, or sometimes $N(X, Z)$, for its normal bundle. Recall that a morphism of schemes $X \to S$ is *lci* (= a local complete intersection) if it admits, Zariski-locally on the source, a factorization $X \xrightarrow{i} Y \xrightarrow{p} S$, where p is smooth and i is a regular closed immersion. Note that any quasi-projective lci morphism admits such a factorization *globally* on the source: in fact, it factors globally as a regular closed immersion into an open subscheme of a projective bundle. In particular, any quasi-projective lci morphism is an s-morphism.

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[3] Using the language of higher category theory, the operations $(f^!, f_!)$ can be extended to the case where f is locally of finite type; assuming this extension, the reader can globally redefine the term "s-morphism" as "locally of finite type morphism".

We say that a cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is *tor-independent*, or that p is *transverse* to f, if the groups $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'})$ vanish for $i > 0$. Recall that if p is flat, then this condition is automatic.

We will work with the stable homotopy category of schemes $S\mathscr{H}(S)$, equipped with the six functors formalism of Grothendieck. See [CD12] or [Ayo07], and [Kha16] or [Hoy15, Appendix C] for the extension to quasi-compact quasi-separated base schemes. We will write $\mathbb{S}_S \in S\mathscr{H}(S)$ for the motivic sphere spectrum over a scheme S. For a virtual vector bundle e over S, we will write $\Sigma^e := \mathrm{Th}_S(e) \otimes -$ for the endofunctor of $S\mathscr{H}(S)$ given by tensoring with the Thom space. Given a motivic spectrum $\mathbb{E} \in S\mathscr{H}(S)$, we will often write $\mathbb{E}_X := f^*(\mathbb{E})$ for any morphism $f : X \to S$. By the term *motivic ring spectrum* over S we will mean a motivic spectrum $\mathbb{E} \in S\mathscr{H}(S)$ equipped with a structure of commutative monoid (in the triangulated category $S\mathscr{H}(S)$).

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2. Basic definitions and notations

2.1. **Bivariant theories.** In this section we construct the *bivariant theory* represented by a motivic spectrum $\mathbb{E}$, and state its main properties. This is simply a variation on the construction of [Dég17, Def. 1.2.2] where we consider gradings by the Picard groupoid of *virtual vector bundles*.

2.1.1. Let S be a base scheme. Let $K(S)$ denote the algebraic K-theory space of perfect complexes on S, and $\underline{K}(S) = \tau_{\leq 1}K(S)$ its 1-truncation, i.e. the Picard groupoid of virtual vector bundles on S in the sense of Deligne [Del87, Section 4]. Recall that the assignment $E \mapsto \mathrm{Th}_S(E)$, sending a vector bundle E/S to its Thom space $\mathrm{Th}_S(E) \in S\mathscr{H}(S)$, induces a functor of Picard groupoids:

$$\mathrm{Th}_S : (\underline{K}(S), \oplus) \to \mathrm{Pic}(S\mathscr{H}(S), \otimes),$$

natural in S with respect to the obvious contravariant functoriality, where the right hand side is the Picard groupoid of $\otimes$-invertible motivic spectra over S; see [CD12, Remark 2.4.15].

In particular, any perfect complex $\mathcal{E}$ on S defines a virtual vector bundle $\langle \mathcal{E} \rangle \in \underline{K}(S)$ and thus a Thom space $\mathrm{Th}_S(\mathcal{E}) \in S\mathscr{H}(S)$. For example, if $\mathcal{E}$ is a bounded complex of vector bundles E^n, then $\mathrm{Th}_S(\mathcal{E})$ is the alternating tensor product of the Thom spaces $\mathrm{Th}_S(E^n)$.

We are now ready to define the *bivariant theory* represented by a motivic spectrum.

Definition 2.1.2. Let $\mathbb{E} \in \mathscr{SH}(S)$ be a motivic spectrum. Given an s-morphism $p : X \to S$ and any pair $(n, v) \in \mathbb{Z} \times \underline{K}(X)$, we define the *(twisted) bivariant theory* of X/S in bidegree (n, v), with coefficients in $\mathbb{E}$, as the abelian group:

$$\mathbb{E}_n(X/S, v) := [\mathrm{Th}_X(v)[n], p^!(\mathbb{E})] = [p_! \,\mathrm{Th}_X(v)[n], \mathbb{E}].$$

The *cohomology theory* represented by $\mathbb{E}$ is defined by the formula:

$$\mathbb{E}^n(X, v) := \mathbb{E}_{-n}(X/X, -v) = \big[\mathbb{S}_X, \mathbb{E}_X \otimes \mathrm{Th}_X(v)[n]\big]$$

for any scheme X over S and any pair $(n, v) \in \mathbb{Z} \times \underline{K}(X)$.

Remark 2.1.3. Note that the bivariant theory (resp. cohomology theory) represented by $\mathbb{E}$ is contravariantly (resp. covariantly) functorial with respect to isomorphisms of virtual vector bundles v/S.

Notation 2.1.4. In order to simplify notation, we will adopt the following convention. Given an s-scheme X/S and a virtual bundle v over X, we will write (when no confusion can arise):

$$\mathbb{E}_n(Y/S, v) := \mathbb{E}_n(Y/S, f^*v)$$

for any morphism $f : Y \to X$ of s-schemes over S.

In the special case where $\mathbb{E}$ is the sphere spectrum, we will use the following notation:

Notation 2.1.5. We set

$$H_n(X/S, v) := \mathbb{S}_n(X/S, v) = [\mathrm{Th}_X(v)[n], p^!(\mathbb{S}_S)]$$

for any s-morphism $p : X \to S$ and any pair $(n, v) \in \mathbb{Z} \times \underline{K}(X)$. We will refer to this simply as the *bivariant* $\mathbb{A}^1$*-theory*. We also set $H(X/S, v) := H_0(X/S, v)$ for the case $n = 0$.

Similarly, we set $H^n(X, v) := \mathbb{S}^n(X, v)$ and refer to this as the $\mathbb{A}^1$*-cohomology*.

2.1.6. The bivariant theory represented by a motivic spectrum $\mathbb{E} \in \mathscr{SH}(S)$ satisfies the following axioms, which are simply $\underline{K}$-graded versions of the axioms of Fulton and MacPherson [FM81]:

- base change for a cartesian square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "p"] \\ T \arrow[r, "f"'] & S, \end{tikzcd}$$

  one gets

$$\Delta^* : \mathbb{E}_n(X/S, v) \to \mathbb{E}_n(Y/T, g^*v).$$

  This map is obtained by applying the functor $g^* : \mathscr{SH}(X) \to \mathscr{SH}(Y)$ and using the exchange transformation $Ex^{*!} : g^*p^! \to q^!f^*$ associated with the square Δ.
- covariance for a proper S-morphism $f : Y \to X$:

$$f_* : \mathbb{E}_n(Y/S, f^*v) \to \mathbb{E}_n(X/S, v).$$

  This covariance is obtained using the unit map $f_!f^! \to \mathrm{Id}$ and the fact $f_! = f_*$ as f is proper.

- contravariance for an étale S-morphism $f : Y \to X$:
$$f^! : \mathbb{E}_n(X/S, v) \to \mathbb{E}_n(Y/S, f^*v).$$
This contravariance is obtained by applying the functor $f^! : S\mathscr{H}(X) \to S\mathscr{H}(S)$ and using the purity isomorphism $f^! \simeq f^*$ as f is étale.
- products: if $\mathbb{E}$ is a motivic *ring* spectrum, then for s-schemes $Y \xrightarrow{q} X \xrightarrow{p} S$, integers (n, m) and virtual vector bundles v/X and w/Y, one gets a multiplication map
$$\mathbb{E}_m(Y/X, w) \otimes \mathbb{E}_n(X/S, v) \to \mathbb{E}_{m+n}(Y/S, w + q^*v).$$
Given maps $y : \mathrm{Th}_Y(w)[m] \to q^!\mathbb{E}_X$ and $x : \mathrm{Th}_X(v)[n] \to p^!\mathbb{E}_S$, the product $y.x$ is defined as follows:
$$\begin{aligned}\mathrm{Th}_Y(w + q^*v)[m + n] &\xrightarrow{y \otimes \mathrm{Id}} q^!\mathbb{E}_X \otimes \mathrm{Th}_Y(q^*v)[n] \xrightarrow{Ex^{!*}_{\otimes}} q^!(\mathbb{E}_X \otimes \mathrm{Th}_Y(v)[n]) \\ &\xrightarrow{q^!(\mathrm{Id} \otimes x)} q^!(\mathbb{E}_X \otimes p^!\mathbb{E}_S) \xrightarrow{Ex^{!*}_{\otimes}} q^!p^!(\mathbb{E}_S \otimes \mathbb{E}_S) \xrightarrow{\mu_{\mathbb{E}}} q^!p^!(\mathbb{E}_S) = (pq)^!(\mathbb{E}_S),\end{aligned}$$
where $\mu_{\mathbb{E}}$ denotes the multiplication map of $\mathbb{E}_S$.

These structures satisfy the usual properties stated by Fulton and MacPherson (associativity, base change formula both with respect to base change and étale contravariance, compatibility with pullbacks and projection formulas; see [Dég17, 1.2.8] for the precise formulation).

Remark 2.1.7. Our main result, Theorem 3.3.2, is concerned with extending the contravariant functoriality for *étale* morphisms to *quasi-projective lci* morphisms.

Remark 2.1.8. The morphism $Ex^{!*}_{\otimes}$ appearing in the definition of the product has the following general form. For any s-morphism $f : X \to S$ and any pair of objects $\mathbb{E}$ and $\mathbb{F}$ in $S\mathscr{H}(S)$, there is a canonical morphism
$$Ex^{!*}_{\otimes} : f^!(\mathbb{E}) \otimes f^*(\mathbb{F}) \to f^!(\mathbb{E} \otimes \mathbb{F})$$
induced by adjunction from the projection formula. For a fixed $\mathbb{E}$ (resp. a fixed $\mathbb{F}$), these define a natural transformation.

Let us also observe that if $\mathbb{F}$ is $\otimes$-*invertible*, then the natural transformation
$$f^!(-) \otimes f^*(\mathbb{F}) \to f^!(- \otimes \mathbb{F})$$
is *invertible*. Indeed it suffices by adjunction to note that its left transpose
$$f_!(-) \otimes \mathbb{F}^{\otimes -1} \to f_!(f^*(\mathbb{F}^{\otimes -1}) \otimes -)$$
is invertible, by the projection formula.

Remark 2.1.9. Note that a particular case of the above product is the cap-product:
$$\mathbb{E}^n(X, v) \otimes \mathbb{E}_m(X/S, w) \to \mathbb{E}_{m-n}(X/S, w - v),$$
for any s-scheme X over S.

Central to our constructions is the following long exact sequence, which is a direct corollary of the localization triangle[4] in the six functors formalism for $S\mathscr{H}$:

Proposition 2.1.10. *Let $\mathbb{E} \in S\mathscr{H}(S)$ be a motivic spectrum. For any closed immersion $i : Z \to X$ of s-schemes over S, with quasi-compact complementary open immersion $j : U \to X$, there exists a canonical* localization long exact sequence *of the form:*

$$\cdots \to \mathbb{E}_n(Z/S,e) \xrightarrow{i_*} \mathbb{E}_n(X/S,e) \xrightarrow{j^*} \mathbb{E}_n(U/S,e) \xrightarrow{\partial_i} \mathbb{E}_{n-1}(Z/S,e) \to \cdots \tag{2.1.10.a}$$

for any $e \in \underline{K}(X)$, which is natural with respect to the contravariance in S, the contravariance in X/S for étale S-morphisms and the covariance in X/S for proper S-morphisms.

We will need more specifically the following two properties of localization long exact sequences.

Proposition 2.1.11. *Let $\mathbb{E} \in S\mathscr{H}(S)$ be a motivic spectrum. Consider a commutative square:*

$$\begin{tikzcd}
T \arrow[r, "l"] \arrow[d, "k"'] & Z' \arrow[d, "j"] \\
Z \arrow[r, "i"] & X
\end{tikzcd}$$

of closed immersions of s-schemes over S. Given a virtual vector bundle e over X, consider the following diagram:

$$\begin{tikzcd}
\mathbb{E}_n(T/S,e) \arrow[r, "l_*"] \arrow[d, "k_*"'] & \mathbb{E}_n(Z'/S,e) \arrow[r, "l'^*"] \arrow[d, "j_*"] & \mathbb{E}_n(Z'-T/S,e) \arrow[r, "\partial_l"] \arrow[d, "\tilde{j}_*"'] & \mathbb{E}_{n-1}(T/S,e) \arrow[d, "k'_*"] \\
\mathbb{E}_n(Z/S,e) \arrow[r, "i_*"] \arrow[d, "k'^*"'] & \mathbb{E}_n(X/S,e) \arrow[r, "i'^*"] \arrow[d, "j'^*"] & \mathbb{E}_n(X-Z/S,e) \arrow[r, "\partial_i"] \arrow[d, "\tilde{j}'^*"'] & \mathbb{E}_{n-1}(Z/S,e) \arrow[d, "k'^*"] \\
\mathbb{E}_n(Z-T/S,e) \arrow[r, "\tilde{i}_*"] \arrow[d, "\partial_k"'] & \mathbb{E}_n(X-Z'/S,e) \arrow[r, "\tilde{i}'^*"] \arrow[d, "\partial_j"] & \mathbb{E}_n(X-Z\cup Z'/S,e) \arrow[r, "\partial_{\tilde{i}}"] \arrow[d, "\partial_{\tilde{j}}"'] \arrow[dr, phantom, "(*)"] & \mathbb{E}_{n-1}(Z-T/S,e) \arrow[d, "\partial_k"] \\
\mathbb{E}_{n-1}(T/S,e) \arrow[r, "l_*"] & \mathbb{E}_{n-1}(Z'/S,e) \arrow[r, "l'^*"] & \mathbb{E}_{n-1}(Z'-T/S,e) \arrow[r, "\partial_l"] & \mathbb{E}_{n-2}(T/S,e)
\end{tikzcd}$$

where a letter with a prime denotes the complementary open immersion of the corresponding closed immersion, and $\tilde{j}$, $\tilde{i}$ denote the obvious closed immersions obtained by restriction.

Then all the squares of the diagram are commutative except square $(*)$ which is anti-commutative.

Proof. The commutativity of the three squares in the first line (resp. in the first column) follows from the naturality of localization long exact sequences with respect to proper covariance. That of the three squares in the second line (resp. in the second column) follows from the naturality of localization long exact sequences with respect to étale contravariance.

To get the anti-commutativity of square $(*)$, instead of looking at the preceding diagram, we look at the corresponding diagram made of the analogous localization distinguished triangles in $S\mathscr{H}(X)$. The resulting diagram is made of nine squares as in the above one. As above, we see that all squares, except the analogue of square $(*)$, commute. Then the

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[4] See [Kha16, Cor. 7.4.7] or [Hoy15, Prop. C.10] for the non-noetherian setting.

resulting anti-commutativity follows from Verdier's octahedral axiom, together with the unicity of boundary maps in distinguished triangles (see e.g. [CD12, Prop. 2.3.3]).[5] ☐

Proposition 2.1.12. *Let $\mathbb{E} \in S\mathscr{H}(S)$ be a motivic ring spectrum. Consider cartesian squares of s-schemes over S:*

$$\begin{tikzcd} T \arrow[r, "k"] \arrow[d] \arrow[rd, phantom, "\Delta_Z"] & Y \arrow[d] \arrow[rd, phantom, "\Delta_U"] & V \arrow[l, "k'"'] \arrow[d] \\ Z \arrow[r, "i"'] & X & U \arrow[l, "i'"] \end{tikzcd}$$

such that i and k are closed immersions with quasi-compact complementary open immersions i' and k', respectively. Consider an element $\pi \in \mathbb{E}_r(Y/X, e')$ with e' a virtual bundle over Y, and put:

$$\pi_Z = \Delta_Z^*(\pi) \in \mathbb{E}_r(T/Z, e'_T), \pi_U = \Delta_U^*(\pi) \in \mathbb{E}_r(V/U, e'_V).$$

Then the following diagram is commutative:

$$\begin{tikzcd} \mathbb{E}_n(Z/S, e) \arrow[r, "i_*"] \arrow[d, "\gamma_{\pi_Z}"'] & \mathbb{E}_n(X/S, e) \arrow[r, "i'^*"] \arrow[d, "\gamma_\pi"] & \mathbb{E}_n(U/S, e) \arrow[r, "\partial_i"] \arrow[d, "\gamma_{\pi_U}"'] & \mathbb{E}_{n-1}(Z/S, e) \arrow[d, "\gamma_{\pi_Z}"] \\ \mathbb{E}_{n+r}(T/S, e+e') \arrow[r, "k_*"] & \mathbb{E}_{n+r}(Y/S, e+e') \arrow[r, "k'^*"] & \mathbb{E}_{n+r}(V/S, e+e') \arrow[r, "\partial_k"] & \mathbb{E}_{n+r-1}(T/S, e+e') \end{tikzcd}$$

where γ_x denotes multiplication by $x = \pi, \pi_Z, \pi_U$.

