

# ON THE CONIVEAU SPECTRAL SEQUENCE OF A FIBRATION

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ABSTRACT. In this work, we build a spectral sequence in motivic homotopy that is analogous to both the Serre spectral sequence in algebraic topology and the Leray spectral sequence in algebraic geometry. Here, we focus on laying the foundations necessary to build the spectral sequence and give a convenient description of its  $E_2$ -page. Our description of the  $E_2$ -page is in terms of homology of the local system of fibers, which is given using a theory similar to Rost's cycle modules. We close by providing some sample applications of the spectral sequence and some hints at future work.

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## 1. INTRODUCTION

The goal of this paper is to study an algebro-geometric version of the Leray–Serre spectral sequence for generalized cohomology theories. To explain the setup, recall that given a topological space  $X$ , a sheaf  $\mathcal{F}$  of abelian groups on  $X$  and a filtration

$$X_\bullet = X_0 \subset \dots \subset X_p \subset X_{p+1} \subset \dots \subset X_n = X$$

by closed subsets, the technique of exact couples yields a spectral sequence of the form:

$$E_1^{p,q}(X_\bullet, \mathcal{F}) = H^{p+q}(X_p \setminus X_{p-1}, \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

If  $X_\bullet$  is cellular with respect to  $\mathcal{F}$ , i.e., if  $H^i(X_p \setminus X_{p-1}, \mathcal{F}) = 0$  for all  $i \neq p$ , then there are isomorphisms of the form  $H^k(X, \mathcal{F}) \cong E_2^{k,0}(X_\bullet, \mathcal{F})$ . If  $f : X \rightarrow B$  is a continuous map of topological spaces and  $B_\bullet$  is a filtration of  $B$  that is cellular with respect to the direct image sheaves  $R^q f_* \mathcal{F}$  for all  $q$ , then the  $E_2$ -term of Leray spectral sequence

$$E_2^{p,q} = H^p(B, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

is isomorphic to  $E_2^{p,q}(X_\bullet, \mathcal{F})$ , where  $X_\bullet$  is the inverse image of  $B_\bullet$ .

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On the other hand, suppose

$$F \longrightarrow X \xrightarrow{f} B$$

is a Serre fibration of topological spaces, where  $B$  has the homotopy type of a (connected) finite CW complex, and  $E$  is a (generalized) cohomology theory in the sense of classical stable homotopy theory. One may consider an associated Atiyah–Hirzebruch spectral sequence (see, e.g., [DK01, §9.2-9.5]): the  $E_2$ -page of this spectral sequence is given in terms of the ordinary (co)homology of  $B$  with coefficients in local systems attached to the  $E$ -(co)homology of  $F$  and converges to the  $E$ -(co)homology of the  $X$ . When  $E$  is ordinary cohomology and  $\mathcal{F}$  is the constant sheaf  $\mathbb{Z}_X$ , the two spectral sequences coincide. Indeed, Serre showed that the direct image sheaves  $R^q f_* \mathbb{Z}_X$  are local systems on  $B$  depending only on the cohomology of the fiber  $F$  and the action of the fundamental group of  $B$  on these cohomology groups.

I think we need to explain the refinement better. For Arapura’s result, one needs some kind of homotopy invariance statement for the cohomology under consideration (else one could not reduce to the affine case). On the other hand, sheaf cohomology is not homotopy invariant in general (e.g., intersection cohomology is not homotopy invariant). Homotopy invariance is, according to my understanding, central to our discussion. I realize that we need some kind of “perverse” truncation in what we do, but there is something disquieting about the juxtaposition of these facts. Indeed Arapura’s main theorem (3.1 in loc. cit.) is stated for complex varieties and cohomology with integral coefficients. I adapted the statements. In addition I added a few words on Rost’s spectral sequence.

To explain our algebro–geometric analog, recall that Arapura showed the Leray spectral sequence is “motivic” [Ara05]. In more detail, suppose  $f : X \rightarrow B$  is a projective morphism of complex quasi–projective algebraic varieties. After reducing to a situation where  $B$  is affine (via the Jouanolou trick), work of Beilinson and Nori [Nor02] shows that any constructible sheaf on  $B$  can be made cellular with respect to a filtration by closed algebraic subsets. In this situation, Arapura compares the Leray spectral sequence

$$E_2^{a,b}(f) = H^a(B, R^q f_* \mathbf{Z}) \Rightarrow H^{a+b}(X, \mathbf{Z})$$

and the spectral sequence associated with the skeletal filtration and uses this to show that the Leray spectral sequence essentially reflects suitably functorial algebro–geometric structure present on cohomology groups (e.g., it lifts to Nori’s category of mixed motives). A similar result holds for the perverse Leray spectral sequence

$$E_2^{a,b} = H^a(B, {}^p R^b f_* \mathbf{Z}) \Rightarrow H^{a+b}(X, \mathbf{Z})$$

that is obtained by replacing the classical truncation functor  $\tau$  by the perverse truncation  ${}^p \tau$  [dCM10].

Here, we consider a variant of this setup; we expose here a simple case of the discussion. Suppose  $k$  is a field,  $B$  is a finite dimensional irreducible smooth  $k$ -variety, and  $f : X \rightarrow B$  is a smooth morphism of  $k$ -varieties. As usual,  $f$  may be thought of as an étale locally trivial fibration in algebraic geometry. We write  $B^{(n)}$  for the set of codimension  $n$  points of  $B$ . If  $B$  has dimension  $d$ , we view the collection  $B^{(i)}$  as  $i$  ranges from 0 through  $d$  as an algebro–geometric analog of the skeletal filtration of  $B$ . In this situation, the scheme-theoretic fiber  $X_b$  for  $b \in B^{(i)}$  is then a smooth variety over  $b$ .

There is a comparison theorem between the Leray spectral sequence and the spectral sequence associated to the Bloch–Ogus complex. If  $H^*(X, A)$  is a suitable cohomology theory defined using a Grothendieck topology that is finer than the Zariski topology (e.g. étale cohomology, or Betti cohomology over  $\mathbf{C}$ ), one can consider the Leray spectral sequence associated with the morphism of sites  $\pi : X_{\text{fine}} \rightarrow X_{\text{zar}}$ . The higher direct image sheaf  $R^q \pi_* A$  is the Zariski sheaf  $\mathcal{H}^q$  associated to the presheaf  $U \mapsto H^q(\pi^{-1}(U), A)$ . Bloch and Ogus show that the Leray spectral sequence  $E_2^{p,q} = H^p(X, \mathcal{H}^q) \Rightarrow H^{p+q}(X, A)$  can be identified from the  $E_2$ -term onward with the spectral sequence

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} H^{q-p}(k(x)) \Rightarrow N^\bullet H^{p+q}(X)$$

that gives the coniveau filtration on cohomology. (Their argument uses Deligne’s technique of “décalage”, see [Par96].) There is a version of the previous constructions where one replaces cohomology by Borel–Moore homology and works with the niveau (rather than coniveau) filtration. More generally, one can work with the bivariant theory

$$H^i(X \xrightarrow{f} Y) = \text{Hom}(Rf_! \mathbf{Q}_X, \mathbf{Q}_Y[i])$$

that generalizes both Borel–Moore homology and cohomology.

Given a suitable notion of cohomology theory  $\mathbb{E}$  on the category of schemes, one may build “local coefficient systems” on  $B$ ; this construction originates from the work of Bloch and Ogus, but is explicitly a generalization of work of Rost [Ros96] (much of the paper will be devoted to presenting axiomatically properties of  $\mathbb{E}$  necessary to make this precise, but we leave it vague for the moment). Given a morphism  $f : X \rightarrow B$  and a “cycle module”  $M$  on  $X$ , Rost defined Chow groups  $A_*(X, M)$  with values in  $M$  and constructed a spectral sequence

$$E_{p,q}^2 = A_p(B, A_q(X, M)) \Rightarrow A_{p+q}(X, M)$$

where  $A_q(X, M)$  is a cycle module on  $B$  obtained by taking homology of the fibers. If we take the  $\mathbb{E}$ -cohomology of  $X_b$  as  $b$  varies through the  $B^{(i)}$ , then one may define complexes of sheaves on  $B$  with differentials defined in terms of residue maps constructed from points of different codimension. These complexes of sheaves on  $B$  may be thought of, by analogy with the topological situation, as a local coefficient system on  $B$ , though equipped with an explicit Gersten-style resolution. Taking ordinary (sheaf) cohomology of these complexes on  $B$ , one obtains a spectral sequence whose  $E_2$ -page is essentially the sheaf cohomology of  $B$  with coefficients in the  $\mathbb{E}$ -cohomology of the fibers and converging to the  $\mathbb{E}$ -cohomology of  $X$ , and thus may be viewed as an analog of the (relative) Atiyah–Hirzebruch spectral sequence described above. In much of the setup we may work with much more general schemes than those mentioned above. Moreover, the cohomology theory  $\mathbb{E}$  we consider will be represented by a  $\mathbb{P}^1$ -spectrum in the sense of motivic stable homotopy theory. We shall work with a different  $t$ -structure, the homotopy  $t$ -structure (defined in section 2) and replace Rost’s cycle modules by homotopy modules. At the end of the paper we present some applications that illustrate how the general theory works: in the homotopical case (5.1-5.2) we discuss  $\mathbf{A}^1$ -trivial fibrations, Gysin and Wang sequences and in the motivic case (5.3) the degeneration of the Leray-type spectral sequence for relative cellular spaces.

#### NOTATIONS AND CONVENTIONS

I use everywhere the terminology  $\delta$ -homotopy. I wonder if one needs not just use the shorter terminology “homotopy”. The dimension function  $\delta$  does not change the categories up to canonical equivalence. It is just there to fix conventions. homotopy  $t$ -structure seems fine to me; maybe we should add a short explanation in the introduction.

**Geometry.** All schemes in this paper will be noetherian, finite dimensional and assumed to come equipped with a dimension function, usually denoted  $\delta$ . While we fix such a dimension function throughout, and while it may even appear explicitly in various notions we use, we emphasize that the choice is inessential in the sense that the relevant notions do not depend on the fixed choice of dimension function up to a suitable notion of canonical equivalence; see Remark 2.1.6 for a more precise statement. For example, the dimension function is used to fix conventions regarding degrees of (co)homology as in 2.1.10.

Fix a base scheme  $S$ . An  $S$ -scheme  $X$  will be said to have essentially finite type or, equivalently,  $X \rightarrow S$  has essentially finite type, if  $X$  can be written as a (co)filtered limit a  $S$ -schemes of finite type with affine étale transition morphisms. By a point of a scheme  $X$ , we will mean a map  $x : \text{Spec}(K) \rightarrow X$  where i)  $K$  is a field and ii) if  $k$  is the residue field of the image of  $x$  in  $X$ , then the extension  $K/k$  is finitely generated; equivalently, the morphism  $x : \text{Spec}(K) \rightarrow X$  has essentially finite type.

The dimension function  $\delta$  on  $S$ , is extended to a dimension function on  $X$  as follows:

$$\begin{aligned} \forall x/s \in X/S, \delta(x) &:= \delta(s) + \text{trd}(\kappa(x)/\kappa(s)). \\ \delta(X) = \delta_+(X) &:= \max_{x \in X}(\delta(x)). \\ \delta_-(X) &:= \min_{x \in X}(\delta(x)). \end{aligned}$$

Two examples to keep in mind include:

- (S1) if  $S$  is the spectrum of a field  $k$ ,  $\delta = 0$ , then  $\delta(x) = \text{trd}(\kappa(x)/k)$ ;
- (S2) if  $S$  is an excellent regular scheme of dimension less or equal than 3 and  $\delta$  is the Krull dimension.

In both cases, if  $X/S$  has finite type, then the integer  $\delta(X)$  coincides with the Krull dimension of  $X$ .

Finally, the following formulas will be used in the paper:

- (D1) if  $S$  is regular connected,  $\delta = d - \text{codim}_S$  where  $d = \delta(S)$ ;
- (D2) if the morphism  $f : X \rightarrow S$  has essentially finite type, and is lci with cotangent complex  $L_f$ , for any point  $x \in X$ ,  $\delta(x) = \text{rk}(L_{f,x}) + \delta(s)$ . Moreover, if  $X$  and  $S$  are irreducible, and  $d = \dim(f)$  is the rank of  $L_f$ , one has:

$$\delta(X) = d + \delta(S).$$

**Motivic stable homotopy.** We fix a motivic triangulated category  $\mathcal{T}$  in the sense of [CD09, Def. 2.4.45]. In brief,  $\mathcal{T}$  consists of the following data: for any scheme  $S$ , a triangulated closed symmetric monoidal category  $\mathcal{T}(S)$ ; for any morphism of schemes  $f$  and any separated morphism of finite type  $p$ , pairs of adjoint functors  $(f^*, f_*)$ ,  $(p_!, p^!)$  satisfying the so-called Grothendieck six functor formalism (see [CD09, Th. 2.4.50] for a precise statement). Following the terminology from stable homotopy theory objects of  $\mathcal{T}(S)$  will be called  $\mathcal{T}$ -spectra over  $S$ .

We write  $\mathbb{1}_S$  for the monoidal unit in  $\mathcal{T}(S)$ , and  $\mathbb{1}_S(1)$  for the Tate twist. In the sequel, various combinations of Tate twists and shifts naturally arise, and we introduce a separate notation for these twists:

$$\mathbb{1}_S\{1\} := \mathbb{1}_S(1)[1], \quad \mathbb{1}_S\langle 1 \rangle := \mathbb{1}_S(1)[2].$$

Given an object  $\mathbb{E}$  in  $\mathcal{T}(S)$ , and a morphism  $f : X \rightarrow S$  (resp. a separated morphism having finite type) we may define  $\mathbb{E}$ -cohomology and bivariate  $\mathbb{E}$ -theory by means of the formulas:

- (Cohomology)  $\mathbb{E}^{n,i}(X) = \mathrm{Hom}_{\mathcal{T}(X)}(\mathbb{1}_X, f^*\mathbb{E}(i)[n])$ .
- (Bivariate theory)  $\mathbb{E}_{n,i}(X/S) = \mathrm{Hom}_{\mathcal{T}(X)}(\mathbb{1}_X(i)[n], f^!\mathbb{E})$ ;

When considering the homotopy Leray spectral sequence, the notation  $H^{**}(X, \mathbb{E})$ ,  $H_{**}(X/S, \mathbb{E})$  seems better. May we should indicate the two possible notations? Or always use the second one?

The definition of the bivariate theory attached to  $\mathbb{E}$  will be extended to the situation where  $X$  has essentially finite type over  $S$  (see Paragraph 2.1.1). The indexing for cohomology used above may also be expressed in terms of the other conventions for twists mentioned above, with notation changed accordingly: one has equalities of the form

$$\mathbb{E}^{n,i}(X) = \mathbb{E}^{n-i, \langle i \rangle}(X) = \mathbb{E}^{n-2i, \langle i \rangle}(X);$$

similar notation will be used for the associated bivariate theory. When  $\mathbb{E} = \mathbb{1}_S$ , the various cohomology groups will be referred to as  $\mathcal{T}$ -cohomology and bivariate  $\mathcal{T}$ -theory respectively, and we write  $H^{n,i}(X, \mathcal{T})$  (resp.  $H_{n,i}^{BM}(X/S, \mathcal{T})$ ) for these groups.