Proof. The proof follows the same pattern as that of the preceding proposition. Indeed, the first square is commutative due to the projection formula, as is the second square since products are compatible with base change; then the third square is automatically commutative once we interpret the formula in the stable homotopy category $S\mathscr{H}(X)$, using the unicity of boundary maps in localization distinguished triangles. ☐

2.1.13. The following further properties, while not part of the formalism of Fulton and MacPherson, are in fact consequences of Grothendieck's six functors formalism:

Commutativity.– Let us consider s-schemes $p : X/S$ and $q : Y/S$ as well as classes $x \in \mathbb{E}_n(X/S, v)$ and $y \in \mathbb{E}_m(Y/S, w)$. Let $\epsilon : q^*w + p^*x \to p^*x + q^*w$ be the isomorphism of virtual vector bundles over $X \times_S Y$ which commutes the factors. Then the following formula holds in $\mathbb{E}_{n+m}(X \times_S Y/S, q^*w + p^*x)$:

$$q^*(x).y = (-1)^{n+m}.\epsilon^*\big(p^*(y).x\big).$$

This formula just follows from a careful unfolding of the definition.

Nisnevich and cdh descent.– Consider a cartesian square of s-schemes over S:

$$\begin{tikzcd} Y' \arrow[r, "k"] \arrow[d, "g"'] & X' \arrow[d, "f"] \\ Y \arrow[r, "i"] & X. \end{tikzcd}$$

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[5] Recall more precisely that the sign comes from the fact the permutation on composite functor $-[1][1]$ acts as -1.

First assume that the square is Nisnevich distinguished.[6] Then we get a long exact sequence:

$$\dots \mathbb{E}_n(X/S,v) \xrightarrow{f^*+i^*} \mathbb{E}_n(X'/S,v) \oplus \mathbb{E}_n(Y/S,v) \xrightarrow{k^*-g^*} \mathbb{E}_n(Y'/S,v) \to \mathbb{E}_{n-1}(X/S,v) \dots$$

Second assume the square is cdh distinguished.[7] Then we get a long exact sequence:

$$\dots \mathbb{E}_n(Y'/S,v) \xrightarrow{k_*-g_*} \mathbb{E}_n(X'/S,v) \oplus \mathbb{E}_n(Y/S,v) \xrightarrow{f_*+i_*} \mathbb{E}_n(X/S,v) \to \mathbb{E}_{n-1}(Y'/S,v) \dots$$

These two formulas respectively follow from the Nisnevich and cdh descent properties of $S\mathscr{H}$.

2.2. **Orientations and systems of fundamental classes.** Following Fulton–MacPherson, we now introduce the notion of *orientation* of a morphism f. As we recall, any choice of orientation gives rise to a Gysin map in bivariant theory (Paragraph 2.2.5). The fundamental classes we construct in Section 3 will be examples of orientations.

For simplicity, throughout this discussion we will restrict our attention to the bivariant theory represented by the sphere spectrum $\mathbb{S}$, as in Notation 2.1.5.

Definition 2.2.1. Let $f : X \to S$ be an s-morphism.

(1) An *orientation* of f is a class:

$$\eta_f \in H(X/S, e_f)$$

for a given virtual vector bundle e_f over X.

(2) We will say η_f is *strong* if for any pair (n, v), the cap-product with η_f

$$\gamma_{\eta_f} : H^n(X, v) \to H_{-n}(X/S, v + e_f), \quad x \mapsto x.\eta_f$$

is an isomorphism. In that case, γ_{η_f} will be called the *duality isomorphism* associated with the strong orientation η_f.

(3) We will say that the orientation η_f is *universally strong* if the map $\eta_f : \mathrm{Th}_X(e_f) \to f^!\mathbb{S}_S$ is an isomorphism.

Remark 2.2.2. We warn the reader that the above use of the term "orientation", which we have taken from [FM81], is unrelated to the notion of "oriented motivic spectrum" (in the sense that will be used in Paragraph 4.1.4).

Example 2.2.3. Let $f : X \to S$ be a smooth s-morphism with tangent bundle T_f. The purity theorem [CD12, Thm. 2.4.50] provides an invertible natural transformation

$$\mathfrak{p}_f : \Sigma^{T_f} f^* \xrightarrow{\sim} f^!$$

which induces in particular a canonical isomorphism

$$\eta_f : \mathrm{Th}_X(T_f) \xrightarrow{\sim} f^!(\mathbb{S}_S).$$

This defines a (universally strong) orientation $\eta_f \in H_0(X/S, \langle T_f \rangle)$ that will be called the *fundamental class* of the smooth morphism f.

---

[6] *i.e.* i open immersion, f étale and isomorphism above the complement of i.

[7] *i.e.* i closed immersion, f proper and isomorphism over the complement of i.

Recall from [CD12, Def. 2.4.25, Cor. 2.4.37] that the transformation $\mathfrak{p}_f$ can be described as follows. Consider the cartesian square Δ:

$$\begin{tikzcd} X \times_S X \arrow[r, "f_1"] \arrow[d, "f_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

and let $\delta : X \to X \times_S X$ be the diagonal embedding. Then $\Sigma^{-T_f}\mathfrak{p}_f$ is inverse to the composite:

(2.2.3.a) $$\Sigma^{-T_f} f^! \xrightarrow{\sim} \delta^! f_1^* f^! \xrightarrow{Ex^{*!}} \delta^! f_2^! f^* = f^*$$

Here the first isomorphism $\Sigma^{-T_f} \to \delta^! f_1^*$ is dual to the relative purity isomorphism $(f_1)_\sharp \delta_* \to \Sigma^{T_f}$ of Morel–Voevodsky (modulo the canonical identification $N_\delta = T_f$), and $Ex^{*!}$ is the exchange transformation associated to the square Δ, and is invertible because f is smooth.

The following lemma explains the terminology "universally strong".

Lemma 2.2.4. *Let $f : X \to S$ be an s-morphism. Let η_f be an orientation for f as in the above definition. Suppose that η_f is universally strong. Then for any cartesian square*

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "g"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d, "f"] \\ T \arrow[r, "p"'] & S \end{tikzcd}$$

with p smooth, the orientation $\Delta^*(\eta_f)$ of g is strong.

Proof. We use the fact that for any smooth $p : T \to S$, the map

$$[p_\sharp(\mathbb{S}_T), \mathrm{Th}_X(e_f + e)[n]] \to [p_\sharp(\mathbb{S}_T), f^!(\mathbb{S}_S) \otimes \mathrm{Th}_X(e)[n]]$$

can be identified with the cap-product by $p^*(e_f)$. This identification holds as p is smooth so that the exchange transformation $Ex^{*!} : p^* f^! \to g^! q^*$ associated with the square Δ is an isomorphism. □

Definition 2.2.5. Let $f : Y \to X$ be a morphism of s-schemes over S. Then any orientation $\eta_f \in H(X/S, e_f)$ gives rise to *Gysin homomorphisms*:

$$f^!(\eta_f) : H_n(X/S, e) \to H_n(Y/S, e_f + f^*e), \quad x \mapsto \eta_f.x$$

using the product structure in bivariant $\mathbb{A}^1$-theory, for all $n \in \mathbb{Z}$ and $e \in \underline{K}(X)$.

For example, if f is smooth, then the fundamental class η_f (Example 2.2.3) gives rise to Gysin homomorphisms

$$f^! : H_n(X/S, e) \to H_n(Y/S, \langle T_f \rangle + f^*e).$$

This extends the contravariant functoriality from étale morphisms to smooth morphisms.

We now make a quick digression to make a few simple observations about Gysin homomorphisms of smooth morphisms. First we consider the special case of vector bundles:

Lemma 2.2.6. *Let X be an s-scheme over S and let $p : E \to X$ be a vector bundle, with tangent bundle $T_p = p^*E$. Then the Gysin homomorphism:*

$$p^! : H_n(X/S, e) \to H_n(E/S, p^*e + p^*\langle E\rangle)$$

is an isomorphism.

Proof. In view of the construction of the Gysin homomorphism, the claim follows from directly from the facts that the morphism $\eta_p : \mathrm{Th}_X(p^*E) \to p^!(\mathbb{S}_X)$ is invertible, and the functor $p^* : \mathscr{SH}(X) \to \mathscr{SH}(E)$ is fully faithful (by $\mathbb{A}^1$-homotopy invariance). □

Definition 2.2.7. In the context of Lemma 2.2.6, we define the *Thom isomorphism*[8]

$$\phi_{E/X} : H_*(E/S, e) \to H_*(X/S, e - \langle E\rangle),$$

associated with E/X, as the inverse of the Gysin homomorphism $p^! : H_*(X/S, e - \langle E\rangle) \to H_*(E/S, e)$.

Remark 2.2.8. Like its classical counterpart, the Thom isomorphism satisfies the properties of compatibility with base change and with direct sums ($\phi_{E\oplus F/X} = \phi_{E\oplus F/F} \circ \phi_{F/X}$). These follow respectively from the compatibility of the Gysin morphisms $p^!$ with base change and with composition.

We conclude this digression by recording the naturality of the localization sequences (Proposition 2.1.10) with respect to Gysin homomorphisms of smooth morphisms.

Proposition 2.2.9. *Consider cartesian squares of S-schemes*

$$\begin{tikzcd}
T \arrow[r, "k"] \arrow[d, "g"'] & Y \arrow[d, "f"] & V \arrow[l, "l"'] \arrow[d, "h"] \\
Z \arrow[r, "i"] & X & U \arrow[l, "j"']
\end{tikzcd}$$

such that f is smooth, i is a closed immersion and j is the complementary open immersion. Let e be a virtual bundle over X. Then the following diagram is commutative:

$$\begin{tikzcd}
H_n(Z/S, e_Z) \arrow[r, "i_*"] \arrow[d, "g^!"'] & H_n(X/S, e) \arrow[r, "j^!"] \arrow[d, "f^!"'] & H_n(U/S, e_U) \arrow[r, "\partial_k"] \arrow[d, "h^!"'] & H_n(Z/S, e_Z) \arrow[d, "g^!"] \\
H_n(T/S, e_T + \langle T_g\rangle) \arrow[r, "k_*"] & H_n(Y/S, e_Y + \langle T_f\rangle) \arrow[r, "l^!"] & H_n(V/S, e_V + \langle T_h\rangle) \arrow[r, "\partial_i"] & H_n(T/S, e_T + \langle T_g\rangle)
\end{tikzcd}$$

where we have used the canonical isomorphisms of vector bundles:

$$T_g \simeq k^{-1}T_f, \quad T_h \simeq l^{-1}T_f. \tag{2.2.9.a}$$

Proof. The diagram of the proposition is induced by a diagram in $\mathscr{SH}(S)$ involving the localization triangles associated with i and k. Then the commutativity of the analogue of square (2) simply follows from the compatibility of Gysin morphisms with composition. The commutativity of the analogue of square (1) follows, by construction of the fundamental classes η_f and η_g (Example 2.2.3), from the naturality of the relative purity isomorphism of Morel–Voevodsky [Hoy15, Prop. A.4]. Then, as explained in the proof of 2.1.11, the

---

[8] Not to be confused with the Thom isomorphism of (4.1.4.a).

commutativity of the analogue of square (3) automatically follows (using the uniqueness of boundary maps in the localization distinguished triangles). ☐

We now come to the final subject of this subsection. The following definition will be a convenient way to express the functoriality and base change properties of the Gysin homomorphisms in the language of orientations.

Definition 2.2.10. Let $\mathscr{C}$ be a (not necessarily full) subcategory of the category of (quasi-compact and quasi-separated) schemes.

1. A *system of fundamental classes* for $\mathscr{C}$ (in bivariant $\mathbb{A}^1$-theory) is the data of, for each morphism $f : X \to Y$ in $\mathscr{C}$, a virtual bundle $e_f \in \underline{K}(X)$ and an orientation $\eta_f^{\mathscr{C}} \in H_0(f, e_f)$ such that the following relations hold:
   - (a) *Normalisation.* If $f = \mathrm{Id}_S$, then $e_f = 0$ and the orientation $\eta_f^{\mathscr{C}} \in H_0(S/S, 0)$ is given by the identity $\mathrm{Id} : \mathbb{S}_S \to \mathbb{S}_S$.
   - (b) *Associativity formula.* For any composable morphisms f and g in $\mathscr{C}$, one has an isomorphism:

$$e_{f\circ g} \simeq e_g + g^* e_f \tag{2.2.10.a}$$

   and modulo this identification, the following relation holds:

$$\eta_g^{\mathscr{C}}.\eta_f^{\mathscr{C}} = \eta_{f\circ g}^{\mathscr{C}}.$$

2. Suppose that the category $\mathscr{C}$ admits fibred products. We say that a system of fundamental classes $(\eta_f^{\mathscr{C}})_f$ is *stable under transverse base change* if it satisfies the following condition: for any cartesian square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & T \arrow[d, "p"] \\ X \arrow[r, "f"'] & S. \end{tikzcd}$$

   such that f and g are in $\mathscr{C}$ and p is transverse to f, then one has $e_g = q^*(e_f)$ and the following formula holds in $H_0(X/S, e_g)$: $\Delta^*(\eta_f^{\mathscr{C}}) = \eta_g^{\mathscr{C}}$.

Remark 2.2.11. The previous definition admits an obvious extension to general bivariant theories (i.e., the contexts of Definition 2.1.2, Paragraphs 4.1.1 and 4.2.1), and we will freely use this extension. Then our definition is both a generalization of [FM81, I, 2.6.2] and of [Dég17, 2.1.9].

Example 2.2.12. It follows from [Ayo07, 1.7.3] that the family of orientations η_f for f smooth (Example 2.2.3) form a system of fundamental classes for the category $\mathscr{C}$ of all schemes with smooth s-morphisms. Moreover, this system is stable under transverse[9] base change: see the proof of [Dég17, Lem. 2.3.13], where the right-hand square (3) can be ignored.

---

[9] Of course, any morphism is transverse to a smooth morphism.

Example 2.2.13. In Section 3, we will extend Example 2.2.12 to the category of schemes with *quasi-projective lci* morphisms.

Recall from [Ill06] that an lci morphism $f : X \to S$ admits a perfect cotangent complex $L_f = L_{X/S}$, which induces a virtual vector bundle $\langle L_f \rangle \in \underline{K}(X)$ (which is nothing else than the "virtual tangent bundle" of f in the sense of [BGI71, Exp. VIII]). For example, if f is smooth then $\langle L_f \rangle = \langle T_f \rangle$ is the class of the relative tangent bundle; if f is a regular closed immersion then $\langle L_f \rangle = -\langle N_f \rangle$, where $\langle N_f \rangle$ is the class of the normal bundle. Given lci morphisms $Y \xrightarrow{g} X \xrightarrow{f} S$, the composite $f \circ g$ is lci with virtual tangent bundle canonically identified in $\underline{K}(Y)$ with

$$\langle L_{fg} \rangle \simeq \langle L_g \rangle + g^* \langle L_f \rangle. \tag{2.2.13.a}$$

The fundamental class of a quasi-projective lci morphism $f : X \to S$ will then be an orientation $H_0(X/S, \langle L_f \rangle)$ (see Theorem 3.3.2).

Example 2.2.14. Suppose that $\mathbb{E}$ is an *oriented* motivic spectrum over S. In [Dég17] the first-named author constructed a system of fundamental classes for quasi-projective lci morphisms: any quasi-projective lci morphism $f : X \to S$ of relative virtual dimension $n = \chi(L_f)$ admits an orientation in $\mathbb{E}_0(X/S, \langle n \rangle)$ (in the sense of Remark 2.2.11). See Paragraph 4.1.4 for the comparison with our construction mentioned in Example 2.2.13.

2.3. **Purity transformations.**

The notion of orientation seen in the preceding section is part of the (twisted) bivariant formalism of Fulton and MacPherson. We state in this subsection a variant, or a companion, of this notion in the spirit of Grothendieck' six functors formalism.