Write  $\underline{\mathbb{K}}(S)$  for the category of virtual vector bundles over  $S$ . The Thom space construction may be viewed as a functor

$$\mathrm{Th}_S : \underline{\mathbb{K}}(S) \rightarrow \mathcal{T}(S)$$

that sends sums to tensor products. Following [DJK18], we may twist cohomology and bivariate theories by pairs  $(n, v) \in \mathbf{Z} \times \underline{\mathbb{K}}(X)$ ; we use the following notation for these twists:

$$\begin{aligned} \mathbb{E}^n(X, v) &= \mathrm{Hom}_{\mathcal{T}(X)}(\mathbb{1}_X, f^*\mathbb{E} \otimes \mathrm{Th}_X(v)[n]) \\ \mathbb{E}_n(X/S, v) &= \mathrm{Hom}_{\mathcal{T}(X)}(\mathrm{Th}_X(v)[n], f^!\mathbb{E}). \end{aligned}$$

Given an integer  $i \in \mathbf{Z}$ , we denote by  $\langle i \rangle$  the unique free virtual vector bundle of rank  $i$  – the underlying scheme is implicit – so that this notation is compatible with our conventions on twists. If  $K$  is a perfect complex over  $X$ , we denote by  $\langle K \rangle$  the associated virtual bundle. Concretely,  $K$  is Zariski locally isomorphic to a bounded complex of vector bundles.

A perfect complex is only locally quasi-isomorphic to a complex of vector bundles, if you want to write  $K$  as below, you should assume  $K$  is strictly perfect. Nevertheless it does not matter since you can use the local presentation below and Zariski descent for Thom spaces. Is it worth keeping the footnote?<sup>1</sup> I agree the footnote is too loopy. We can refer to Bachmann-Hoyois for a sophisticated version; or make the footnote more precise about the necessary use of Zariski descent: the Thom space of a perfect complex should be well defined at the level of Nisnevich sheaves; no need for  $\infty$ -categories here! :) Since this is supposed to be an announcement, perhaps it is better to just leave it imprecise for the moment?

In order to be able to apply the construction of the  $\delta$ -homotopy t-structure of [BD17], we will require that  $\mathcal{T}$  satisfies the following assumptions:

- (T1)  $\mathcal{T}$  is generated by Tate twists of smooth schemes: more precisely, for any scheme  $S$ ,  $\mathcal{T}(S)$  is generated as a triangulated category by objects of the form  $M_S(X)(i)$  with  $X/S$  smooth and  $i \in \mathbf{Z}$  generates  $\mathcal{T}(S)$ .<sup>2</sup>
- (T2)  $\mathcal{T}$  is continuous with respect to Tate twists: see [CD09, 4.3.2].<sup>3</sup>

<sup>1</sup>Write  $K$  as a complex of vector bundles  $\dots K^i \rightarrow K^{i+1} \rightarrow \dots$  and put  $\langle K \rangle = \sum_i (-1)^i \langle K^i \rangle$ .

<sup>2</sup>All objects in  $\mathcal{T}(S)$  are obtained by taking extensions of arbitrary coproducts. Equivalently, an object  $K$  of  $\mathcal{T}(S)$  is zero if and only if:

$$\forall X/S \text{ smooth}, (n, i) \in \mathbf{Z}^2, \mathrm{Hom}_{\mathcal{T}(S)}(M_S(X)(i)[n], K) = 0.$$

<sup>3</sup>Recall that this expresses the compatibility of  $\mathcal{T}$  with projective limits.

(T3)  $\mathcal{T}$  is *homotopically compatible*: see [BD17, 3.2.12].<sup>4</sup>

(T4) A suitable form of resolution of singularities holds: property (Resol) of [BD17, 2.4.1].<sup>5</sup>

**Examples.** For the sake of concreteness, we observe that the following examples will have all the structure described above.

- (1) **Motivic case.** If  $R$  is a ring of coefficients, then we consider one of the following situations:
  - assuming (S1) holds, the characteristic exponent of  $k$  is invertible in  $R$  and  $\mathcal{T} = \mathrm{DM}(-, R)$  is Voevodsky's cdh-local category of triangulated mixed motives, introduced in [CD15].
  - assuming (S2) holds,  $R$  is a  $\mathbf{Q}$ -algebra and  $\mathcal{T} = \mathrm{DM}(-, R)$  is one of the many models of the triangulated category of  $R$ -motives introduced in [CD09].
- (2) **Homotopical case.** The assumption (S1) holds and  $\mathcal{T} = \mathrm{SH}[p^{-1}]$  is the  $\mathbf{Z}[p^{-1}]$ -linearization of the Morel-Voevodsky stable homotopy category of  $\mathbb{P}^1$ -spectra. Particular motivic ring spectra of interest to us will include:
  - $S_S^0 = \mathbb{1}_S$  the sphere spectrum.
  - $\mathbf{H}R_S$  the motivic Eilenberg-MacLane ring spectrum with coefficients in  $R$ .<sup>6</sup>
  - $\mathbf{H}\tilde{R}_S$  the Milnor–Witt motivic ring spectrum with coefficients in  $R$ . See [DF17].

## 2. HOMOTOPY T-STRUCTURE AND DUALITY

### 2.1. Recollections on the homotopy t-structure.

**2.1.1.** For any  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over a scheme  $S$ , bivariant  $\mathbb{E}$ -theory extends canonically from the category of separated  $S$ -schemes of finite type to that of separated  $S$ -schemes essentially of finite type. Indeed, suppose  $X$  is a separated  $S$ -scheme essentially of finite type. By assumption, there exists a pro-scheme  $(X_\lambda)_\lambda$  with affine and étale transition morphisms, such that each  $X_\lambda$  is separated and has finite type over  $S$ , and with limit  $X$ . We set:

$$\mathbb{E}_{n,i}(X/S) := \varinjlim_\lambda \mathbb{E}_{n,i}(X_\lambda/S).$$

This definition is independent of the pro-scheme presenting  $X$  as a limit [GD67, §8.2]. The resulting definition presents a canonical extension of the original functor by the continuity assumption on  $\mathcal{T}$  (i.e., property (T1)). Moreover, when  $X$  has finite type over  $S$  the new definition agrees with the old definition by using our continuity assumption on  $\mathcal{T}$  (i.e., property (T1) of our conventions on motivic categories).

**Proposition 2.1.2.** *If  $\mathbb{E}$  is a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$ , then the following conditions are equivalent.*

(i) *For any separated scheme  $X/S$  of finite type, one has:*

$$\begin{aligned} \mathbb{E}_{n,i}(X/S) &= 0 \text{ when } n - i < \delta_-(X), \\ &\text{respectively } n - i > \delta_+(X). \end{aligned}$$

(ii) *For any point  $x \in S(K)$ , one has:*

$$\begin{aligned} \mathbb{E}_{n,i}(x) &= 0 \text{ when } n - i < \delta(x), \\ &\text{respectively } n - i > \delta(x). \end{aligned}$$

*Proof.* The equivalence of (i) and (ii) follows by combining Theorem 3.3.1 and Corollary 3.3.5 of [BD17]. Alternatively, it is a straightforward consequence of the existence and convergence of the  $\delta$ -niveau spectral sequence ([BD17, Def. 3.1.5] or Paragraph 3.2.2 in this paper).  $\square$

<sup>4</sup>This condition is automatically verified (see [BD17, 3.2.13]) if  $\mathcal{T}$  satisfies absolute purity and the following vanishing statement holds:

$$\forall \text{fields } E, \forall n > m, H^{n,m}(\mathrm{Spec}(E), \mathcal{T}) = 0.$$

<sup>5</sup>Under two geometric assumptions, this means that:

- In case (S1), we will require that, for  $p$  the characteristic exponent of  $k$ ,  $\mathcal{T}$  is  $\mathbf{Z}[1/p]$ -linear.
- In case (S2), we require that  $\mathcal{T}$  is  $\mathbf{Q}$ -linear.

<sup>6</sup>Recall this is obtained as the image of the constant motive under the canonical map:  $K \circ \gamma_* : \mathrm{DM}(S, \mathbf{Z}[p^{-1}]) \rightarrow \mathrm{SH}(S)[p^{-1}]$ , obtained by forgetting the transfer and then taking the Nisnevich Eilenberg-MacLane functor. See for example [CD09].

*Remark 2.1.3.* (1) In [BD17], the extension  $\mathbb{E}_{**}$  to separated schemes essentially of finite type was denoted by  $\hat{\mathbb{E}}_{**}$ . Since, according to paragraph 2.1.1, this extension is unique and well-defined – using in particular the assumption (T2) for coherence, we will not follow this notational convention here (and we caution the reader that the decoration  $\hat{\mathbb{E}}$  is used with a different meaning in this paper. )

- (2) If we use the  $\delta$ -niveau spectral sequence, then the proof of the previous proposition does not use the assumptions (T1), (T3) and (T4). Therefore, the preceding proposition is true without assuming these conditions.

We can now state the main theorem of [BD17] (see *loc. cit.* Th. 3.3.1 and Cor. 3.3.5).

**Theorem 2.1.4.** *Given any scheme  $S$ , there exists a  $t$ -structure on  $\mathcal{T}(S)$  whose homologically non-negative (resp. non-positive) objects are the  $\mathcal{T}$ -spectra  $\mathbb{E}$  over  $S$  satisfying the equivalent conditions (i) and (ii) of the above proposition. This  $t$ -structure is, moreover, non-degenerate and satisfies gluing in the sense of [BBD82, 1.4.10] (see also Remark 2.1.12).*

**Definition 2.1.5.** Given any scheme  $S$ , the  $t$ -structure on  $\mathcal{T}(S)$  of the above theorem will be called the  $\delta$ -homotopy  $t$ -structure. Objects of the heart of this  $t$ -structure, denoted by  $\mathcal{T}(S)^\heartsuit$ , will be called  $\delta$ -homotopy modules.

*Remark 2.1.6.* The  $t$ -category  $\mathcal{T}(S)$  is “independent” of the choice of  $\delta$  in a sense we now explain. Given another choice  $\delta'$ , we know that the function  $\delta' - \delta$  is constant on connected components of  $S$ . As  $\mathcal{T}(S \sqcup S') = \mathcal{T}(S) \oplus \mathcal{T}(S')$ , we may obviously assume  $S$  is connected so that  $\delta' = \delta + n$ . It follows from the above definition that

$$\phi_{\delta, \delta'} : (\mathcal{T}(S), t_\delta) \rightarrow (\mathcal{T}(S), t_{\delta'}), \mathbb{E} \mapsto \mathbb{E}[n]$$

is an equivalence of  $t$ -categories. It may be useful to remember the formula:

$$(2.1.6.a) \quad \tau_{\geq p}^{\delta+n} = \tau_{\geq p+n}^\delta$$

(see the following conventions). **I assume this is referring to Par. 2.1.8.**

*Example 2.1.7.* (1) Let  $k$  be a perfect field and assume  $\delta$  is the obvious dimension function on  $k$ . In the motivic case, the  $\delta$ -homotopy  $t$ -structure on  $\mathrm{DM}(k, R)$  coincides with the stable version of Voevodsky’s homotopy  $t$ -structure introduced in [Dég11, Sec. 5.2]. See [BD17, Ex. 2.3.5] for details. Moreover, a  $\delta$ -homotopy module  $\mathbb{E}$  over  $k$  is just a  $\mathbf{Z}$ -graded homotopy invariant sheaf with transfers equipped with an isomorphism:

$$(\mathbb{E}_{n+1})_{-1} \simeq \mathbb{E}_n.$$

We refer the reader to [Dég11, 1.17] for more details.

- (2) In the homotopical case, given a field  $k$  (**The assumption  $k$  perfect is unnecessary.**) with the obvious dimension function, the  $\delta$ -homotopy  $t$ -structure on  $\mathrm{SH}(k)$  coincides with Morel’s homotopy  $t$ -structure (see [Mor03, Sec. 5.2]). We refer the reader again to [BD17, Ex. 2.3.5] for more details. In this case, a  $\delta$ -homotopy module  $\mathbb{E}$  over  $k$  is a  $\mathbf{Z}$ -graded strictly  $\mathbf{A}^1$ -invariant Nisnevich sheaf over smooth  $k$ -schemes with a given isomorphism:

$$(\mathbb{E}_{n+1})_{-1} \simeq \mathbb{E}_n.$$

Recall also that a spectrum  $\mathbb{E}$  is called *orientable* if it admits the structure of a module over the ring spectrum  $\mathbf{MGL}$ . Orientability turns out to be equivalent to requiring that the Hopf map  $\eta$  (an element in the graded endomorphisms of the motivic sphere spectrum) acts trivially. In fact, orientability is also equivalent to requiring that  $\mathbb{E}$  admits transfers, in which case these transfers are unique (see [Dég13, 4.1.5, 4.1.7] for further details).

- (3) Over a general base  $S$ , in both the homotopical and motivic cases, the  $\delta$ -homotopy  $t$ -structure can be compared with the *perverse homotopy  $t$ -structure* defined by Ayoub in [Ayo07, §2.2.4]. This comparison requires an appropriate choice of  $\delta$ , and we refer the reader to [BD17, 2.3.11] for details.

**2.1.8. Homological conventions.**– For the most part, we adopt homological conventions, as they are better suited to issues that arise involving singularities. We will write  $\mathbb{E} \geq_\delta n$  (resp.  $\mathbb{E} \leq_\delta n$ ) to say that  $\mathbb{E}$  is concentrated in homological degree below  $n + 1$  (resp. above  $n - 1$ ) **Shouldn’t the two statements be interchanged ?** and denote by  $\tau_{\geq n}^\delta$  (resp.  $\tau_{\leq n}^\delta$ ) the corresponding homological truncation functor. Using homological conventions, the truncation triangles read:

$$\tau_{\geq 0}^\delta(\mathbb{E}) \rightarrow \mathbb{E} \rightarrow \tau_{< 0}^\delta(\mathbb{E}) \xrightarrow{+1}$$

and  $\mathrm{Hom}(\mathbb{E}, \mathbb{F}) = 0$  if  $\mathbb{E} \geq_\delta 0$  and  $\mathbb{F} <_\delta 0$ .

We denote by  $H_n^\delta$  the  $n$ -th homology functor with respect to the  $t_\delta$ -homotopy  $t$ -structure. Finally, we summarize conventions with respect to suspensions:

$$H_n^\delta(\mathbb{E}[i]) = H_{n-i}^\delta(\mathbb{E}), \quad \tau_{\geq n}^\delta(\mathbb{E}[i]) = \tau_{\geq n-i}^\delta(\mathbb{E})[i], \quad \tau_{\leq n}^\delta(\mathbb{E}[i]) = \tau_{\leq n-i}^\delta(\mathbb{E})[i].$$

*Remark 2.1.9.* We may pass from homological to cohomological indexing by writing indices as super-scripts and reversing signs. In formulas:

$$H_\delta^n(\mathbb{E}) = H_{-n}^\delta(\mathbb{E}), \quad \tau_\delta^{\leq n} = \tau_{\geq -n}^\delta, \quad \tau_\delta^{> n} = \tau_{< -n}^\delta.$$

We now give a number of equivalent characterizations of positivity or negativity with respect to the  $\delta$ -homotopy  $t$ -structure. The following proposition is an immediate consequence of the definition of Proposition 2.1.2 using the above conventions and the construction in Paragraph 2.1.1.