2.3.1. Let us fix an s-morphism $f : X \to S$, and an orientation $\eta_f \in H(X/S, e_f)$ of f for a given virtual vector bundle e_f over X. According to our definitions, this class can be seen as a morphism in $S\mathscr{H}(X)$:

$$\eta_f : \mathrm{Th}_X(e_f) \to f^!(\mathbb{S}_S).$$

This gives rise to a natural transformation

$$\mathfrak{p}(\eta_f) : \Sigma^e f^* \to f^!$$

associated to the orientation η_f, defined as the following composite[10]:

$$f^*(-) \otimes \mathrm{Th}_X(e_f) \xrightarrow{\mathrm{Id} \otimes \eta_f} f^*(-) \otimes f^!(\mathbb{S}_S) \xrightarrow{Ex^{!*}_{\otimes}} f^!(- \otimes \mathbb{S}_S) \simeq f^!.$$

Remark 2.3.2. (1) Consider the above notations and assume that f is smooth. According to Example 2.2.3, one has a canonical orientation η_f associated with f, and $e_f = T_f$ is the tangent bundle of f. Moreover, it follows from the above constructions that the purity transformation $\mathfrak{p}(\eta_f)$ associated with η_f in the above definition is the natural transformation (2.2.3.a), denoted by $\mathfrak{p}_f$. In particular, $\mathfrak{p}(\eta_f)$ is an isomorphism in this case.

---

[10] See Remark 2.1.8 for the notation $Ex^{!*}_{\otimes}$.

(2) For non smooth morphisms f, the purity transformation of the above definition is in general not an isomorphism. A very interesting question is to know when it becomes an isomorphism when applied to a specific object (see Definitions 4.2.5 and 4.2.9).
(3) The data of an orientation or of the associated purity transformation are equivalent.
(4) An analogue of the purity transformation, for non necessarily smooth morphism f, can be found in [AGV73, VIII, §3.2, (3.2.1.2)]. Two notable differences must be taken into account. First, the later construction holds for flat separated morphisms of finite type. Second, it only involves Tate twists rather than a Thom space. This is because the theory developed in SGA4 is oriented (see also Paragraph 4.2.3).

2.3.3. Consider the notations of the previous definition. Then one associates to $\mathfrak{p}(\eta_f)$, using the adjunction properties, two natural transformations:

$$\mathrm{tr}_f : f_!\Sigma^{e_f} f^* \to \mathrm{Id}$$
$$\mathrm{cotr}_f : \mathrm{Id} \to f_*\Sigma^{-e_f} f^!.$$

The first (resp. second) natural transformation will be called the *trace map* (resp. *cotrace map*) associated with the orientation η_f, following the classical usage in the literature. These two maps are a functorial incarnation of the Gysin map defined earlier (Paragraph 2.2.5). This fact will be exemplified in the applications (see 4.2.2).

2.3.4. The notion of a system of fundamental classes (Definition 2.2.10) was introduced to reflect the functoriality of Gysin morphisms. For completeness, we now formulate the analogous functoriality property for the associated purity transformations.

Let $\mathscr{C}$ be a (not necessarily full) subcategory of the category of (qcqs) schemes, and let $(\eta_f)_f$ denote a system of fundamental classes for $\mathscr{C}$ (Definition 2.2.10). For each morphism f in $\mathscr{C}$, set $f^{L*} := \Sigma^{L_f} f^*$ so that we get from Paragraph 2.3.1 a natural transformation:

$$\mathfrak{p}(\eta_f) : f^{e*} \to f^!.$$

Let $\mathscr{T}ri$ denote the $(2,1)$-category of large triangulated categories, triangulated functors, and invertible triangulated natural transformations. Let $S\mathscr{H}^!$ denote the contravariant pseudofunctor from the category $\mathscr{C}$ to the $(2,1)$-category $\mathscr{T}ri$, given by the assignments $S \mapsto S\mathscr{H}(S)$, $f \mapsto f^!$.

Proposition 2.3.5. *Consider the above notations.*
(i) *The assignments $S \mapsto S\mathscr{H}(S)$, $f \mapsto f^{e*}$ define a contravariant pseudofunctor:*

$$S\mathscr{H}^{e*} : \mathscr{S} \to \mathscr{T}ri.$$

(ii) *The assignment $f \mapsto \mathfrak{p}(\eta_f)$ defines a natural transformation of pseudofunctors*

$$\mathfrak{p} : S\mathscr{H}^{e*} \to S\mathscr{H}^!$$

on the category $\mathscr{C}$.

(iii) *Suppose that the system of fundamental classes $(\eta_f)_f$ is stable under transverse base change (Definition 2.2.10). Then for any tor-independent cartesian square of the form*

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "v"] & T \arrow[d, "u"] \\
X \arrow[r, "f"] & S,
\end{tikzcd}$$

where $f : X \to Y$ is in $\mathscr{C}$, the diagram

$$\begin{tikzcd}
v^*\Sigma^{e_f} f^* \arrow[r, "v^*{*}\mathfrak{p}_f"] \arrow[d, "\sim"] & v^* f^! \arrow[dd, "Ex^{*!}"] \\
\Sigma^{e_g} v^* f^* \arrow[d, "\sim"] & \\
\Sigma^{e_g} g^* u^* \arrow[r, "\mathfrak{p}_g{*}u^*"] & g^! u^*
\end{tikzcd}$$

commutes.

Proof. Let f and g be composable morphisms in $\mathscr{S}$. The isomorphism of virtual vector bundles (2.2.10.a) induces a canonical isomorphism of Thom spaces

$$\begin{aligned}
\mathrm{Th}(e_{fg}) &\simeq \mathrm{Th}(g^* e_f) \otimes \mathrm{Th}(e_g) \\
&\simeq g^* \mathrm{Th}(e_f) \otimes \mathrm{Th}(e_g)
\end{aligned}$$

in $S\mathscr{H}(S'')$. In particular we get a canonical 2-isomorphism

$$(fg)^{e*} \simeq g^{e*} f^{e*},$$

which shows claim (i). Claims (ii) and (iii) follow formally from the associativity and transverse base change axioms for the system $(\eta_f)_f$, respectively; we leave the details to the reader. □

We can also formulate the property of base change for tor-independent squares in terms of the (co)trace maps.

Proposition 2.3.6. *Consider the above notations. Let $f : X \to Y$ be morphism in $\mathscr{C}$. For any morphism $u : Y' \to Y$, form the cartesian square*

$$\begin{tikzcd}
X' \arrow[r, "g"] \arrow[d, "v"] & Y' \arrow[d, "u"] \\
X \arrow[r, "f"] & Y.
\end{tikzcd}$$

If u is transverse to f and proper, then we have a commutative diagram

$$\begin{tikzcd}
f_!\Sigma^{e_f} f^* u_! \arrow[r, "\sim"] \arrow[d, "\mathrm{tr}_f{*}u_!"] & f_!\Sigma^{e_f} v_! g^* \arrow[r, "\sim"] & f_! v_! \Sigma^{e_g} g^* \arrow[r, "\sim"] & u_! g_! \Sigma^{e_g} g^* \arrow[d, "u_!{*}\mathrm{tr}_g"] \\
u_! \arrow[rrr, equal] & & & u_!
\end{tikzcd}$$

Dually, we have a commutative diagram

$$\begin{tikzcd}
u^! \arrow[rrr, equal] \arrow[d, "\mathrm{cotr}_g * u^!"] & & & u^! \arrow[d, "u_! * \mathrm{cotr}_f"] \\
g_* \Sigma^{-e_g} g^! u^! \arrow[r, "\sim"] & g_* \Sigma^{-e_g} v^! f^! \arrow[r, "\sim"] & g_* v^! \Sigma^{-e_f} f^! \arrow[r, "\sim"] & u^! f_* \Sigma^{-e_f} f^!
\end{tikzcd}$$

obtained by passing to right adjoints.

Proof. It suffices to consider the first square. The properness assumption on u implies that the functor $u_! \simeq u_*$ is right adjoint to u^* (and similarly for v). The statement is then a formal consequence of Proposition 2.3.5(iii); we leave the details to the reader. ☐

3. Construction of fundamental classes

3.1. **Euler classes.**

3.1.1. We will use the notion of Euler classes in the followings. Our basic definition is very simple and can be formulated unstably.

Let X be a scheme and E be a vector bundle over X. Recall that the Thom space of E/X (relative to X) is the Nisnevich sheaf of sets:

$$\mathrm{Th}(E) := \mathrm{coKer}(E^\times \to E)$$

where $E^\times$ is complement of the zero section. It can be seen as a pointed sheaf where the base point corresponds to $E^\times$, and we will consider it as an object of the pointed $\mathbb{A}^1$-homotopy category $\mathscr{H}_\bullet(X)$ over X.

Thom spaces are functorial. Given a monomorphism of vector bundles $\nu : F \to E$ over X, one gets a canonical morphism of pointed sheaves:

$$\nu_* : \mathrm{Th}(F) \to \mathrm{Th}(E).$$

Our definition of Euler classes is very simple.

Definition 3.1.2. Let E/X be a vector bundle over a scheme X, and s be its zero section. We can look at s as a monomorphism of vector bundles $s : 0_X \to E$ from the null vector space over X to E. We define the *Euler class* $e(E)$ of E/X as the map in $\mathscr{H}_\bullet(X)$:

$$s_* : X_+ = \mathrm{Th}(0_X) \to \mathrm{Th}(E).$$

We can see the class $e(E)$ as an element of the (twisted) unstable cohomotopy group $\pi^0(X, E) = [X_+, \mathrm{Th}(E)]$.

Remark 3.1.3. This definition may seem inappropriate for the reader. However, the class $e(E)$ defined above has natural realization, by the universality of the unstable homotopy category. In what follows, we will look at its $\mathbb{P}^1$-stable version. With the notations of Definition 2.1.2, this is class in the $\mathbb{A}^1$-cohomotopy $H^0(X, \langle E \rangle)$. It can then be realized as a class in the (twisted) cohomology of any ring spectra $\mathbb{E}$. We will prove below that our Euler class do correspond to the classical Euler class in the Chow-Witt group when $\mathbb{E}$ is the unramified Milnor-Witt sheaf and to the top Chern class when $\mathbb{E}$ is oriented.

3.1.4. Consider the notations of the previous definition. By construction, the Thom space fits into a cofiber sequence of pointed spaces over X:

$$(E^\times)_+ \xrightarrow{j_*} E_+ \xrightarrow{\pi} \mathrm{Th}(E).$$

According to the $\mathbb{A}^1$-homotopy property, the projection map $p : E \to X$ induces an isomorphisms of pointed spaces, whose inverse is induces by the zero section $s : X \to E$. It follows from our construction that the following diagram commutes:

$$\begin{tikzcd}
(E^\times)_+ \arrow[r, "j_*"] \arrow[d, equal] & E_+ \arrow[r, "\pi"] & \mathrm{Th}(E) \arrow[d, equal] \\
(E^\times)_+ \arrow[r, "q_*"] & X_+ \arrow[r, "e(E)"] \arrow[u, "s_*"] & \mathrm{Th}(E)
\end{tikzcd}$$

where $q : E^\times \to X$ is the canonical projection map.

Definition 3.1.5. Consider the above notations. Then the homotopy cofiber sequence of pointed spaces over X:

$$(E^\times)_+ \xrightarrow{q_*} X_+ \xrightarrow{e(E)} \mathrm{Th}(E),$$

seen in $\mathscr{H}_\bullet(X)$, is called the *Euler cofiber sequence.*

The Euler cofiber sequence immediately yields the following characteristic property of Euler classes:

Proposition 3.1.6. *Let X be a vector bundle over X. If E/X has a nowhere vanishing section*[11] *then the Euler class is the zero pointed map.*

This simply follows by looking at the homotopy long exact sequence obtained from the Euler cofiber sequence after applying the functor $[X_+, -]$ and using the fact q is a split epimorphism under the assumptions of the proposition.

3.1.7. It is well known that the Thom space functor sends direct sums to wedge products. One can in fact improve this result. Consider a short exact sequence of vector bundles over a scheme X:

$$(\sigma) : 0 \to F \xrightarrow{\nu} E \to G \to 0.$$

First recall that a section of the exact sequence induces a canonical isomorphism of pointed sheaves:

$$\mathrm{Th}(F) \wedge \mathrm{Th}(G) \xrightarrow{\sim} \mathrm{Th}(E).$$

Let us put $Y = \underline{\mathrm{Hom}}(G, F)$, seen as vector bundle over X with structural map π. Recall we have an adjunction of categories:

$$\pi_\sharp : \mathscr{H}_\bullet(X) \leftrightarrows \mathscr{H}_\bullet(Y) : \pi^*$$

and that the $\mathbb{A}^1$-homotopy property implies that the adjunction map $\mathrm{Id} \to \pi_\sharp\pi^*$ is an isomorphism. The pullback over Y of the exact sequence (σ) is split so we get a canonical isomorphism:

$$\phi : \pi^*(\mathrm{Th}(F)) \wedge \pi^*(\mathrm{Th}(G)) \xrightarrow{\simeq} \pi^*(\mathrm{Th}(E))$$

---

[11] In particular if it contains $\mathbb{A}^1_X$ as a direct summand.

that we will consider in $\mathscr{H}_\bullet(Y)$. We deduce a canonical isomorphism in $\mathscr{H}_\bullet(X)$:

$$\begin{aligned}\epsilon_\sigma : \mathrm{Th}(F) \wedge \mathrm{Th}(G) &\xrightarrow{\sim} \mathrm{Th}(F) \wedge \pi_\sharp\pi^*(\mathrm{Th}(G)) \xrightarrow{\sim} \pi_\sharp(\pi^*(\mathrm{Th}(F)) \wedge \pi^*(\mathrm{Th}(G)) \\ &\xrightarrow{\phi} \pi_\sharp\pi^*(\mathrm{Th}(E)) \xrightarrow{\sim} \mathrm{Th}(E)\end{aligned}$$

where the second map is obtained by the projection formula.[12]

Lemma 3.1.8. *Consider the exact sequence* (σ) *of vector bundles and the above notations. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Th}(F) \arrow[r, "1\wedge e(G)"] \arrow[d, equal] & \mathrm{Th}(F) \wedge \mathrm{Th}(G) \arrow[d, "\epsilon_\sigma", "\sim"'] \\
\mathrm{Th}(F) \arrow[r, "\nu_*"] & \mathrm{Th}(E)
\end{tikzcd}$$

After pullback along $\pi : Y \to X$, we reduce to the split case $E = F \oplus G$, for which the result follows from the constructions.

The additivity property of Euler classes is then a direct corollary.

Proposition 3.1.9. *Consider the exact sequence* (σ) *of vector bundles. Then the following diagram is commutative:*

$$\begin{tikzcd}
X_+ \arrow[r, "e(F)\wedge e(G)"] \arrow[d, equal] & \mathrm{Th}(F) \wedge \mathrm{Th}(G) \arrow[d, "\epsilon_\sigma", "\sim"'] \\
X_+ \arrow[r, "e(E)"] & \mathrm{Th}(E)
\end{tikzcd}$$

Indeed, it suffices to compose the diagram of the preceding lemma with the map $e(F) : X_+ \to \mathrm{Th}(F)$ (on the left).

3.2. **Fundamental classes: regular closed immersions.** In this section we construct the fundamental class of a regular immersion and demonstrate its expected properties. Before proceeding, we make a brief digression to consider a certain preliminary construction.

3.2.1. Let X be a scheme and consider the diagram

$$\begin{tikzcd}
\mathbb{G}_m X \arrow[rr, "j"] \arrow[dr, "\pi"'] & & \mathbb{A}^1 X \arrow[dl, "\bar\pi"] \\
& X &
\end{tikzcd}$$

---

[12] Usually, this isomorphism is considered in the stable homotopy category and gives the construction used in Paragraph 2.1.1.