**Proposition 2.1.10.** *Given a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$  and an integer  $m \in \mathbf{Z}$ , the following conditions are equivalent:*

- (i)  $\mathbb{E} \geq_\delta m$  (resp.  $\mathbb{E} \leq_\delta m$ )
- (ii) For any separated scheme  $X/S$  having finite type, one has:

$$\mathbb{E}_{n,i}(X/S) = 0 \text{ when } n - i < m + \delta_-(X), \\ \text{respectively } n - i > m + \delta_+(X).$$

- (ii') For any separated scheme  $X/S$  being essentially of finite type, one has:

$$\mathbb{E}_{n,i}(X/S) = 0 \text{ when } n - i < m + \delta_-(X), \\ \text{respectively } n - i > m + \delta_+(X).$$

- (iii) For any point  $x \in S(K)$ , one has:

$$\mathbb{E}_{n,i}(x) = 0 \text{ when } n - i < m + \delta(x), \\ \text{respectively } n - i > m + \delta(x).$$

**2.1.11.**  *$t$ -exactness of the six operations.*— Given a functor  $F$  between triangulated categories equipped with  $t$ -structures, one says that  $F$  is left (resp. right)  $t$ -exact if it respects homologically negative (resp. positive) objects; such a functor  $F$  is  $t$ -exact if it is both left and right  $t$ -exact. One says that  $F$  has homological amplitude  $[a, b]$  if for any object  $\mathbb{E}$ :

- $\mathbb{E} \geq 0 \Rightarrow F(\mathbb{E}) \geq a$ .
- $\mathbb{E} \leq 0 \Rightarrow F(\mathbb{E}) \leq b$ .

Let  $f : X \rightarrow S$  be a morphism essentially of finite type, and  $d$  the maximum dimension of its fibers. We consider the  $\delta^f$ -homotopy  $t$ -structure on  $\mathcal{T}(X)$ , where  $\delta^f$  is the dimension function on  $X$  induced by that of  $S$  with respect to the morphism  $f$  (see Notations and conventions page 3). Then one has the following results:

- $f^*[d]$  is right  $t_\delta$ -exact.
- If  $f$  is smooth,  $f^*[d]$  is  $t_\delta$ -exact.
- The functor  $f_*$  has  $t_\delta$ -amplitude  $[0, d]$ .

The first, second and third points are respectively proved in [BD17], 2.1.6(3), 2.1.12, 3.3.7.

If in addition  $f$  is separated of finite type, we get:

- $f_!$  is right  $t_\delta$ -exact.
- $f^!$  is  $t_\delta$ -exact.
- If  $\delta \geq 0$  then  $\otimes$  is right  $t_\delta$ -exact.

These points are respectively proved in [BD17], 2.1.6(1), 3.3.7(4), 2.1.6(2).

*Remark 2.1.12.* As mentioned in the Theorem 2.1.4, the  $\delta$ -homotopy  $t$ -structure satisfies gluing. We recall here precisely what this means. Consider a closed immersion  $i : Z \rightarrow S$  with complementary open immersion  $j : U \rightarrow S$  and a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$ . The following conditions are equivalent:

- $\mathbb{E}$  is homologically non- $t_\delta$ -negative (resp. non- $t_\delta$ -positive).
- $j^*\mathbb{E}$  and  $i^*\mathbb{E}$  are homologically non- $t_\delta$ -negative (resp.  $j^*\mathbb{E}$  and  $i^!\mathbb{E}$  are homologically non- $t_\delta$ -positive).

The equivalence of these conditions can be deduced from the localization property of  $\mathcal{T}$  and the  $t$ -exactness stated above (see [BD17]).

Let us now state explicitly the following consequence of the result on the tensor product.

**Proposition 2.1.13.** *Let  $\mathbb{E}$  be a  $\mathcal{T}$ -spectrum over  $S$ . Then for any pair  $(p, q) \in \mathbf{Z}^2$ , one has a canonical pairing:*

$$\bar{\phi} : \tau_{\geq p}^{\delta}(\mathbb{E}) \otimes \tau_{\geq q}^{\delta}(\mathbb{F}) \rightarrow \tau_{\geq p+q}^{\delta}(\mathbb{E} \otimes \mathbb{F})$$

which is independent of  $\delta$  and bi-functorial in  $\mathbb{E}$ .

*Proof.* Let us first assume that  $\delta$  is non-negative. According to the preceding paragraph, the assumption implies that the tensor product  $\otimes$  in  $\mathcal{T}(S)$  preserves non-negative objects. Consider the canonical map

$$\tau_{\geq p}^{\delta}(\mathbb{E}) \otimes \tau_{\geq q}^{\delta}(\mathbb{F}) \xrightarrow{\phi} \mathbb{E} \otimes \mathbb{F}.$$

According to the preceding assertion, the left hand-side is in homological degrees  $\geq p + q$ . Consider the distinguished triangle:

$$\tau_{\geq p+q}^{\delta}(\mathbb{E} \otimes \mathbb{F}) \xrightarrow{a} \mathbb{E} \otimes \mathbb{F} \xrightarrow{b} \tau_{< p+q}^{\delta}(\mathbb{E} \otimes \mathbb{F}) \xrightarrow{+1}$$

We deduce that the composition  $b \circ \phi$  is zero. So  $\phi$  uniquely factors through  $a$  giving us the desired map. The bifunctoriality of  $\bar{\phi}$  follows from the uniqueness.

This pairing does not depend on  $\delta$  as it is uniquely defined by  $a$ , from Formula (2.1.6.a).

Then, if  $\delta$  takes negative values, we can always replace it by the dimension function  $\delta' = \delta - \delta_-(S)$ . This is well defined as  $S$  is noetherian and finite dimensional according to our conventions.  $\square$

*Example 2.1.14.* Consider the assumptions of the preceding proposition.

The previous pairing is associative in an obvious sense - this follows from the uniqueness of the map  $\bar{\phi}$ . Hence the symmetric monoidal structure on  $\mathcal{T}(X)$  induces a canonical symmetric monoidal structure on  $\mathcal{T}(X)^{\heartsuit}$ , using the formula, for  $\delta$ -homotopy modules  $\mathbb{E}$  and  $\mathbb{F}$ :

$$\mathbb{E} \otimes^H \mathbb{F} := \tau_{\leq 0}^{\delta}(\mathbb{E} \otimes \mathbb{F}).$$

Note in particular that the canonical functor:

$$\mathcal{T}(X)_{\geq s_0} \rightarrow \mathcal{T}(X)^{\heartsuit}, \mathbb{E} \mapsto \tau_{\leq 0}^{\delta}(\mathbb{E}) = H_0^{\delta}(\mathbb{E})$$

is monoidal.

Finally, given two dimension functions  $\delta$  and  $\delta'$  on  $S$ , the equivalence

$$\mathcal{T}(S)_{t_{\delta}=0} \rightarrow \mathcal{T}(S)_{t_{\delta'}=0},$$

induced by Remark 2.1.6, is monoidal. In other words, the monoidal structure on the heart does not depend on the choice of  $\delta$ .

## 2.2. Recollection on purity and duality.

**2.2.1.** Consider a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over a scheme  $S$ .

Let us recall a construction from [DJK18]. Let  $f : Y \rightarrow X$  be a quasi-projective lci morphism of  $S$ -schemes with cotangent complex  $L_f$ . Let  $\mathbb{E}_X$  be the pullback of  $\mathbb{E}$  along  $X/S$ . Then we associate to  $f$  a purity transformation (see [DJK18, 4.2.1]); evaluated at the object  $\mathbb{E}_X$ , it gives a canonical map:

$$\mathfrak{p}_f : f^*(\mathbb{E}_X) \otimes \mathrm{Th}(L_f) \rightarrow f^!(\mathbb{E}_X).$$

where  $\mathrm{Th}(L_f)$  is the Thom space associated with the perfect complex  $L_f$ . The following definition extends classical considerations; in the motivic case, see [DJK18, 4.2.9].

**Definition 2.2.2.** Consider a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over a scheme  $S$ .

We will say that  $\mathbb{E}$  is *absolutely pure* if for any quasi-projective morphism  $f : Y \rightarrow X$  between regular schemes, the map  $\mathfrak{p}_f$  is an isomorphism.

If  $S$  is regular, we will say that  $\mathbb{E}$  is  *$S$ -pure* if for any quasi-projective morphism  $f : X \rightarrow S$  with  $X$  regular,  $\mathfrak{p}_f$  is an isomorphism.

*Example 2.2.3.* (1) In the motivic case, the constant object  $\mathbb{1}_{\Sigma}$  is absolutely pure. Equivalently,  $\mathbb{1}_S$  is  $S$ -pure for any regular scheme  $S$ .

(2) Let us consider the homotopical case,  $\mathcal{T} = \mathrm{SH}[p^{-1}]$ . Then any spectrum  $\mathbb{E}$  over the base field  $k$  is absolutely pure (see [DJK18, 4.2.8(2)]). Equivalently, any spectrum  $\mathbb{E}$  over a regular base  $S$  is  $S$ -pure.

An almost immediate corollary of the  $S$ -purity assumption is the following duality statement.

**Proposition 2.2.4.** *Let  $f : X \rightarrow S$  be a morphism and  $\mathbb{E}$  be a  $\mathcal{T}$ -spectrum over  $S$ . We assume one of the following hypothesis is fulfilled:*

- *$f$  is essentially smooth.*



- $f$  is essentially quasi-projective,  $X$  and  $S$  are regular and  $\mathbb{E}$  is  $S$ -pure.

Then for any pair  $(n, v) \in \mathbf{Z} \times \underline{\mathbf{K}}(X)$ , the map  $\mathbf{p}_f$  gives an isomorphism:

$$(2.2.4.a) \quad \mathbb{E}^n(X, v) \xrightarrow{\sim} \mathbb{E}_{-n}(X/S, \langle L_f \rangle - v)$$

which is contravariantly natural in  $X$  with respect to étale maps.

*Proof.* The case where  $f$  is of finite type is tautological, while the contravariance with respect to étale map follows from the compatibility of  $\mathbf{p}_f$  with étale pullbacks (apply [DJK18, 3.3.4] in the case where  $p$  is étale).

The general case is obtained using the previous one, together with the naturality with respect to étale maps, and the extension of bivariant theory described in 2.1.1.  $\square$

*Remark 2.2.5.* (1) This isomorphism can be described, at least when  $f$  is of finite type, as the cap-product by the fundamental class  $\eta_f \in \mathbb{E}_0(X/S, \langle L_f \rangle)$  of  $f$ : see [DJK18, 4.2.7]. In fact, the description of the above isomorphism in the general case immediately follows when one considers the obvious extension of the notion of fundamental classes to essentially quasi-projective lci morphisms.

- (2) Recall that if either  $\mathbb{E}$  is an object of  $\mathrm{DM}(S, R)$  or if  $E$  is an **MGL**-module<sup>7</sup> over  $S$  then the Thom isomorphism implies we can canonically identify the twist by a virtual vector bundle with the Tate twist by its rank. In particular, the above isomorphism takes the following classical form:

$$(2.2.5.a) \quad \mathbb{E}^{n,i}(X) \xrightarrow{\sim} E_{2d-n, d-i}(X/S)$$

where  $d$  is the relative dimension of  $f$ .

### 3. FIBER HOMOLOGY AND GERSTEN COMPLEXES

#### 3.1. Fiber $\delta$ -homology.

**3.1.1.** Recall from the introduction that we have also considered  $\mathbf{G}_m$ -twists on bivariant theory. Indeed these twists are more natural with respect to the  $\delta$ -homotopy  $t$ -structure, because the functor  $-(1)[1]$  is  $t_\delta$ -exact. Note also that using this grading, one can reformulate Proposition 2.1.2 for a given  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$  as the equivalence of the following conditions:

- (i)  $\mathbb{E} \geq 0$  (resp.  $\mathbb{E} \leq 0$ ).
- (ii) For any separate scheme  $X/S$  essentially of finite type, one has:

$$\mathbb{E}_{n, \{*\}}(X/S) = 0 \text{ when } n < \delta_-(X) \text{ (resp. } n > \delta_+(X)\text{)}.$$

- (iii) For any point  $x \in S(K)$ , one has:

$$\mathbb{E}_{n, \{*\}}(x) = 0 \text{ when } n < \delta(x) \text{ (resp. } n > \delta(x)\text{)}.$$

In view of this characterization of the  $\delta$ -homotopy  $t$ -structure, we adopted the following definition in [BD17].

**Definition 3.1.2.** Let  $\mathbb{E}$  be a  $\mathcal{T}$ -spectrum over  $S$ . We define the *fiber  $\delta$ -homology* of  $\mathbb{E}$  in degree  $n \in \mathbf{Z}$  as the functor

$$\hat{H}_n^\delta(\mathbb{E}) : \mathrm{Pts}(S) \rightarrow \mathcal{A}b^{\mathbf{Z}}, x \mapsto (\mathbb{E}_{\delta(x)+n, \{\delta(x)-r\}}(x))_{r \in \mathbf{Z}}$$

where  $\mathrm{Pts}(S)$  is the discrete category of points of  $S$ . Dually, we define the *fiber  $\delta$ -cohomology* of  $\mathbb{E}$  as  $\hat{H}_\delta^n(\mathbb{E}) = \hat{H}_{-n}^\delta(\mathbb{E})$  (the  $\mathbf{G}_m$ -grading does not change).

Given an object  $\mathbb{F}$  of the  $\delta$ -homotopy heart, we also put  $\hat{\mathbb{F}}_*^\delta := \hat{H}_0^\delta(\mathbb{F})$ .

Therefore  $\mathbb{E} \geq 0$  (resp.  $\mathbb{E} \leq 0$ ) if and only if  $\hat{H}_n^\delta(\mathbb{E}) = 0$  for  $n < 0$  (resp.  $n > 0$ ).

*Remark 3.1.3.* Fiber  $\delta$ -homology is a good approximation of  $\delta$ -homology. In fact, given an object  $\mathbb{F}$  of the heart of the  $\delta$ -homotopy  $t$ -structure on  $\mathcal{T}(S)$ , the functor  $\hat{\mathbb{F}}_*^\delta$  is an approximation of a cycle module in the sense of Rost.

In the motivic case, this can be turned into an equivalence of categories between the  $\delta$ -homotopy heart of  $\mathrm{DM}(S, R)$  and the category of Rost  $R$ -linear cycle modules over  $S$  in the sense of [Ros96]. The details of such a theorem have not yet been written up, but see [Dég14a]. In the homotopical case, a generalization of Rost's theory is in the works ([Fel18]).

<sup>7</sup>in other words, an oriented spectrum; here module is to be understood in the sense of the monoidal category  $\mathrm{SH}(S)$  i.e. in the weak homotopical sense.

In any case, we have already obtained in [BD17, 4.2.2] that, in the general case of an abstract triangulated motivic category  $\mathcal{T}$  satisfying our general assumptions, the functor:

$$\mathcal{T}(S)^{t_\delta=0} \rightarrow \mathrm{PSh}(\mathrm{Pts}(S), R - \mathrm{mod}^{\mathbf{Z}}), \mathbb{F} \mapsto \hat{F}_*^\delta$$

is conservative, exact and commutes with colimits. Similarly, the family of functors  $(\hat{H}_n^\delta)_{n \in \mathbf{Z}}$  is conservative on the whole category  $\mathcal{T}(S)$ .

*Example 3.1.4.* Let  $S$  be a regular connected scheme, and put  $d = \delta(S)$ . We fix a point  $x : \mathrm{Spec}(K) \rightarrow S$ .