For any $n \in \mathbb{Z}$ and $e \in \underline{K}(X)$, we have a commutative diagram

$$\begin{tikzcd}
H_n(\mathbb{A}^1X/X, e) \arrow[r, "j^!"] & H_n(\mathbb{G}_mX/X, e) \\
H_n(X/X, e - \langle 1 \rangle) \arrow[r, "\pi^!"] \arrow[u, "\bar{\pi}^!"] \arrow[d, equal] & H_n(\mathbb{G}_mX/X, e) \arrow[u, equal] \\
H^{-n}(X, e - \langle 1 \rangle) \arrow[r, "\pi^!"] & H^{-n}(\mathbb{G}_mX, e - \langle 1 \rangle) \arrow[u, "\gamma_{\eta_\pi}"']
\end{tikzcd}$$

using the identifications $\langle T_\pi \rangle \simeq \langle 1 \rangle$ in $\underline{K}(\mathbb{G}_mX)$, $\langle T_{\bar{\pi}} \rangle \simeq \langle 1 \rangle$ in $\underline{K}(\mathbb{A}^1X)$. The upper square consists of Gysin maps and commutes by Example 2.2.12; moreover, the morphism $\bar{\pi}^!$ is invertible (Lemma 2.2.6). In the lower square, the morphism γ_{η_π} is the duality isomorphism associated to π (Definition 2.2.1), and the square evidently commutes by construction of the morphisms involved. Since the lower horizontal arrow $\pi^!$ admits a retraction, given by the inverse image by the unit section $s_1 : X \to \mathbb{G}_mX$ in cohomology, it follows that we get a canonical retraction ν_t of the upper arrow $j^!$.

This implies that the localization long exact sequence in bivariant $\mathbb{A}^1$-theory associated with the zero section $s_0 : X \to \mathbb{A}^1_X$ splits into short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & H_{*+1}(\mathbb{A}^1X/X, *) \arrow[r, "j^!"] & H_{*+1}(\mathbb{G}_mX/X, *) \arrow[r, "\partial_{s_0}"] & H_*(X/X, *) \arrow[r] & 0
\end{tikzcd}$$

of modules over the bigraded ring $H^{-*}(X, *) = H_*(X/X, *)$. In particular the retraction ν_t induces a canonical section of ∂_{s_0} which we denote by:

$$\gamma_t : H_*(X/X, *) \to H_{*+1}(\mathbb{G}_mX/X, *).$$

Being $H_*(X/X, *)$-linear, this map is determined uniquely by the element $\{t\} := \gamma_t(1) \in H_1(\mathbb{G}_mX/X, 0)$, or equivalently by the corresponding morphism

$$\{t\} : \mathbb{S}_{\mathbb{G}_mX}[1] \to \pi^!(\mathbb{S}_X)$$

in $S\mathscr{H}(\mathbb{G}_mX)$; that is, one has $\gamma_t(x) = \{t\}.x$ for all $x \in H_*(X/X, *)$. If X is an S-scheme, we will abuse notation and write γ_t also for the map

$$\gamma_t : H_*(X/S, *) \to H_{*+1}(\mathbb{G}_mX/S, *),$$

given again by the assignment $x \mapsto \{t\}.x$.

We now proceed to the construction of the fundamental class.

3.2.2. Let X be a S-scheme, $i : Z \to X$ be a regular closed immersion and e be a virtual bundle over X.

We write D_ZX or $D(X, Z)$ for the (affine) deformation space $B_{Z\times 0}(X \times \mathbb{A}^1) - B_{Z\times 0}(X \times 0)$, as defined by Verdier (denoted $M(Z/X)$ in [Ver76a, §2]); here B_ZX denotes the blow-up of X in Z. There exists a canonical regular function of D_ZX that we denote by:

$$t : D_ZX \to \mathbb{A}^1.$$

Through t, D_ZX is flat over $\mathbb{A}^1$, isomorphic to the open subscheme $\mathbb{G}_mX$ over $\mathbb{G}_m$ and with fiber over 0 being the normal bundle $p : N_ZX \to Z$ associated with the regular closed immersion i. Let $h : \mathbb{G}_mX \to D_ZX$ (resp. $k : N_ZX \to D_ZX$) the corresponding open (resp. closed) immersion. We identify the function t on D_ZX with a parameter of $\mathbb{G}_mX$ via the open immersion h.

In particular we get the following localization sequence (Proposition 2.1.10):

$$\cdots \to H_{n+1}(D_ZX/S, e) \xrightarrow{h^!} H_{n+1}(\mathbb{G}_mX/S, e) \xrightarrow{\partial_{N_ZX/D_ZX}} H_n(N_ZX/S, e) \xrightarrow{k_*} H_n(D_ZX/S, e) \to \cdots$$

Definition 3.2.3. With notation as above, the *specialization to the normal cone* associated with i is the composite map

$$\sigma_{Z/X} : H_*(X/S, *) \xrightarrow{\gamma_t} H_{*+1}(\mathbb{G}_mX/S, *) \xrightarrow{\partial_{N_ZX/D_ZX}} H_*(N_ZX/S, *),$$

where γ_t is the map constructed in Paragraph 3.2.1.

Taking $S = X$, we define the *fundamental class* $\eta_i \in H_0(Z/X, -N_ZX)$ associated with the regular closed immersion i as the image of 1_X by the composite map:

$$H_0(X/X, 0) \xrightarrow{\sigma_{Z/X}} H_0(N_ZX/X, 0) \xrightarrow{\phi_{N_ZX/Z}} H_0(Z/X, -N_ZX)$$

where $\phi_{N_ZX/Z}$ is the Thom isomorphism of $N_ZX \to Z$ (Definition 2.2.7).

Recall that the virtual tangent bundle of i is given by $\langle L_i \rangle = -\langle N_ZX \rangle$. Therefore the class η_i defines an orientation of i (Definition 2.2.1), and corresponds to a map:

$$\eta_i : \mathrm{Th}_Z(L_i) \to i^!(\mathbb{S}_X)$$

in $S\mathscr{H}(Z)$.

Remark 3.2.4.
(1) Obviously, in the notations of the above definition with $X = S$, one has: $\eta_i = (p^!)^{-1}(\sigma_{Z/X}(\{t\}))$.
(2) Our definition of the specialization map is formally very close to the corresponding map in Rost's theory of cycle modules, denoted by $J(X, Z)$ in [Ros96, §11].
(3) Let us assume that X is a S-scheme. Then the orientation η_i gives rise to a Gysin morphism (Definition 2.2.5):

$$i^! : H_n(X/S, e) \to H_n(Z/S, e_Z + \langle L_i \rangle), x \mapsto \eta_i.x.$$

It follows from the definitions that this map can also be described as the composite:

$$H_n(X/S, e) \xrightarrow{\sigma_{Z/X}} H_n(N_ZX/S, e) \xrightarrow{(p^!)^{-1}} H_n(Z/S, e_Z - \langle N_ZX \rangle),$$

therefore comparing our construction with that of Verdier (cf. again [Ver76a]). Note by the way that $\sigma_{Z/X} = p^!i^!$ so that the Gysin map and the specialization map uniquely determined each other.

(4) One can concretely describe the map η_i as follows. Let us recall the deformation diagram:

$$\begin{tikzcd}
\mathbb{G}_m X \arrow[r, hook, "h"] \arrow[rd, "\pi"'] & D_Z X \arrow[d, "r"] & N_Z X \arrow[l, hook', "k"'] \arrow[d, "p"] \\
& X & Z \arrow[l, "i"]
\end{tikzcd}$$

First the map $\{t\} : \mathbb{S}_{\mathbb{G}_m X}[1] \to \pi^!(\mathbb{S}_X)$ corresponds by adjunction and after one desuspension to a map:

$$\sigma_t : \mathbb{S}_D \to h_*\pi^!(\mathbb{S}_X[-1]) = h_*h^*r^!(\mathbb{S}_X[-1]).$$

Then one gets the following composite map:

$$\mathbb{S}_D \xrightarrow{\sigma_t} h_*h^*r^!(\mathbb{S}_X[-1]) \xrightarrow{\text{boundary}} k_!k^!r^!(\mathbb{S}_X) = k_!p^!i^!(\mathbb{S}_X) \simeq k_*p^*\big(Th(N_ZX) \otimes i^!(\mathbb{S}_X)\big)$$

where the last isomorphism uses the purity isomorphism associated with p. Using the fact $\mathbb{S}_D = r^*(\mathbb{S}_X)$ and the adjunction (r^*, r_*) we deduce a map:

$$\mathbb{S}_X \to p_*p^*\big(Th(N_ZX) \otimes i^!(\mathbb{S}_X)\big) \simeq Th(N_ZX) \otimes i^!(\mathbb{S}_X)$$

where the last isomorphism follows from the homotopy invariance of $S\mathscr{H}$ with respect to the vector bundle N_ZX/Z. The latter compositum is the map $\mathrm{Th}_Z(-N_ZX) \otimes \eta_i$.

3.2.5. Consider a cartesian square

$$\begin{tikzcd}
T \arrow[r, "k"] \arrow[d, "q"'] \arrow[rd, phantom, "\Delta"] & Y \arrow[d, "p"] \\
Z \arrow[r, "i"'] & X
\end{tikzcd}$$

such that i and k are regular closed immersions.

Then we get a morphism of deformation spaces $D_TY \to D_ZX$ and similarly a morphism of vector bundles:

$$N_TY \xrightarrow{\nu} q^{-1}N_ZX \to N_ZX$$

where ν is in general a monomorphism of vector bundles (*i.e.* the codimension of T in Y can be strictly smaller than that of Z in X: there is excess of intersection). We put $\xi = q^{-1}N_ZX/N_TY$, the excess intersection bundle.

Proposition 3.2.6 (Excess intersection formula)**.** *Under the preceding hypothesis, the following formula holds in $H_0(T/Y, -q^{-1}N_ZX)$:*

$$\Delta^*(\eta_i) = e(\xi).\eta_k$$

where we have considered the action of the Euler class $e(\xi) \in \pi^0(T, \xi)$ (Definition 3.1.2) via the canonical map $\pi_0(T, \xi) \to H^0(T, \langle\xi\rangle) \simeq H_0(T/T, -\langle\xi\rangle)$ and the product on bivariant $\mathbb{A}^1$-theory, using the identification $\langle\xi\rangle + \langle N_TY\rangle = \langle q^{-1}N_ZX\rangle$.

Proof. Let us put $D'_T Y = D_Z X \times_X Y$ and $N'_T Y = q^{-1} N_Z X$. Then we get the following commutative diagram of schemes, in which each square is cartesian:

$$\begin{tikzcd}
N_T Y \arrow[r, hook] \arrow[d] & D_T Y \arrow[d] & \mathbb{G}_m Y \arrow[l, hook'] \arrow[d, equal] \\
N'_T Y \arrow[r, hook] \arrow[d] & D'_T Y \arrow[d] & \mathbb{G}_m Y \arrow[l, hook'] \arrow[d] \\
N_Z X \arrow[r, hook] & D_Z X & \mathbb{G}_m X \arrow[l, hook']
\end{tikzcd}$$

Therefore, one gets the following commutative diagram:

$$\begin{tikzcd}
H_1(\mathbb{G}_m Y/Y, 0) \arrow[r, "\partial_{T/Y}"] \arrow[d, equal] \arrow[rd, phantom, "(1)"] & H_0(N_T Y/Y, 0) \arrow[r, "\phi_{N_T Y/T}"] \arrow[d] \arrow[rd, phantom, "(3)"] & H_0(T/Y, -N_T Y) \arrow[d, "\nu_*"] \\
H_1(\mathbb{G}_m Y/Y, 0) \arrow[r] \arrow[rd, phantom, "(2)"] & H_0(N'_T Y/Y, 0) \arrow[r, "\phi_{N'_T Y/T}" description] \arrow[rd, phantom, "(4)"] & H_0(T/Y, -N'_T Y) \\
H_1(\mathbb{G}_m X/X, 0) \arrow[r, "\partial_{Z/X}"'] \arrow[u, "\Delta^*"] & H_0(N_Z X/Z, 0) \arrow[r, "\phi_{N_T Y/T}"'] \arrow[u, "\Delta^*"'] & H_0(Z/X, -N_Z X). \arrow[u, "\Delta^*"']
\end{tikzcd}$$

Here the arrow labelled Δ^* on the right-hand side denotes as usual the base change functoriality associated to the square Δ; we have abused notation by also writing Δ^* for the two analogous maps on the left and middle (induced by the obvious cartesian squares). Square (1) (resp. (2)) is commutative because of the naturality of localization long exact sequences with respect to the proper covariance (resp. base change); Square (3) is commutative by definition of ν_*, and Square (4) by compatibility of Thom isomorphisms with respect to base change.

Then, by looking at the image of $\sigma_\pi^X \in H_1(\mathbb{G}_m X/X, 0)$ through the morphisms of the latter commutative diagram, we deduce the relation:

$$\Delta^*(\eta_i) = \nu_*(\eta_k)$$

and we conclude using Lemma 3.1.8. □

Example 3.2.7. We get the following usual applications of the preceding formula.

(1) If we assume that p is transverse to i, then ν is an isomorphism and the excess bundle vanishes. Thus we get the relation: $\Delta^* \eta_i = \eta_k$.

(2) If we apply the formula to the cartesian square

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] \arrow[rd, phantom, "\Delta"] & Z \arrow[d, "i"] \\
Z \arrow[r, "i"'] & X
\end{tikzcd}$$

where the excess bundle $\xi = N_Z X$, we get the *self-intersection formula*:

$$\Delta^*(\eta_i) = e(N_Z X) \tag{3.2.7.a}$$

in $H_0(Z/Z, -N_Z X) = H^0(Z, N_Z X)$.

(3) In the case where $p : Y \to X$ is the blow-up along Z, we obtain a generalization of the "Key formula" for blow-ups in [Ful98, 6.7].

We will now state good properties of our constructions of orientations for regular closed immersions, culminating in the associativity formula. Note all these formulas will be subsumed once we will get our final construction.

3.2.8. First consider a cartesian square of S-schemes:

$$\begin{tikzcd}
T \arrow[r, "k"] \arrow[d, "g"'] & Y \arrow[d, "f"] \\
Z \arrow[r, "i"] & X
\end{tikzcd}$$

such that i is a regular closed immersion and f is smooth. The isomorphisms of vector bundles $T_g \simeq T_f|_T$ and $N_T Y \simeq N_Z X|_T$ induce an identification of virtual vector bundles

$$\langle T_g \rangle - \langle N_Z X|_T \rangle \simeq \langle L_{T/X} \rangle \simeq \langle T_f|_T \rangle - \langle N_T Y \rangle \tag{3.2.8.a}$$

in $\underline{K}(T)$, where $\langle L_{T/X} \rangle$ is the virtual tangent bundle of T over X.

Lemma 3.2.9. *With notation as above, one has the commutative square*

$$\begin{tikzcd}
H_*(X/S,*) \arrow[r, "\sigma_{Z/X}"] \arrow[d, "f^!"] & H_*(N_Z X/S,*) \arrow[d, "N_g(f)^!"] \\
H_*(Y/S,*+\langle T_f \rangle) \arrow[r, "\sigma_{T/Y}"] & H_*(N_T Y/S,*+\langle T_f \rangle).
\end{tikzcd}$$

Proof. It suffices to show that both squares in the following diagram commute:

$$\begin{tikzcd}
H_*(X/S,*) \arrow[r, "\gamma_t"] \arrow[d, "f^!"'] \arrow[rd, phantom, "(1)"] & H_{*+1}(\mathbb{G}_m X/S,*) \arrow[r, "\partial_{N_Z X/D_Z X}"] \arrow[d, "(1\times f)^!"] \arrow[rd, phantom, "(2)"] & H_*(N_Z X/S,*) \arrow[d, "N_g(f)^!"] \\
H_0(Y/S,*+\langle T_f \rangle) \arrow[r, "\gamma_t"'] & H_{*+1}(\mathbb{G}_m Y/S,\langle T_f \rangle) \arrow[r, "\partial_{N_T Y/D_T Y}"'] & H_*(N_T Y/S,\langle T_f \rangle)
\end{tikzcd}$$

In fact, the commutativity of (1) (where we have denoted the canonical functions of $D_Z X$ and $D_T Y$ by the same letter t) is obvious, and (2) follows from Proposition 2.2.9. □

Lemma 3.2.10. *With notation as above, one has the formula*

$$\eta_g . \eta_i = \eta_k . \eta_f$$

in the group $H_0(T/X, \langle L_{T/X} \rangle)$.