- (1) Abstractly, using any  $S$ -pure spectrum  $\mathbb{E}$ , one obtains, because of the duality isomorphism (2.2.4.a) and relation (D2) of dimension functions, a canonical isomorphism:

$$\hat{H}_n^\delta(\mathbb{E})_r(x) \simeq \mathbb{E}^{-n-r}(\mathrm{Spec}(K), \langle L_{x/S} \rangle - \langle \delta(x) \rangle + \langle r \rangle).$$

- (2) Assume we are in the motivic case. Then one obtains using the previous computation a canonical isomorphism:

$$\hat{H}_\delta^n(\mathbb{1}_S)_r(x) = H_M^{r-n-2d, r-d}(\mathrm{Spec}(K), R).$$

This is because under our assumptions, motivic cohomology satisfies absolute purity, and is oriented.

As a first observation, we thus get that:

$$\hat{H}_d^\delta(\mathbb{1}_S)_* = \hat{H}_\delta^{-d}(\mathbb{1}_S)_* = K_*^M|_S,$$

the restriction of the Milnor K-theory functor to the discrete category  $\mathrm{Pts}(S)$ .

Then the only vanishing that we have is:  $\hat{H}_n^\delta(\mathbb{1}_S) = 0$  if  $n > d$ . In other words,  $\mathbb{1}_S$  is concentrated in  $t_\delta$ -homological degrees  $] -\infty, d]$ . On the other hand, we know for a fact, at least when  $S$  is a complex scheme, that for all  $n \leq d$ ,  $\hat{H}_n^\delta(S) \neq 0$  (this is due to the existence and non-triviality of polylogarithm elements; see [BD17, Ex. 3.3.2]). So  $\mathbb{1}_S$  is  $t_\delta$ -unbounded below.

- (3) Assume we are in the homotopical case. Then one obtains, using the computation of point (i), a non-canonical isomorphism:

$$\hat{H}_n^\delta(\mathbb{1}_S)_r(x) \simeq \pi_{n+d}^{\mathbf{A}^1}(S_K^0)_{r-d}[1/p].$$

This is because absolute purity is automatic under the geometric assumption (S1).

In particular, Morel's computation of the stable  $\mathbf{A}^1$ -homotopy groups of spheres gives an isomorphism, again non-canonical:

$$\hat{H}_d^\delta(\mathbb{1}_S)_r(x) \simeq K_r^{MW}(K)$$

where the right hand side is the  $r$ -th Milnor-Witt cohomology group of the field  $K$ .

Moreover, the zero sphere spectrum  $\mathbb{1}_S$  is concentrated in homological degrees  $] -\infty, d]$  for the  $\delta$ -homotopy  $t$ -structure.

However, there is not much that can be said as, when  $S$  is a complex scheme, the rational homology of the zero sphere spectrum agrees with the rational homology of the motivic spectrum, which is non trivial. This implies  $\mathbb{1}_S$  is unbounded below for the  $\delta$ -homotopy  $t$ -structure.

- (4) Again in the homotopical case, we can consider the Milnor-Witt motivic ring spectrum  $\mathbf{H}\tilde{R}_S$ . Then the known computations of Milnor-Witt cohomology imply, as in the case of motivic cohomology, that  $\mathbf{H}\tilde{R}_S$  is concentrated in homological degrees  $] -\infty, d]$ . Besides, the unit of this ring spectrum induces a canonical isomorphism:

$$\hat{H}_d^\delta(\mathbb{1}_S)_* \xrightarrow{\sim} \hat{H}_d^\delta(\mathbf{H}\tilde{R}_S)_*.$$

*Remark 3.1.5.* In the non-oriented context, it appears more natural to introduce another twist for a virtual vector bundle  $v$  over a scheme  $X$ , namely:

$$\mathbb{1}_X\{v\} := \mathrm{Th}_X(v)[- \mathrm{rk}(v)].$$

This is coherent with the notation:  $\mathbb{1}_X\{r\}$  for the  $\mathbf{G}_m$ -twist.

Then, assuming  $\delta(S) = 0$  to simplify, the isomorphism of point (1) can be rewritten as:

$$\hat{H}_n^\delta(\mathbb{E})_r(x) \simeq \mathbb{E}^{-n}(\mathrm{Spec}(K), \{L_{x/S}\} + \{r\}).$$

The unboundedness of the constant object, observed in Points (2), (3), (4) of the previous example, appears to be a consequence of the stable context – recall that stability with respect to Tate twists is among the axioms of triangulated motivic categories used in [CD09]. It can be corrected as follows.

**Definition 3.1.6.** (see [BD17, Def. 2.2.1]) Let  $S$  be an arbitrary scheme. We define the triangulated category of  $\delta$ -effective  $\mathcal{T}$ -spectra over  $S$ , denoted by  $\mathcal{T}^{\delta\text{-eff}}(S)$ , as the full localizing subcategory of  $\mathcal{T}(S)$  generated by objects of the form:

$$f_!(\mathbb{1}_X)(n)$$

where  $f : X \rightarrow S$  is separated of finite type and  $\delta(X) \geq n$ .

Given a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$ , we define its *effective fiber  $\delta$ -homology*  $H_n^{\delta\text{eff}}(\mathbb{E})$  as the negatively graded functor obtained from  $\hat{H}_n^{\delta}(\mathbb{E})$  by restricting the  $\mathbf{G}_m$ -grading to negative integers.

Then it follows from [BD17, 3.1.1] that one can define an effective version of the  $\delta$ -homotopy  $t$ -structure.

**Theorem 3.1.7.** *Consider the notations of the previous definition. Then for any scheme  $S$ , there exists a  $t$ -structure on  $\mathcal{T}^{\delta\text{eff}}(S)$  whose homologically non-negative (resp. non-positive) objects are those  $\mathcal{T}$ -spectra  $\mathbb{E}$  over  $S$  such that  $\hat{H}_n^{\delta\text{eff}}(\mathbb{E}) = 0$  for  $n < 0$  (resp.  $n > 0$ ).*

*This  $t$ -structure is non-degenerate and satisfies glueing (recall: [BBD82, 1.4.10], Remark 2.1.12).*

*Example 3.1.8.* Let  $S$  be a regular connected scheme, and put  $d = \delta(S)$ .

- (1) Assume we are in the motivic case. Then the computation done in Example 3.1.4(1) shows that  $\hat{H}_n^{\delta}(\mathbb{1}_S)_{\leq 0} = 0$  if  $n \neq d$ . In other words,  $\mathbb{1}_S[-d]$  is in the heart of the effective  $\delta$ -homotopy  $t$ -structure. This is exactly what happens for the perverse  $t$ -structure.
- (2) Assume we are in the homotopical case. The preceding example shows that the Eilenberg-MacLane motivic ring spectrum  $\mathbf{H}R_S[-d]$  is in the heart of the effective  $\delta$ -homotopy  $t$ -structure.

The computations of higher stable homotopy groups of spheres do not allow us at the moment to conclude anything about the sphere spectrum. However, the computations of Milnor-Witt motivic cohomology imply, as in the motivic case, that  $\mathbf{H}_{MW}R_S[-d]$  is in the heart of the effective  $\delta$ -homotopy  $t$ -structure.

**3.1.9.** The  $\delta$ -effective categories  $\mathcal{T}^{\delta\text{eff}}(S)$ , with the  $t$ -structure just defined, satisfy good properties. We refer the reader to [BD17] for the following facts. First, by construction, one has a pair of adjoint functors:

$$s : \mathcal{T}^{\delta\text{eff}}(S) \rightleftarrows \mathcal{T}(S) : w$$

such that  $s$  is fully faithful and  $w$  is  $t_{\delta}$ -exact.