Proof. Consider the following diagram:

$$\begin{tikzcd}
H_*(X/X,*) \arrow[r, "i^!"] \arrow[d, "f^!"] & H_*(Z/X,*-\langle N_Z X \rangle) \arrow[r, "p^!_{N_Z X/Z}"] \arrow[d, "g^!"] & H_*(N_Z X/X,*) \arrow[d, "N_g(f)^!"] \\
H_*(Y/X,*+\langle T_f \rangle) \arrow[r, "k^!"] & H_*(T/X,*+\langle L_{T/X} \rangle) \arrow[r, "p^!_{N_T Y/T}"] & H_*(N_T Y/X,\langle T_f \rangle)
\end{tikzcd}$$

The right-hand square commutes by the associativity formula for Gysin morphisms associated with smooth morphisms (Example 2.2.12). Furthermore, the horizontal arrows $p^!_{N_Z X/Z}$ and $p^!_{N_T Y/T}$ are invertible (Lemma 2.2.6), so it suffices to show that the composite square commutes. But the upper and lower composites are the respective specialization maps $\sigma_{Z/X}$ and $\sigma_{T/Y}$, so we conclude by Lemma 3.2.9. □

3.2.11. Second we consider a commutative diagram of schemes:

$$\begin{tikzcd}
Z \arrow[rr, "i"] \arrow[dr, "q"'] & & X \arrow[dl, "p"] \\
& S &
\end{tikzcd}$$

such that i is a closed immersion and p, q are smooth. In this situation, the canonical exact sequence of vector bundles over Z

$$0 \to T_q \to T_p|_Z \to N_Z X \to 0$$

gives rise to an identification of virtual vector bundles

$$\langle T_p|_Z \rangle - \langle N_Z X \rangle \simeq \langle T_q \rangle. \tag{3.2.11.a}$$

in $\underline{K}(Z)$.

Lemma 3.2.12. *With notation as above, one has the formula*

$$\eta_p.\eta_i = \eta_q$$

in the group $H_0(Z/S, \langle T_q \rangle)$.

Proof. Consider the cartesian square Δ

$$\begin{tikzcd}
D_Z X \arrow[d] & N_Z X \arrow[l, "k"'] \arrow[d, "\pi"] \\
\mathbb{A}^1_S & S \arrow[l, "s"']
\end{tikzcd}$$

where s is the zero section and π is the composite map $N_Z X \xrightarrow{p_N} Z \xrightarrow{q} S$. The claim follows from the commutativity of the following diagram:

$$\begin{tikzcd}
H_0(S/S,0) \arrow[r, "\gamma_t"] \arrow[d, "p^!"'] \arrow[dr, phantom, "(1)"] & H_1(\mathbb{G}_m S/S,0) \arrow[r, "\partial_s"] \arrow[d, dashed, "(1\times p)^!"] \arrow[dr, phantom, "(2)"] & H_0(S/S,0) \arrow[r, equal] \arrow[d, dashed, "\pi^!"] \arrow[dr, phantom, "(3)"] & H_0(S/S,0) \arrow[d, "q^!"] \\
H_0(X/S,T_p) \arrow[r, "\gamma_t"'] & H_1(\mathbb{G}_m X/S,T_p) \arrow[r, "\partial_{N_T Y/D_T Y}"'] & H_0(N_Z X/s,T_p) & H_0(Z/s,e). \arrow[l, "p_N^!"]
\end{tikzcd}$$

by considering the image of $1 \in H_0(S/S,0)$ (recall that we have $\partial_s \circ \gamma_t = 1$ by construction, see Paragraph 3.2.1).

The commutativity of square (1) is obvious, that of (2) follows from Proposition 2.2.9 applied to the cartesian square Δ, and that of (3) follows from the associativity of Gysin morphisms associated with smooth morphisms (Example 2.2.12). □

Before going to the third lemma, it is worth to draw out the following corollaries of the previous lemma.

Corollary 3.2.13. *Consider the assumptions of the previous lemma. Then the orientation* η_i *is universally strong. In other words the map* $\eta_i : \mathrm{Th}_Z(-N_i) \to i^!(\mathbb{S}_X)$ *is an isomorphism.*

This simply follows by going back to the definition of the product on bivariant $\mathbb{A}^1$-theory and from the fact the maps η_p and η_q are isomorphims (Example 2.2.3).

Corollary 3.2.14. *Let $p : X \to S$ be a smooth morphism and $i : S \to X$ a section of p. Then one gets the relation: $\eta_i.\eta_p = 1_S$, up to the identification $i^{-1}T_p = N_i$.*

Example 3.2.15. Let X be an S-scheme. Consider a vector bundle $p : E \to X$, $s_0 : X \to E$ its zero section and e a virtual vector bundle over X. Then the associated Gysin map:

$$s^! : H_n(E/S, e) \to H_*(X/S, e - E)$$

is precisely the Thom isomorphism (Definition 2.2.7). In more classical terms, we also get the following tautological Thom isomorphism:

$$H^n(X, e) = H^n(E, e) = H_{-n}(E/E, -e) \xrightarrow{s^!} H_{-n}(X/E, -e - E) = H^n_X(E, e + E).$$

We now proceed towards the formulation and proof of the associativity formula for regular closed immersions.

3.2.16. Consider regular closed immersions:

$$Z \xrightarrow{k} Y \xrightarrow{i} X.$$

Recall that there is a short exact sequence

$$0 \to N_Z Y \to N_Z X \to N_Y X|_Z \to 0$$

of vector bundles over Z, whence an identification $\langle N_Z X\rangle \simeq \langle N_Z Y\rangle + \langle N_Y X|_Z\rangle$ in $\underline{K}(Z)$. There is also a canonical isomorphism of vector bundles

$$N(N_Z X, N_Z Y) \simeq N(N_Y X, N_Y X|_Z)$$

over Z; we will abuse notation and write N for both.

As in the classical case, the key lemma for the associativity formula relies on the double deformation space whose formula is similar to that of the normal bundle N just considered:

$$D = D(D_Z X, D_Z X|_Y).$$

We also refer the reader to [Ros96, §10] for this space. Note this is a scheme over X with a canonical flat map to $\mathbb{A}^2$; we write s and t for the first and second coordinates, respectively. We put:

$$D_1 = D|_{\{0\}\times\mathbb{A}^1}, D_2 = D|_{\mathbb{A}^1\times\{0\}}, D_0 = D|_{\{0\}\times\{0\}}.$$

and recall the following table computing the various pullbacks of D over some subscheme of $\mathbb{A}^2$:

$\{0\} \times \mathbb{A}^1$	$\mathbb{A}^1 \times \{0\}$	$\mathbb{G}_m \times \mathbb{A}^1$	$\mathbb{A}^1 \times \mathbb{G}_m$
$D(N_Z X, N_Z Y)$	$D(N_Y X, N_Y X\|_Z)$	$D - D_1 = \mathbb{G}_m \times D_Z X$	$D - D_2 = D_Y X \times \mathbb{G}_m$

$\mathbb{G}_m \times \mathbb{G}_m$	$\mathbb{G}_m \times \{0\}$	$\{0\} \times \mathbb{G}_m$	$\{0\} \times \{0\}$
$X \times \mathbb{G}_m \times \mathbb{G}_m$	$D_2 - D_0 = \mathbb{G}_m \times N_Z X$	$D_1 - D_0 = N_Y X \times \mathbb{G}_m$	N

We first demonstrate a formula for the specialization maps.

Lemma 3.2.17. *Consider the above assumptions and notations. Then given any virtual bundle e over X, the following diagram is commutative:*

$$\begin{tikzcd}
H_*(X/X,e) \arrow[r, "\sigma_{Y/X}"] \arrow[d, "\sigma_{Z/X}"'] & H_*(N_YX/X,e) \arrow[d, "\sigma_{N_YX|_Z/N_YX}"] \\
H_*(N_ZX/X,e) \arrow[r, "\sigma_{N_ZY/N_ZX}"] & H_*(N/X,e)
\end{tikzcd}$$

Proof. By construction of the specialization maps, the square in question factors as in the following diagram:

$$\begin{tikzcd}
H_*(X/X,e) \arrow[r, "\gamma_s"] \arrow[d, "\gamma_t"'] \arrow[dr, phantom, "(1)"] & H_*(X\mathbb{G}_m^s/X,e) \arrow[r, "\partial_{N_YX/D_YX}"] \arrow[d, "\gamma_t"] \arrow[dr, phantom, "(2)"] & H_*(N_YX/X,e) \arrow[d, "\gamma_t"] \\
H_*(X\mathbb{G}_m^t/X,e) \arrow[r, "\gamma_s"] \arrow[d, "\gamma_t"'] \arrow[dr, phantom, "(3)"] & H_*(X\mathbb{G}_m^s\mathbb{G}_m^t/X,e) \arrow[r, "\partial_{N_YX\mathbb{G}_m^t/D_YX\mathbb{G}_m^t}"] \arrow[d, "\partial_{\mathbb{G}_m^sN_ZX/\mathbb{G}_m^sD_ZX}"] \arrow[dr, phantom, "(4)"] & H_*(N_YX\mathbb{G}_m^t/X,e) \arrow[d, "\partial_{N/D_1}"] \\
H_*(N_ZX/X,e) \arrow[r, "\gamma_s"'] & H_*(\mathbb{G}_m^sN_ZX/X,e) \arrow[r, "\partial_{N/D_2}"'] & H_*(N/X,e)
\end{tikzcd}$$

Some remarks on the notation are in order. First of all we have omitted the symbol × in the diagram. We have also used exponents s and t to indicate that $\mathbb{G}_m^s$, resp. $\mathbb{G}_m^t$, is viewed as a subset of the s-axis, resp. t-axis, in $\mathbb{A}^2$. Finally, we have written γ_u for multiplication with the class $\sigma_\pi \in H_1(\mathbb{G}_m^u/\mathbb{Z}, 0)$ with $u = s, t$.

Now observe that squares (2) and (3) commute by Proposition 2.1.12. Square (1) is *anti*-commutative due to the relation: $\gamma_t.\gamma_s = -\gamma_s.\gamma_t$ (see Paragraph 2.1.13). Applying Proposition 2.1.11 to the commutative square

$$\begin{tikzcd}
D_0 \arrow[r] \arrow[d] & D_1 \arrow[d] \\
D_2 \arrow[r] & D
\end{tikzcd}$$

we deduce that square (4) is also anti-commutative, whence the claim. □

Theorem 3.2.18. *The orientations η_i for a regular closed immersion $i : Z \to X$ (Definition 3.2.3) form a system of fundamental classes. In other words, given regular closed immersions $Z \xrightarrow{k} Y \xrightarrow{i} X$, the associativity formula holds: $\eta_k.\eta_i = \eta_{ik}$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
H(X,0) \arrow[r, "\sigma_{Y/X}"'] \arrow[d, "\sigma_{Z/X}"] \arrow[rr, bend left, "i^!"] \arrow[dd, bend right=60, "(ik)^!"'] \arrow[rd, phantom, "(1)"] & H(N_Y X,0) \arrow[d, "\sigma_{N_Y X|_Z/N_Y X}"'] \arrow[rd, phantom, "(2)"] & H(Y,-N_Y X) \arrow[l, "p^!_{N_Y X/Y}"] \arrow[d, "\sigma_{Z/Y}"] \arrow[dd, bend left=60, "k^!"] \\
H(N_Z X,0) \arrow[r, "\sigma_{N_Z Y/N_Z X}"] \arrow[rd, phantom, "(3)"] & H(N,0) \arrow[rd, phantom, "(4)"] & H(N_Y X|_Z,-N) \arrow[l, "p^!_{N/N_Y X|_Z}"'] \\
H(Z,-N_Z X) \arrow[r, "p^!_{N_Z Y/Z}"'] \arrow[u, "p^!_{N_Z X/Z}"'] & H(N_Z Y,-N) \arrow[u, "p^!_{N/N_Z Y}"] & H(Z,-N_Z Y-N_Y X|_Z). \arrow[l, "p^!_{N_Z Y/Z}"] \arrow[u, "p^!_{N_Y X|_Z/Z}"]
\end{tikzcd}$$

where we have put $H(-,-)=H_*(-/X,-)$. Since the maps $p^!_{N_Z Y/Z}$ and $p^!_{N/N_Z Y}$ are invertible (Lemma 2.2.6), it suffices to show that each square commutes:

(1) Apply Lemma 3.2.17.

(2) Apply Lemma 3.2.9 to the cartesian square:

$$\begin{tikzcd}
N_Y X|_Z \arrow[r] \arrow[d, "p_{N_Y X|_Z/Z}"'] & N_Y X \arrow[d, "p_{N_Y X/Y}"] \\
Z \arrow[r, "k"] & Y
\end{tikzcd}$$

(3) This square factors into two triangles:

$$\begin{tikzcd}
H(N_Z X,0) \arrow[r, "\sigma_{N_Z Y/N_Z X}"] \arrow[rd, "N_Z(i)^!"] & H(N,0) \\
H(Z,-N_Z X) \arrow[u, "p^!_{N_Z X/Z}"] \arrow[r, "p^!_{N_Z Y/Z}"'] & H(N_Z Y,-N) \arrow[u, "p^!_{N/N_Z Y}"']
\end{tikzcd}$$

The upper-right triangle commutes by construction of $(N_Z(i))^!$, the Gysin map associated to $N_Z(i) : N_Z Y \to N_Z X$. The lower-left triangle commutes by Lemma 3.2.12 applied to the commutative diagram:

$$\begin{tikzcd}
N_Z Y \arrow[rr, hook] \arrow[rd, "p_{N_Z Y/Z}"'] & & N_Z X. \arrow[ld, "p_{N_Z X/Z}"] \\
& Z &
\end{tikzcd}$$

(4) Apply the associativity of Gysin morphisms associated with smooth morphisms (Example 2.2.12).

□

3.3. **Fundamental classes: general case.** Recall that a system of fundamental classes is roughly a family of orientations, relative to the bivariant $\mathbb{A}^1$-theory, that satisfies the associativity formula: see Definition 2.2.10.

Recall also that we have defined the fundamental class in the cases of smooth s-morphisms (Example 2.2.3) and regular closed immersions (Definition 3.2.3). The following lemma states the compatibility between these two definitions:

Lemma 3.3.1. *Let $i : X \to Y$ be a regular closed immersion and $f : Y \to S$ be a smooth morphism such that the composition $i' = f \circ i : X \to S$ is a regular closed immersion. Then the following associativity formula holds: $\eta_i.\eta_f = \eta_{i'}$.*

Proof. We have the following commutative diagram

$$\begin{tikzcd}
 & X \arrow[r, "i'"] & S \\
X \arrow[ur, equal] \arrow[r, "\Gamma_i"] \arrow[rr, bend right, "i"'] & X \times_S Y \arrow[u, "p_1"'] \arrow[r, "p_2"] & Y \arrow[u, "f"]
\end{tikzcd}$$

where $\Gamma_i : X \to X \times_S Y$ is the graph of i, $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ are the canonical projections and the square is Cartesian. By Lemma 3.2.10, Corollary 3.2.14 and Theorem 3.2.18 we have

$$\eta_i.\eta_f = \eta_{\Gamma_i}.\eta_{p_2}.\eta_f = \eta_{\Gamma_i}.\eta_{p_1}.\eta_{i'} = \eta_{i'},$$

and the result follows. ☐

We are now ready to state the main theorem, extending fundamental classes to the case of quasi-projective lci morphisms:

Theorem 3.3.2. *There exists a unique system of fundamental classes $\eta_f \in H_0(X/S, \langle L_f \rangle)$ associated with the class of quasi-projective lci morphisms f such that:*

1. *For any smooth s-morphism f, the class η_f agrees with the fundamental class defined in Example 2.2.3;*
2. *For any regular closed immersion $i : Z \to X$, the class η_i agrees with the fundamental class defined in Definition 3.2.3.*

Furthermore, we can replace the condition (1) *by the following weaker version:*

- (1') *For the following smooth s-morphisms f, the class η_f agrees with the fundamental class defined in Example 2.2.3:*
  - (a) *open immersions;*
  - (b) *the projection of a projective space $\mathbb{P}^n_S \to S$ for some S.*

Proof. Let $f : X \to S$ be a quasi-projective lci morphism factored as $X \xrightarrow{i} Y \xrightarrow{p} S$ where i is a regular closed immersion and p is a smooth morphism. We define the class $\eta_f = \eta_i.\eta_p$ as an element of $H_0(X/S, \langle L_f \rangle)$. This class does not depend on the factorization chosen: indeed, if $X \xrightarrow{i'} Y' \xrightarrow{p'} S$ is another such factorization, by comparing both factorizations with the diagonal

$$\begin{tikzcd}
 & Y \arrow[r, "p"] & S \\
X \arrow[ur, "i"] \arrow[r, "{(i,i')}"] \arrow[rr, bend right, "i'"'] & Y \times_S Y' \arrow[u, "p_1"'] \arrow[r, "p_2"] & Y', \arrow[u, "p'"]
\end{tikzcd}$$

we conclude by applying Lemma 3.3.1.

To show that the classes η_f form a system of fundamental classes, it remains to show the associativity formula with respect to compositions of quasi-projective lci morphisms. If $f : X \to Y$ and $g : Y \to Z$ are two composable quasi-projective lci morphisms, there exists a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "i_1"] \arrow[dr, "f"'] & P \arrow[r, "i_3"] \arrow[d, "p_1"] & R \arrow[d, "p_3"] \\
& Y \arrow[r, "i_2"] \arrow[dr, "g"'] & Q \arrow[d, "p_2"] \\
& & Z,
\end{tikzcd}$$

where all i_k's are closed immersions and p_k's are smooth morphisms, and the square is cartesian. By Example 2.2.12, Lemma 3.2.10 and Theorem 3.2.18 we have

$$\eta_f.\eta_g = \eta_{i_1}.\eta_{p_1}.\eta_{i_2}.\eta_{p_2} = \eta_{i_1}.\eta_{i_3}.\eta_{p_3}.\eta_{p_2} = \eta_{i_3 \circ i_1}.\eta_{p_2 \circ p_3} = \eta_{g \circ f},$$

which gives the desired associativity formula. The uniqueness statement is clear from the construction.

For (1'), since every quasi-projective lci morphism factors as a regular closed immersion into an open subscheme of a projective space, a system of fundamental classes is uniquely determined by its values in these cases, and the result follows. ☐

3.3.3. We also have an excess intersection formula in this case. Consider a cartesian square

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & T \arrow[d, "p"] \\
X \arrow[r, "f"'] & S
\end{tikzcd}$$

where f and g are lci morphisms. Factor $f = \phi \circ i$ as a closed immersion followed by a smooth morphism and consider the Cartesian diagram

$$\begin{tikzcd}
Y \arrow[r, "k"] \arrow[d, "q"'] & Q \arrow[r, "\psi"] \arrow[d, "r"'] & T \arrow[d, "p"] \\
X \arrow[r, "i"] & P \arrow[r, "\phi"] & S
\end{tikzcd}$$

where k and i are regular closed immersions and ψ and ϕ are smooth. By 3.2.5 there is a canonical monomorphism of Y-vector bundles $N_Y Q \xrightarrow{\nu} q^{-1} N_X P$. We let ξ be the quotient bundle.

Proposition 3.3.4. *Consider the above notations and assumptions. Then the following formula holds in* $H_0(Y/T, \langle q^* L_f \rangle)$*:*

$$\Delta^*(\eta_f) = e(\xi).\eta_g$$

where we have considered, as in Proposition 3.2.6, the action of the Euler $e(\xi) \in \pi^0(Y, \xi)$ *on (Definition 3.1.2) via the canonical map* $\pi_0(Y, \xi) \to H^0(Y, \langle \xi \rangle) \simeq H_0(Y/Y, -\langle \xi \rangle)$ *and the product on bivariant* $\mathbb{A}^1$*-theory, using the identification* $\langle q^* L_f \rangle = \langle L_g \rangle - \langle \xi \rangle$*.*

The proof is omitted here, which reduces to that of Proposition 3.2.6 using the fact that fundamental classes for smooth morphisms are compatible with any base change (see [Dég17, 2.3.13]).

4. Main results and applications

4.1. Ring spectra and Gysin morphisms.

4.1.1. Let $\mathscr{S}$ be a full subcategory of the category of (qcqs) schemes.[13] As in [Dég14, Dég17], an $\mathscr{S}$-absolute ring spectrum $\mathbb{E}$ is a cartesian section over $\mathscr{S}$ of the category of commutative monoids in $S\mathscr{H}$. In other words, for any scheme S in $\mathscr{S}$, we are given a motivic ring spectrum $\mathbb{E}_S$ over S, and for any morphism $f : T \to S$, a base change isomorphism $\tau_f : f^*(\mathbb{E}_S) \to \mathbb{E}_T$ (satisfying the cocycle condition). A morphism of $\mathscr{S}$-absolute ring spectra $\phi : \mathbb{E} \to \mathbb{F}$ is the data, for all schemes S in $\mathscr{S}$, of morphisms of ring spectra $\phi_S : \mathbb{E}_S \to \mathbb{F}_S$ compatible with the structural base change isomorphisms.

To such an absolute ring spectrum, we associate a *(twisted) bivariant theory* which to any s-morphism $f : X \to S$ in $\mathscr{S}$ and any pair $(n, v) \in \mathbb{Z} \times \underline{K}(X)$ associates the bivariant group defined in 2.1.2 with respect to $\mathbb{E}_S$:

$$\mathbb{E}_n(X/S, v) := [p_! \operatorname{Th}_X(v)[n], \mathbb{E}_S]. \tag{4.1.1.a}$$

The reader is refered to Paragraph 2.1.6 for the properties of this bivariant theory.[14]

Note this construction is obviously natural in the absolute ring spectrum $\mathbb{E}$. In particular, using the unit map of the ring spectrum $\mathbb{E}$, we get a natural transformation of bivariant theories:

$$\rho_{X/S} : H_n(X/S, v) \to \mathbb{E}_n(X/S, v),$$

that we will call the $\mathbb{A}^1$*-regulator map.*

Definition 4.1.2. Consider the above notations. Then for any lci quasi-projective morphism $f : X \to S$ with cotangent complex L_f, we define the $\mathbb{A}^1$*-fundamental class of f with coefficients in* $\mathbb{E}$ as the image $\eta_f^{\mathbb{E}}$ of the class η_f of Theorem 3.3.2 under the $\mathbb{A}^1$-regulator map:

$$\rho_{X/S} : H_n(X/S, \langle L_f \rangle) \to \mathbb{E}_n(X/S, \langle L_f \rangle).$$

Note that this definition is obviously functorial in the absolute ring spectrum $\mathbb{E}$. Moreover, all the formulas of the fundamental classes of the form η_f immediately induce formulas for the classes $\eta_f^{\mathbb{E}}$. We only draw here the corollaries obtained for the induced Gysin morphisms

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[13] In our main examples, $\mathscr{S}$ will be either the category of all schemes, or the category of schemes over a given field.

[14] Note also that one has functorial properties with respect to the target of f. For example, given an tale morphism $p : S \to T$, one gets an isomorphism:

$$\mathbb{E}_n(X/S, v) \xrightarrow{\sim} \mathbb{E}_n(X/T, v)$$

using the fact $p^! = p^*$.

Proposition 4.1.3. *Consider the assumptions of the previous definitions. Then for any lci quasi-projective morphism of s-morphisms over S, with cotangent complex L_f, for any pair $(n,v) \in \mathbb{Z} \times \underline{K}(X)$, there exists a Gysin morphism:*

$$f^! : \mathbb{E}_n(X/S,v) \to \mathbb{E}_n(Y/S, f^*v + \langle L_f \rangle), \quad x \mapsto \eta_f^{\mathbb{E}}.x$$

using the product of the bivariant theory $\mathbb{E}_*(-,*)$. These Gysin morphisms satisfy the following formulas:

(i) Functoriality*: for composable morphisms f and g, one has $(fg)^! = g^! f^!$.*

(ii) Base change*: Consider a cartesian square*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & T \arrow[d, "p"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

such that f is quasi-projective lci and transverse to p. Then the following formula holds:

$$f^! p_* = q_* g^!.$$

Everything follows from Theorem 3.3.2 and Proposition 3.3.4 (applied to the square Δ).

4.1.4. *Oriented ring spectra.*– Suppose the absolute ring spectrum $\mathbb{E}$ admits an (global) orientation[15] c. Then our construction gives back the constructions of [Dég17] as follows.

Let v be a virtual vector bundle over a scheme X of virtual rank r. According to [Dég17, Rem. 2.2.22], one gets the so-called *Thom isomorphism*[16] canonically determined by the orientation c:

$$\tau_v^c : \mathbb{E}_*(X/S,v) \simeq \mathbb{E}_*(X/S,\langle r \rangle) \tag{4.1.4.a}$$

where $\langle n \rangle$ denotes the trivial virtual vector bundle of rank n. Note that this isomorphism is in fact functorial with respect to $v \in \underline{K}(X)$ and compatible with pullbacks and with the addition law of the Picard category $\underline{K}(X)$ (see in particular [Dég17, 2.2.17]).

Then for any lci quasi-projective morphism $f : Y \to X$ in $\mathscr{S}$ with cotangent complex L_f and virtual dimension $d_f = \chi(L_f)$, the fundamental class $\eta_f \in \mathbb{E}_*(X/S, \langle L_f \rangle)$ gives a fundamental class:

$$\eta_f^c = \tau_{L_f}^c(\eta_f) \in \mathbb{E}_*(Y/X, \langle d_f \rangle)$$

which agree with that defined in [Dég17, 2.5.3].[17] Note these classes form a system of fundamental classes as in Definition 2.2.10.

---

[15] *i.e.* for any scheme S in $\mathscr{S}$, the ring spectrum $\mathbb{E}_S$ admits an orientation c_S and the base change isomorphism preserves these orientations; see [Dég14, Dég17]

[16] It is interesting to note that the data of Thom isomorphisms are in fact equivalent to the data of a system of fundamental classes τ_p for the category where morphisms are given by the projection $p : E \to X$ of vector bundles, where the twisting virtual bundle e_p is the rank of p, such that τ_p is strong — in fact the associated purity transformation $\mathfrak{p}_{\tau_p}$ evaluated at $\mathbb{E}$ is an isomorphism.

[17] One reduces to the case of regular closed immersion and smooth morphisms. The case of smooth morphisms is obvious (which reduces to reduces to the six functors formalism). For the case of regular closed immersions, using the deformation to the normal cone and the compatibility of the two fundamental

When X/S is an s-scheme, the induced Gysin morphism

$$f_c^! : \mathbb{E}_n(X/S, \langle r \rangle) \to \mathbb{E}_n(Y/S, \langle r + d_f \rangle), x \mapsto \eta_f.x$$

is determined by the Gysin morphism of the previous Proposition according to the following commutative diagram:

$$\begin{tikzcd}
\mathbb{E}_n(X/S, v) \arrow[r, "f^!"] \arrow[d, "\tau_v^c"', "\sim"] & \mathbb{E}_n(Y/S, f^*v + \langle L_f \rangle) \arrow[d, "\tau^c_{f^*v+L_f}", "\sim"'] \\
\mathbb{E}_n(X/S, \langle r \rangle) \arrow[r, "f_c^!"] & \mathbb{E}_n(Y/S, \langle r + d_f \rangle).
\end{tikzcd}$$

Note finally that we deduce from that commutative diagram a very neat proof of the Grothendieck-Riemann-Roch formula (cf. [Dég17, 3.2.6 and 3.3.10] for the formulation in $\mathbb{A}^1$-homotopy). Indeed, it boils down to the definition of the Todd class (cf. [Dég17, 3.2.4 and 3.3.5]) and compare to *loc. cit.* we do not need to consider any factorization of f.[18]

4.1.5. The natural cohomology theory associated with the bivariant theory $\mathbb{E}_*(-,*)$ is the cohomology represented by $\mathbb{E}$:

$$\mathbb{E}^n(X, v) := \mathbb{E}_{-n}(X/X, -v) = [\mathbb{S}_X, \mathbb{E}_S \otimes \mathrm{Th}_X(v)[n]].$$

We also get an unstable $\mathbb{A}^1$-regulator map:

$$\pi^n(X, E) = [\mathbb{S}_X, \mathbb{S}_X[n] \wedge \mathrm{Th}(E)] \xrightarrow{\rho_{X/S} \circ \Sigma^\infty} \mathbb{E}^n(X, v)$$

Definition 4.1.6. Consider the above notations. Then for any vector bundle E/X, one defines the *Euler class of E/X with coefficients in* $\mathbb{E}$ as the image $e(E, \mathbb{E}) \in \mathbb{E}^0(X, \langle E \rangle)$ of the class $e(E)$ of Definition 3.1.2 under the unstable $\mathbb{A}^1$-regulator map.

4.1.7. Using the previous definition, the base change formula can be extended to an excess intersection formula. We consider a cartesian square: Consider a cartesian square

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & T \arrow[d, "p"] \\
X \arrow[r, "f"'] & S
\end{tikzcd}$$

such that f and g are lci quasi-projective, and we let ξ be the excess bundle as in Paragraph 3.3.3.

Proposition 4.1.8 (Excess intersection)**.** *Consider the above assumptions, and the notations of Definitions 4.1.2 and 4.1.6. Then the following relations hold:*

$$\Delta^*(\eta_f) = e(\xi, \mathbb{E}).\eta_g$$

in $\mathbb{E}_0(Y/T, \langle L_f \rangle)$*, and*

$$f^! p_*(x) = q_*(e(\xi, \mathbb{E}).g^!(x))$$

---

classes with transverse base change, we reduce to the case of the zero section of a vector bundle. This case follows because both fundamental classes gives the (refined) Thom class of the vector bundle, by design.

[18] For the record, Grothendieck mentioned that such a direct proof of his formula, without going through a factorisation and the use of a blow-up, should exist.

in $\mathbb{E}_*(X/S, * + \langle L_f \rangle)$*, for any* $x \in \mathbb{E}_*(T/S, *)$.

This is a direct corollary of Proposition 3.3.4.

Applying Proposition 4.1.8 to the cartesian square

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\Delta"] & Z \arrow[d] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

we obtain the following:

Corollary 4.1.9. *Let* $i : Z \to X$ *be a regular closed immersion of s-schemes over* S*. Then we have the following relations:*

$$\Delta^*(\eta_i) = e(N_Z X, \mathbb{E})$$

in $\mathbb{E}_0(Z/Z, -\langle N_Z X \rangle)$*, and*

$$i^! i_*(x) = e(N_Z X, \mathbb{E}).x$$

in $\mathbb{E}_*(Z/S, * - \langle N_Z X \rangle)$*, for any* $x \in \mathbb{E}_*(Z/S, *)$.

Remark 4.1.10.

1. We do not state this in a separate proposition, but we also get Gysin morphism in (twisted) cohomology. Given an absolute ring spectrum $\mathbb{E}$ as previously, and considering a quasi-projective lci morphism $f : Y \to X$, with cotangent complex L_f, one gets for any pair $(n, v) \in \mathbb{Z} \times \underline{K}(X)$, a Gysin morphism:
$$f_! : \mathbb{E}_n(Y, f^* v - \langle L_f \rangle) \to \mathbb{E}_n(X, v), y \mapsto f_*(y.\eta_f^{\mathbb{E}}),$$
where we have used the product and the direct image of the bivariant theory $\mathbb{E}_*(-/S, *)$. Moreover, the functoriality, base change and excess intersection formulas formally follow from Theorem 3.3.2 and Proposition 3.3.4.
2. Applying Corollary 4.1.9 to the zero section $s : X \to E$ of a vector bundle, we obtain the following formula to compute Euler classes:
$$e(E, \mathbb{E}) = s^* s_!(1) \tag{4.1.10.a}$$
in $\mathbb{E}^0(X, \langle E \rangle)$.
3. When the ring spectrum $\mathbb{E}$ is oriented (Paragraph 4.1.4), we get back the usual Gysin morphism in cohomology ([Nav16]), for projective lci morphisms, after taking care of the Thom isomorphism in cohomology (*i.e.* isomorphism 4.1.4.a with $X = S$).
4. We do not need the ring structure on the absolute ring spectrum $\mathbb{E}$ to get Gysin morphisms. Indeed, if $\mathbb{E}$ is an $\mathscr{S}$-absolute spectrum (*i.e.* a cartesian section of the fibered category $S\mathscr{H}$ over $\mathscr{S}$), one can still define its bivariant theory using formula (4.1.1.a). This theory has no product, but still satisfy the basic functoriality described in Paragraph 2.1.6. Instead of having the $\mathbb{A}^1$-regulator map, we get a canonical action:
$$H_n(Y/X, w) \otimes \mathbb{E}_m(X/S, v) \to \mathbb{E}_{m+n}(Y/S, w + q^* v)$$

using the formula for products of Paragraph 2.1.6 but replacing the product of μ by the obvious map $\mathbb{S}_X \otimes \mathbb{E} \to \mathbb{E}$.[19] This allows to get Gysin morphisms both for the previous bivariant theory, but also for the cohomology (without product) represented by $\mathbb{E}$. Again, the functoriality, base change and excess intersection formulas extend to that case.

Example 4.1.11. Let $\mathscr{S}$ be the category of all schemes.
The main new examples are given by either the absolute ring spectrum corresponding to the sphere spectrum (the theory used in the previous sections), or the ring spectrum $\mathsf{N}R$ representing $\mathbb{A}^1$-homology with coefficients in any ring R. That is for any scheme S, we consider the adjunction:

$$K : S\mathscr{H}(S) \to \mathrm{D}_{\mathbb{A}^1}(S, R) : N$$

induced by the Dold-Kan correspondance (see for example [CD12, 5.3.35]). Then we put:

$$\mathsf{N}R_S = N(R_S).$$

One checks that this in fact defines an absolute ring spectrum.

In particular, we obtain Gysin morphisms for the associated bivariant theories and cohomologies. Note that if we restrict to cohomology and to finite lci morphisms, this gives a very general notion of transfer maps extending the definitions of Morel in [Mor12].[20]

Example 4.1.12. (1) *Hermitian K-theory.* Let $\mathscr{S}$ be the category of $\mathbb{Z}[1/2]$-schemes. According to [PW10], for any regular scheme S in $\mathscr{S}$, there exists a ring spectrum $\mathbf{BO}_S$ over S that represents Hermitian K-theory of smooth S-schemes. According to the geometric model of $\mathbf{BO}$ (denoted by $\mathbf{BO}^{geom}$ in *loc.cit.*, these ring spectra are compatible with base change. Therefore, we can view $\mathbf{BO}_{\mathbb{Z}[1/2]}$ as an absolute ring spectrum that we denote simply $\mathbf{BO}$ with the property that the cohomology represented by $\mathbf{BO}$ over a regular scheme S in $\mathscr{S}$ is Hermitian K-theory.[21]

Note that for non regular scheme, $\mathbf{BO}$-cohomology is a homotopy invariant version of hermitian K-theory, on the model of [Cis13], though this notion has not been introduced and worked out as far as we know.

The twisted bivariant theory associated with $\mathbf{BO}$ as above is new. The Gysin morphisms that one gets on $\mathbf{BO}$-cohomology are also new, at least in the generality of arbitrary projective lci morphisms, between arbitrary schemes (eventually singular and without a base field). In the case of regular schemes, our construction for some part of Hermitian K-theory (namely that which compare to Balmer's higher Witt groups) should be compared to that of [CH11]. This would require a similar discussion to that of Paragraph 4.1.4 as, according to Panin and Walter, hermitian K-theory has a special kind of orientation which allows to consider only twists by line bundles (see also the next example).

We expect to come back to these questions in a future work.

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[19] See also [Dég17, 1.2.1].

[20] Morel defines transfer only for finite field extensions, but he works unstably.

[21] The paper of Panin and Walter is not yet published. If one agrees to restrict to k-schemes for a field k of characteristic different from 2, one can take the ring spectrum constructed in [Hor05].

(2) *Milnor-Witt motivic cohomology.* Introduced by Barge and Morel, the theory of Chow-Witt groups was fully developed by Fasel ([Fas07, Fas08]). More recently, the theory was extended to "higher Chow-Witt groups", in a series of work [CF14, DF17a, DF17b]. In particular, given any coefficient ring R, there exists a canonical ring spectrum $\mathbf{H}_{\mathrm{MW}}$ in $S\mathscr{H}(k)$, called the *Milnor-Witt ring spectrum* (cf. [DF17b, 3.1.2]). We view it as an $\mathscr{S}$-absolute ring spectrum and denote by $H^{MW}_*(X/S,v)$ (resp. $H^*_{MW}(X,v)$) its associated bivariant theory (resp. cohomology). This definition coincides with that of [DF17b].

Then it follows from [DF17a, 4.2.6, 4.2.7] that for any smooth k-scheme X and any virtual bundle v over X of virtual rank r over X, one has a canonical isomorphism, contravariantly functorial in X and covariantly functorial in v.

$$H^0_{MW}(X,v) \simeq \widetilde{\mathrm{CH}}^r(X,\det(v)).$$

Note in particular that one deduces that the ring spectrum $\mathbf{H}_{\mathrm{MW}}$ is symplectically oriented in the sense of Panin and Walter [PW10]. Note also that we prove that, when X is singular but separated of finite type over k, the bivariant theory $H^{MW}_0(X/k,v)$ can be computed by a Gersten complex with coefficients in the Milnor-Witt ring of the residue fields, so we can put:

$$H^{MW}_0(X/k,v) = \widetilde{\mathrm{CH}}_r(X,\det(v))$$

and view this as the Chow-Witt group of the scheme X. Similarly, the groups $H^{MW}_i(X/k,v)$ for $i \geq 0$ can be viewed as the higher Chow-Witt groups. In fact, we have canonical maps:

$$\varphi_X : H^{MW}_i(X/k,v) \to \mathrm{CH}_n(X,i)$$

where $n = \chi(v)$ is the rank of the virtual bundle v, which is functorial in X with respect to proper pushforward (resp. pullback along open immersions).

Thus the construction of the present paper gives Gysin morphisms on these higher Chow-Witt groups, for any lci quasi-projective morphisms. Besides, by construction, the map $\varphi_?$ is compatible with Gysin morphisms. And finally, we get a (twisted) bivariant higher Chow-Witt groups.

4.2. **Purity, traces and duality.**

4.2.1. Let $\mathscr{S}$ be a subcategory of the category of all schemes stable under base change along morphisms of finite type and quasi-projective extensions. Let $\mathscr{T}$ be a motivic triangulated category over $\mathscr{S}$ (cf. [CD12, 2.4.45]) and consider a premotivic adjunction:

$$\varphi^* : S\mathscr{H} \leftrightarrows \mathscr{T} : \varphi_*.$$

Then we can define a twisted bivariant theory associated with $\mathscr{T}$, using the analogue of the formula of Definition 2.1.2. For any s-morphism $p : X \to S$ and any pair $(n,v) \in \mathbb{Z} \times \underline{K}(X)$, we put:

$$H_n(X/S,v,\mathscr{T}) := \mathrm{Hom}_{\mathscr{T}(S)}(\mathrm{Th}_X(v,\mathscr{T})[n], p^!(\mathbb{1}_S)] = [p_! \,\mathrm{Th}_X(v,\mathscr{T})[n], \mathbb{1}_S]$$

where we have put: $\mathrm{Th}_X(v,\mathscr{T}) = \varphi^*(\mathrm{Th}_X(v))$. This bivariant theory formally satisfies all the properties stated in Paragraph 2.1.6. Besides, the premotivic adjunction (φ^*,φ_*) induces a natural transformation of twisted bivariant theories:

$$\varphi^* : H_n(X/S,v) \to H_n(X/S,v,\mathscr{T}), \tag{4.2.1.a}$$

which can also be called the $\mathbb{A}^1$-*regulator map*. In particular, the fundamental classes constructed in Theorem 3.3.2 induces fundamental classes in $\mathscr{T}$. For any quasi-projective morphism $f : X \to S$ in $\mathscr{S}$, we get:

$$\eta_f^{\mathscr{T}} : \mathrm{Th}_X(L_f,\mathscr{T}) \to p^!(\mathbb{1}_S).$$

All the properties of the fundamental classes in $S\mathscr{H}$ are formally translated into formulas in for the classes $\eta_f^{\mathscr{T}}$. We are interested here in the purity transformation this class induces in $\mathscr{T}$, according to the formula of Paragraph 2.3.1, applied inside $\mathscr{T}$. This is the natural transformation:

$$\mathfrak{p}_f^{\mathscr{T}} : f^{L*} = f^*(-) \otimes \mathrm{Th}_X(\langle L_f\rangle) \xrightarrow{\mathrm{Id}\otimes\eta_f^{\mathscr{T}}} f^*(-)\otimes f^!(\mathbb{1}_S) \xrightarrow{Ex^{!*}_{\otimes}} f^!(-\otimes \mathbb{1}_S) \simeq f^!. \tag{4.2.1.b}$$

Then we can deduce by adjunction from this purity transformation trace and cotrace maps as in Paragraph 2.3.3:

$$\mathrm{tr}_f : f_!(f^*(-)\otimes \mathrm{Th}_X(L_f)) \to \mathrm{Id}$$
$$\mathrm{cotr}_f : \mathrm{Id} \to f_*(f^!(-)\otimes \mathrm{Th}_X(-L_f)).$$

Example 4.2.2. Consider the previous notations and let $\mathbb{E}$ be a cartesian section of the fibered category $\mathscr{T}$ over $\mathscr{S}$ (that is an $\mathscr{S}$-absolute $\mathscr{T}$-spectrum). Then, on the model of [VSF00, chap. 4, section 9], one associates to $\mathscr{T}$ four theories, for an s-morphism $f : X \to S$ in $\mathscr{S}$ (in the case of cohomology, we take $X = S$), indexed by couple $(n,v) \in \mathbb{Z}\times \underline{K}(X)$:

- *Cohomology.–*

$$\mathbb{E}^n(X,v) := \mathrm{Hom}_{\mathscr{T}(X)}(\mathbb{1}_X, \mathbb{E}_X\otimes \mathrm{Th}_X(v)[n]) = \mathrm{Hom}_{\mathscr{T}(S)}\big(\mathbb{1}_S, f_*(f^*(\mathbb{E}_S)\otimes \mathrm{Th}_X(v)[n])\big).$$

- *Bivariant theory aka Borel-Moore (relative) homology.–*

$$\mathbb{E}_n(X/S,v) := \mathrm{Hom}_{\mathscr{T}(S)}\big(f_!(\mathrm{Th}_X(v,\mathscr{T})[n], \mathbb{E}_S\big) = \mathrm{Hom}_{\mathscr{T}(S)}\big(\mathbb{1}_S, f_*(f^!(\mathbb{E}_S)\otimes \mathrm{Th}_X(-v))[-n]\big).$$

- *Cohomology with proper support.–*

$$\mathbb{E}^n_c(X/S,v) := \mathrm{Hom}_{\mathscr{T}(S)}\big(\mathbb{1}_S, f_!(\mathbb{E}_X\otimes \mathrm{Th}_X(v))[n]\big) = \mathrm{Hom}_{\mathscr{T}(S)}\big(\mathbb{1}_S, f_!(f^!(\mathbb{E}_S)\otimes \mathrm{Th}_X(v))[n]\big).$$

- *Bivariant theory with proper support aka (relative) homology.–*

$$\mathbb{E}^c_n(X/S,v) := \mathrm{Hom}_{\mathscr{T}(S)}\big(\mathbb{1}_S, f_!(f^!(\mathbb{E}_S)\otimes \mathrm{Th}_X(-v))[-n]\big).$$

If then immediately follows that the purity transformation induces Gysin morphisms on each of this four theories: contravariantly (resp. covariantly) for Borel-Moore relative homology (resp. cohomology with compact support) with respect to quasi-projective lci morphisms, covariantly (resp. contravariantly) for cohomology (resp. relative homology) with respect to projective lci morphisms. This obviously coincides with the Gysin morphisms defined in the previous section (through the sole algebraic structure of the bivariant theory associated with $\mathbb{E}$).

These considerations apply in particular to the ring spectra of Examples 4.1.11 and 4.1.12.

Remark 4.2.3. *Orientations.* Consider again the assumptions of the paragraph 4.2.1. Recall we say that $\mathscr{T}$ is oriented ([CD12, 2.4.38, 2.4.40]) if for any scheme S, the cohomology theory on smooth S-schemes associated with $\mathscr{T}$:

$$H^n(X, v, \mathscr{T}) = H_{-n}(X/X, v, \mathscr{T})$$

admits an orientation c_S in the classical sense (which amounts to ask that the ring spectrum $\varphi_*(\mathbb{S}_S)$ is oriented) and these orientations for various S are compatible with base change.

Then in this case, one gets a Thom isomorphism for any virtual bundle v over X of virtual rank r:

$$\mathrm{Th}_X(v) \to \mathbb{1}_X(r)[2r]$$

which is functorial and additive in v. In particular, for any lci quasi-projective morphism f of dimension d, the purity transformation can be written, in a more classical way:

$$\mathfrak{p}_f^c : f^*(-)(d)[2d] \to f^! \tag{4.2.3.a}$$

and similarly for the trace and cotrace maps.

Example 4.2.4. Let $\mathscr{S}$ be the category of $\mathbb{Z}[1/\ell]$-schemes for a prime ℓ and let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$. Then the constructible derived category of étale Λ-sheaves $D^b_c(X_{\text{ét}}, \Lambda)$ on a scheme S in $\mathscr{S}$ defines a motivic triangulated category (see [AGV73], [Eke90], [CD14]). According to [CD14], we have a canonical realisation functor:

$$\rho_\ell^* : S\mathscr{H} \to \mathrm{DM}_h(-, \mathbb{Z}) \to D^b_c(-_{\text{ét}}, \Lambda).$$

In particular, the previous construction defines purity transformations of the form (4.2.3.a) for the motivic triangulated category $D^b_c(-_{\text{ét}}, \Lambda)$, generalizing the previously known constructions.

Definition 4.2.5. Let $f : X \to Y$ a quasi-projective lci morphism in $\mathscr{S}$. We say that $\mathbb{E} \in \mathscr{T}(S)$ is *f-pure* if the canonical morphism

$$(\mathfrak{p}_f^{\mathscr{T}})_{\mathbb{E}} : f^*(\mathbb{E}) \otimes \mathrm{Th}_X(L_f) \to f^!(\mathbb{E})$$

is invertible.

Note in particular that, for $\mathscr{T} = S\mathscr{H}$, the orientation η_f is universally strong (Definition 2.2.1) if and only if $\mathbb{S}_S$ is f-pure.

Remark 4.2.6.
1. If $f : X \to S$ is *smooth*, one can check using part (2) of Theorem 3.3.2 that the morphism $\eta_f^{\mathbb{E}}$ is the purity isomorphism of the motivic category $\mathscr{T}$. In particular, every object $\mathbb{E} \in \mathscr{T}(S)$ is f-pure.
2. Our definition extends similar considerations previously introduced by several authors ([ILO14, XVI, 3.1.5], [BD17, 4.4.2], [Pep15, 1.7]).
3. Given a a quasi-projective lci morphism f in $\mathscr{S}$, the category of f-pure objects satisfies good formal properties: it is stable under direct factors, extensions and tensor products with invertible objects, or more generally strongly dualizable objects.

4.2.7. *Duality.* Consider a cartesian section $\mathbb{E}$ of $\mathscr{T}$ over $\mathscr{S}$ as in Example 4.2.2. Let $f : X \to S$ be a quasi-projective lci morphism. If $\mathbb{E}_S$ is f-pure, then we immediately get duality isomorphisms:

$$\mathbb{E}^n(X, v) \xrightarrow{(\mathfrak{p}_f^{\mathscr{T}})_{\mathbb{E}}} \mathbb{E}_n(X/S, \langle L_f\rangle - v),$$

$$\mathbb{E}^n_c(X/S, v) \xrightarrow{(\mathfrak{p}_f^{\mathscr{T}})_{\mathbb{E}}} \mathbb{E}^c_n(X/S, \langle L_f\rangle - v).$$

The first isomorphism can be expressed in more classical terms as follows. Via the $\mathbb{A}^1$-regulator map (4.2.1.a), we get an action of the bivariant $\mathbb{A}^1$-theory:

$$\mathbb{E}_n(Y/X, v) \otimes H_m(X/S, w) \to \mathbb{E}_{n+m}(Y/S, v + f^*w)$$

as in point (3) of Remark 4.1.10. Then it rightly follows from definition that the first purity isomorphism can be written as the cup-product with the fundamental class $\eta_f \in H_0(X/S, \langle L_f\rangle)$:

$$\mathbb{E}^n(X, v) = \mathbb{E}_{-n}(X/X, -v) \to \mathbb{E}^n(X/S, \langle L_f\rangle - v), x \mapsto x.\eta_f.$$

Proposition 4.2.8. *Consider the assumptions of 4.2.1. Let $f : X \to Y$ be a quasi-projective morphism of S-schemes and assume that one of the following conditions are satisfied:*
(i) *X and Y are smooth over S.*
(ii) *X and Y are regular, and S is the spectrum of a field. The motivic category $\mathscr{T}$ is continuous (in the sense of* [CD12, Def. 4.3.3] *with respect to Tate twists).*
Then the morphism f is lci, and any motivic spectrum $\mathbb{E} \in \mathscr{T}(S)$ is f-pure.

Proof. Since f factors through a closed immersion and a smooth morphism, we may reduce to the case of closed immersions, using the associativity formula and the fact that $\mathfrak{p}_p^{\mathscr{T}}$ is invertible for p smooth. Moreover, in both cases f is automatically a *regular* closed immersion. The second case reduces to the first by using the continuity property of $\mathscr{T}$ together with Popescu's theorem [Swa98]. For the first case, let $p : X \to S$ and $q : Y \to S$ denote the smooth structural morphisms. By construction we have a commutative diagram

$$\begin{tikzcd}
\mathbb{E}_X \otimes \mathrm{Th}_X(L_f) \arrow[r, "\eta_f^{\mathbb{E}}"] \arrow[d, "\mathrm{Id} \otimes \eta_f"'] & f^!(\mathbb{E}_Y) \\
\mathbb{E}_X \otimes f^!(\mathbb{S}_Y) \arrow[ru, "Ex^{!*}_{\otimes}"'] &
\end{tikzcd}$$

where the left-hand vertical arrow is invertible by Lemma 3.2.12 and the fact that η_p is an isomorphism for p smooth (Example 2.2.3). Therefore it suffices to note that the morphism induced by the exchange transformation $Ex^{!*}_{\otimes}$ is invertible. This follows by writing $\mathbb{E}_X = p^*(\mathbb{E})$ and $\mathbb{E}_Y = q^*(\mathbb{E})$, using the purity 2-isomorphisms $\mathfrak{p}_p : p^* \simeq p^!(-) \otimes \mathrm{Th}_X(-T_p)$ and $\mathfrak{p}_q : q^* \simeq q^!(-) \otimes \mathrm{Th}_Y(-T_q)$, and applying Remark 2.1.8 in view of the invertibility of Thom spaces. □

The following definition first appears (as a conjecture) in the context of étale cohomology in [Gro77, I, 3.1.4].

Definition 4.2.9. Let $\mathbb{E}$ be a cartesian section of $\mathscr{T}$ over $\mathscr{S}$. We say that $\mathbb{E}$ satisfies *absolute purity* if if for all quasi-projective lci morphism $f : X \to S$ in $\mathscr{S}$ such that X and S are regular, $\mathbb{E}_S$ is f-pure.

Remark 4.2.10.
(1) According to point (i) of Remark 4.2.6 and given the functoriality property of the purity transformation (Proposition 2.3.5), to check that $\mathbb{E}$ satisfies absolute purity it is sufficient to restrict to the morphism f which are regular closed immersions between regular schemes.
(2) If $\mathscr{S}$ is the category of schemes over some base field and $\mathscr{T}$ is continuous, it follows from 4.2.8 that $\mathbb{E}$ is absolutely pure.

Example 4.2.11. The property of absolute purity in this setting was already introduced in [Dég14] (and later [Dég17]). The above could be regarded as a more precise formulation, though in fact it is not difficult to see that both definitions are equivalent (see [Dég17, 4.2.2]). It is known that the motivic spectra $\mathbf{KGL}$, $\mathbf{H}\mathbb{Q}$, $\mathbf{MGL} \otimes \mathbb{Q}$, over $\mathrm{Spec}(\mathbb{Z})$, satisfy absolute purity (see [Dég14, Rem. 1.3.5]). It was conjectured in [Dég14, Conjectures B and C] that $\mathbf{MGL}$ and $\mathbb{S}$ also satisfy absolute purity.

4.3. **Refined fundamental classes and specializations.** The notion of refined Gysin morphisms in Fulton's treatment of intersection theory (cf. [Ful98, 6.2]) has a very natural analogue in terms of fundamental classes.

Definition 4.3.1. Consider a cartesian square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\Delta"] & T \arrow[d, "p"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

where f is a quasi-projective lci morphisms. Then we associate to Δ a *refined fundamental class* $\eta_\Delta \in H_0(Y/T, q^*\langle L_f \rangle)$ according to the formula:

$$\eta_\Delta = \Delta^*(\eta_f)$$

with the notations of Paragraph 2.1.6 and Theorem 3.3.2.

The class η_Δ defines a Gysin morphism in bivariant theory. Assuming S is a s-scheme over a base scheme S_0, p is separated of finite type, and given any spectrum $\mathbb{E}$ in $S\mathscr{H}$ (or more generally in a motivic category $\mathscr{T}$) over S_0, one gets a refined Gysin map:

$$\Delta^! : \mathbb{E}_n(T/S_0, e) \to \mathbb{E}_n(Y/S_0, g^*e + q^*\langle L_f \rangle), t \mapsto \eta_\Delta.t. \tag{4.3.1.a}$$

These refined Gysin maps, as well as the refined fundamental classes of the form η_Δ, satisfy good formal properties, that can be deduced from the properties of our fundamental classes. Compatibility with vertical horizontal composition of squares, compatibility with proper covariance (resp. étale contravariance) of bivariant theory, excess intersection formula.

As shown in the reference book of Fulton, such a formalism has literally tons of applications. We simply give two new applications.

Example 4.3.2. *Specialization of families.* Assume S_0 is the spectrum of a field k. We consider an arbitrary s-morphism $p : X \to S$ of s-schemes over k. Given any ring spectrum $\mathbb{E}$ over k, we view an element $\alpha \in \mathbb{E}_n(X/k, e)$ as a family over S. For any regular rational point $s : \mathrm{Spec}(k) \to S$, we define following [Ful98, 10.1] the specialization of α at s as the element of

$$\alpha_s = \Delta_s^!(\alpha) \in \mathbb{E}_n(X_s/k, e_s + p^*\langle N_s\rangle)$$

where Δ is the obvious cartesian square build out of s and X/S. The properties stated in [Ful98, Prop. 10.1] formally extends to this more general version of specialization.

Moreover, we can apply this definition to the ring spectrum representing Milnor-Witt cohomology (Example 4.1.12). Then one obtains a generalization of the considerations of Fulton for Chow-Witt groups. Beware however that the theory is more complicated than the case of usual Chow groups.

For example, assume S/k is smooth connected of dimension n and p is proper. Consider a Chow-Witt cycle $\alpha \in \widetilde{\mathrm{CH}}_n(X)$. Then given any rational point $s \in S$, one gets a specialization $\alpha_s \in \widetilde{\mathrm{CH}}_0\left(X_s, fg_s^* \det(-N_s)\right)$. As X_s is proper over k, this class admits a degree (the pushforward along the structural map of X_s) in the Grothendieck-Witt group of k:

$$\deg(\alpha_s) \in \mathrm{GW}(k).$$

Then these degree depend in general of the point s, unlike the case of usual cycles. In fact, the Chow-Witt cycle $p_*(\alpha) \in \widetilde{\mathrm{CH}}_n(S)$ corresponds to the class of an unramified quadratic form q in $\mathrm{GW}(\kappa(S))$. Then for any $s \in S(k)$, one can check that the class $\deg(\alpha_s)$ is the specialization of the quadratic form q at s.

Example 4.3.3. *Specialization maps.* Let us fix an ring spectrum $\mathbb{E}$ over a base scheme S. We consider cartesian diagrams

$$\begin{tikzcd}
X_Z \arrow[r, "i_X"] \arrow[d, "f_Z"'] \arrow[dr, phantom, "\Delta"] & X \arrow[d] & X_U \arrow[l, "j_X"'] \arrow[d] \\
Z \arrow[r, "i"'] & S & U \arrow[l, "j"]
\end{tikzcd}$$

such that i is a regular closed immersion, j the inclusion of the open complement and X/S is separated of finite type. We finally consider a pair $(n, e) \in \mathbb{Z} \times \underline{K}(X)$.

Let us make the following assumptions:

(1) The spectrum $\mathbb{E}$ is i-pure (Definition 4.2.5).
(2) the Euler class of the normal bundle $e(N_Z S)$ is zero (by Proposition 3.1.6 this is the case when $N_Z S$ is a trivial bundle).
(3) The boundary map $\mathbb{E}_n(X_U/S, e) \to \mathbb{E}_{n-1}(X_Z/S, e)$ of the localization sequence (2.1.10.a) is zero.

Then we can consider the *refined Gysin morphism*:

$$\Delta^! : \mathbb{E}_n(X/S, e) \to \mathbb{E}_n(X_Z/S, e - f_Z^{-1}N_Z S),$$

where we have written e for the obvious pullback of e. According to the self-intersection formula (Example 3.2.7(2)), the composition

$$\mathbb{E}_n(X_Z/S,e) \xrightarrow{i_{X*}} \mathbb{E}_n(X/S,e) \xrightarrow{\Delta^!} \mathbb{E}_n(X_Z/S,e-f_Z^{-1}N_ZS)$$

is equal to the multiplication by the Euler class $e(f_Z^{-1}N_ZS) = f_Z^*e(N_ZS)$, which is zero according to assumption (2). Thus, according to the localization sequence (2.1.10.a) and assumption (3), the map (4.3.1.a) factors through the group $\mathbb{E}_n(X_U/S,e) = \mathbb{E}_n(X_U/U,e)$, and we obtain a map

$$\sigma : \mathbb{E}_n(X_U/U,e) \to \mathbb{E}_n(X_Z/S,e-f_Z^{-1}N_ZS).$$

Finally, according to assumption (1), we get the following specialization map:

$$\sigma : \mathbb{E}_n(X_U/U,e) \to \mathbb{E}_n(X_Z/Z,e). \tag{4.3.3.a}$$

which is a generalization of the *specialization map* on cycles in [Ful98, 20.3].

Remark 4.3.4.
1. Assumption (3) naturally occurs as follows. Let us assume that S is a k-scheme for a base field k of characteristic exponent p. Let δ be the dimension function given by the Krull dimension. Then we can consider the δ-homotopy t-structure of [BD17] on $\mathscr{SH}[1/p]$. Then according to [BD17, 3.3.5], if $\mathbb{E}$ is non negative with respect to the δ-homotopy t-structure, for any s-scheme X/S, one has $\mathbb{E}_p(X/S,e) = 0$ as soon as $p < \mathrm{rk}(e)$. In particular we can take $n = \mathrm{rk}(e)$ to get assumption (3).[22] This applies in particular when $\mathbb{E}$ is $\mathbb{S}$, or the ring spectra representing $\mathbb{A}^1$-homology, motivic cohomology, Milnor-Witt K-theory.

   If S is in addition regular, we get by looking at the relevant degrees of the bivariant theory represented by the Milnor-Witt ring spectrum a well defined specialization map:

   $$\widetilde{\mathrm{CH}}_0(X_U,\det(L)) \to \widetilde{\mathrm{CH}}_0(X_Z,\det(L)),$$

   for any line bundle L and any s-scheme X/S.
2. Our specialization map (4.3.3) for a smooth family is compatible with Ayoub's motivic nearby cycle functor. Let us be more precise.

   Let k be a field of characteristic 0 and $i : \operatorname{Spec} k \to \mathbb{A}^1_k$ be the zero point with open complement $U = \mathbb{G}_{m,k}$. Let $f : X \to \mathbb{A}^1_k$ be a smooth morphism, e be a virtual vector bundle over X, $\mathbb{E} \in \mathscr{SH}(S)$ and $n \in \mathbb{N}$ such that the above assumptions (1) and (3) are holds. Then the map (4.3.3.a) is induced by the canonical natural transformation $i_X^*j_{X*} \to \Psi_f$, where $\Psi_f : \mathscr{SH}(X_U) \to \mathscr{SH}(X_Z)$ is Ayoub's nearby cycle functor ([Ayo07, 3.5.6]).

4.4. **The motivic Gauss-Bonnet formula.** Let $p : X \to S$ be a smooth proper morphism. Recall that the spectrum $\Sigma^\infty_+(X) \simeq p_!p^!(\mathbb{S}_S)$ is a strongly dualizable object of $\mathscr{SH}(S)$ [CD12, Prop. 2.4.31], so that we may consider the *trace* of its identity endomorphism. This is an endomorphism $\chi^{cat}(X/S) \in \mathrm{Hom}_{\mathscr{SH}(S)}(\mathbb{S}_S,\mathbb{S}_S)$ that we refer to as the *categorical Euler characteristic*; see [Hoy15, § 3] for details. In this subsection, we view

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[22] This corresponds to the fact specialization map exists on 0-cycles.

$\chi^{cat}(X/S)$ as an element of the group $H_0(S/S,0)$, and compute it as the "degree of the Euler class of the tangent bundle":

Theorem 4.4.1. *Let $p : X \to S$ be a smooth proper morphism. Then there is an equality $\chi^{cat}(X/S) = p_*(e(T_{X/S}))$ in the group $H_0(S/S,0)$.*

Remark 4.4.2. Let S be the spectrum of a field of characteristic different from 2, and let $p : X \to S$ be a smooth projective morphism. Under these assumptions, a version of Theorem 4.4.1 was proven recently by Levine [Lev17b, Theorem 1]. The formulation of *loc. cit.* can be recovered from Theorem 4.4.1 by applying the $\mathbb{A}^1$-regulator map (Paragraph 4.1.1).

4.4.3. In order to prove Theorem 4.4.1, we begin by giving a useful intermediate description of $\chi^{cat}(X/S)$. Consider the cartesian square

$$\begin{tikzcd}
X \times_S X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "p"] \\
X \arrow[r, "p"] & S
\end{tikzcd}$$

and let $\delta : X \to X \times_S X$ denote the diagonal (a regular closed immersion).

Lemma 4.4.4. *The endomorphism $\chi^{cat}(X/S) : \mathbb{S}_S \to \mathbb{S}_S$ is obtained by evaluating the following natural transformation at the monoidal unit $\mathbb{S}_S$:*

$$\begin{tikzcd}
\mathrm{Id} \arrow[r] \arrow[dd, dashed] & p_*p^* \arrow[r, equal] & p_*\delta^!(\pi_2)^!p^* \\
 & & p_*\delta^!(\pi_1)^*p^! \arrow[u, "Ex^{*!}", "\sim"'] \arrow[d, "\theta"] \\
\mathrm{Id} & p_*p^! \arrow[l] \arrow[r, equal] & p_*\delta^*(\pi_1)^*p^!
\end{tikzcd}$$

Here the arrow $\mathrm{Id} \to p_*p^*$ *is the unit of the adjunction* (p^*, p_*), *the arrow* $p_*p^! \to \mathrm{Id}$ *is the co-unit of the adjunction* $(p_*, p^!)$, *and* $\theta : \delta^! \to \delta^*$ *is the exchange transformation* $Ex^{*!} : \mathrm{Id}^*\,\delta^! \to \mathrm{Id}^!\,\delta^*$.

Proof. This follows from the description given in [Hoy15, Prop. 3.6], in view of the commutativity of the diagram

$$\begin{tikzcd}
\delta^!(\pi_2)^!p^* \arrow[r, equal] & p^* \arrow[r, equal] & (\pi_1)_*\delta_*\delta^*(\pi_2)^!p^* \arrow[r, "\varepsilon"] & (\pi_1)_*(\pi_2)^!p^* \\
\delta^!(\pi_1)^*p^! \arrow[u, "Ex^{*!}", "\sim"'] \arrow[r, "\theta"] & p^! \arrow[r, equal] & (\pi_1)_*\delta_*\delta^*(\pi_1)^*p^! & (\pi_1)_*(\pi_1)^*p^! \arrow[l, "\eta"'] \arrow[u, "Ex^{*!}", "\sim"']
\end{tikzcd}$$

which the reader will easily verify. ☐

Proof of Theorem 4.4.1. By Lemma 4.4.4, it will suffice to show that the following diagram commutes:

$$\begin{tikzcd}
p^* = \delta^!(\pi_2)^! p^* \arrow[dr, "\Sigma^{-T_p} * \mathfrak{p}_p"'] & \delta^!(\pi_1)^* p^! \arrow[l, "Ex^{*!}"'] \arrow[r, "\theta"] & \delta^*(\pi_1)^* p^! = p^! \\
& \Sigma^{-T_p} p^! \arrow[u, "\mathfrak{p}_\delta"] \arrow[ur, "\xi_p * p^!"'] &
\end{tikzcd}$$

Here we have written ξ_p for the natural transformation $\Sigma^{-T_p} \to \mathrm{Id}$ induced by the Euler class $e(T_p) : \mathbb{S}_X \to \mathrm{Th}_X(T_p)$. The commutativity of the left-hand triangle follows by construction of the fundamental class of p (Example 2.2.3) and Corollary 3.2.14. For the right-hand triangle, commutativity follows immediately from the self-intersection formula (Example 3.2.7(2)), which asserts the commutativity of the square

$$\begin{tikzcd}
\mathrm{Th}_X(-T_p) \arrow[r, "\eta_\delta"] \arrow[d, equal] & \delta^!(\pi_1)^*(\mathbb{S}_X) \arrow[r, "\theta"] & \delta^*(\pi_1)^*(\mathbb{S}_X) \arrow[d, equal] \\
\mathrm{Th}_X(-T_p) \arrow[rr, "e(T_p)"] & & \mathbb{S}_X.
\end{tikzcd}$$

□

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