- If  $\delta \geq 0$ , the subcategory  $\mathcal{T}^{\delta\text{eff}}(S)$  of  $\mathcal{T}(S)$  is stable under tensor products. So  $\mathcal{T}^{\delta\text{eff}}(S)$  becomes a closed symmetric monoidal category with internal Hom given by:

$$\underline{\text{Hom}}_{\mathcal{T}^{\delta\text{eff}}(S)}(M, N) = w \circ \underline{\text{Hom}}_{\mathcal{T}(S)}(M, N).$$

- For  $f : X \rightarrow S$  essentially of finite type with  $\dim(f) \leq d$ ,

$$f^*(d) : \mathcal{T}^{\delta\text{eff}}(S) \rightleftarrows \mathcal{T}^{\delta\text{eff}}(X) : w \circ (f_*(-d)).$$

- For  $f : X \rightarrow S$  separated and essentially of finite type,

$$f_! : \mathcal{T}^{\delta\text{eff}}(X) \rightleftarrows \mathcal{T}^{\delta\text{eff}}(S) : w \circ f^!$$

The last two statements are repeated below, in a more detailed form. Sould we remove them ? Note in particular that  $\mathcal{T}^{\delta\text{eff}}$  satisfies glueing, as it is stable under the operations  $j_!$ ,  $j^!$ ,  $i_*$ ,  $i^*$  for  $i$  a closed immersion with complementary open immersion  $j$ .

Let us fix  $f : X \rightarrow S$  a morphism essentially of finite type. We assume  $X$  is equipped with the dimension function induced by that of  $S$ .

- If  $\delta \geq 0$ , the tensor product on  $\mathcal{T}^{\delta\text{eff}}(S)$  is right  $t_{\delta}$ -exact.
- If  $\dim(f) \leq d$ , then

$$f^*(d)[2d] : \mathcal{T}^{\delta\text{eff}}(S) \rightleftarrows \mathcal{T}^{\delta\text{eff}}(X) : w \circ (f_*(-d)[-2d])$$

is an adjunction of  $t$ -categories. The functor  $w \circ (f_*(d)[d])$  has homological amplitude  $[0, d]$ . When  $f$  is smooth of pure dimension  $d$ ,  $f^*(d)[2d]$  is  $t_{\delta}$ -exact.

- If  $f : X \rightarrow S$  is separated, the functors

$$f_! : \mathcal{T}^{\delta\text{eff}}(X) \rightleftarrows \mathcal{T}^{\delta\text{eff}}(S) : w \circ f^!$$

form an adjunction of  $t$ -categories. Moreover,  $w \circ f^!$  is  $t_{\delta}$ -exact.

**3.2. Gersten complexes.** A good way to define spectral sequences is the theory of exact couples. However, there is no standard conventions and to make our definitions sufficiently precise, we now give our conventions for exact couples.<sup>8</sup>

**Definition 3.2.1.** Let  $\mathcal{A}$  be an abelian category.

A *homological exact couple of degree  $d > 0$*  in  $\mathcal{A}$  is the data of a couple of bigraded objects  $(D, E)$  of  $\mathcal{A}$  together with a triangle of homogeneous maps  $(a, b, c)$

$$\begin{array}{ccc} D & \xrightarrow[\quad a \quad]{(-1,+1)} & D \\ & \searrow c & \swarrow b \\ & & E \end{array} \quad \begin{array}{l} \\ \\ (-d, d-1) \end{array} \quad \begin{array}{l} \\ \\ (0,0) \end{array}$$

such that each consecutive maps form an exact sequence.

It is standard to derive a homological exact sequence from such an exact couple, starting at page  $d$ , with  $d$ -th term equal to the differential bigraded abelian group:  $(E, b \circ c)$ . We refer the reader to [McC01].

**3.2.2.** We first recall  $\delta$ -niveau spectral sequences, in the  $\mathbf{A}^1$ -homotopical setting and using dimension functions following [BD17, 3.1.5].<sup>9</sup>

Let  $X$  be a separated  $S$ -scheme essentially of finite type. Recall from [BD17, 3.1.1] that a  $\delta$ -flag of  $X$  is an increasing sequence of reduced closed subschemes  $Z_* = (Z_p)_{p \in \mathbf{Z}}$  of  $X$  such that  $\delta(Z_p) \leq p$  for all  $p$ . The set  $\mathcal{F}(X)$  of  $\delta$ -flags, ordered by term-wise inclusion, is cofiltered. Given such a  $\delta$ -flag, the classical properties of the bivariant theory  $\mathbb{E}_*(-/-)$  imply that we have a long exact sequence:

$$\mathbb{E}_{p+q}(Z_{p-1}/S) \xrightarrow{a} \mathbb{E}_{p+q}(Z_p/S) \xrightarrow{b} \mathbb{E}_{p+q}(Z_p - Z_{p-1}/S) \xrightarrow{c} \mathbb{E}_{p+q-1}(-Z_{p-1}/S)$$

where  $a$  (resp.  $b$ ) is pushforward (resp. pullback) along the obvious closed (resp. open) immersion, and  $c$  the boundary map. These long exact sequences are covariantly functorial with respect to inclusion of flags. Thus we can combine them into the following homological exact couple of degree 1:

$$\begin{aligned} {}^\delta D_{p,q} &= \varinjlim_{Z_* \in \mathcal{F}(X)} (\mathbb{E}_{p+q}(Z_p/S)), \\ {}^\delta E_{p,q}^1 &= \varinjlim_{Z_* \in \mathcal{F}(X)} (\mathbb{E}_{p+q}(Z_p - Z_{p-1}/S)). \end{aligned}$$

Let us finally observe that one can express the  $E_1$ -term as follows:

$${}^\delta E_{p,q}^1 = \bigoplus_{x \in X_{(p)}} \mathbb{E}_{p+q}(x/S),$$

where

$$X_{(p)} = \{x \in X \mid \delta(x) = p\}.$$

**Definition 3.2.3.** Under the assumptions and notations above, the spectral sequence:

$${}^\delta E_{p,q}^1 = \bigoplus_{x \in X_{(p)}} \mathbb{E}_{p+q}(x/S) \Rightarrow \mathbb{E}_{p+q}(X/S)$$

will be called the  $\delta$ -niveau spectral sequence of  $X/S$  with coefficients in  $\mathbb{E}$ .

The spectral sequence converges to the following filtration, called the  $\delta$ -niveau filtration:

$$N_p \mathbb{E}_*(X/S) = \bigcup_{i: Z \rightarrow X, \delta(Z)=p} \text{Im}(i_* : \mathbb{E}_*(Z/S) \rightarrow \mathbb{E}_*(X/S)),$$

where  $i$  runs over the closed immersions whose source has  $\delta$ -dimension  $p$ .

It is usual to consider this definition with Tate twists. So if we apply this definition to the graded spectrum  $\mathbb{E}(n)$  for an integer  $n \in \mathbf{Z}$ , we get the following form:

$${}^\delta E_{p,q}^1 = \bigoplus_{x \in X_{(p)}} \mathbb{E}_{p+q,n}(x/S) \Rightarrow \mathbb{E}_{p+q,n}(X/S).$$

<sup>8</sup>Note that this convention agrees with the one used in [BD17, 3.1.2].

<sup>9</sup>The main idea is of course classical, introduced by Grothendieck, and thoroughly developed in [BO74]. The originality of [BD17] is to use an arbitrary base scheme  $S$  (rather than a field) and abstract dimension functions. This use of dimension functions can also be found in the homological indexation of Chow groups in [wc15]. See [BD17, 3.1.7] for more on this subject.

In view of Rost's theory of cycle modules, it will be useful to introduce the following definition.

**Definition 3.2.4.** Under the assumptions of the preceding definition, we define the *Gersten* ( $\delta$ -homological) complex of  $X/S$  with coefficients in  $\mathbb{E}$ , denoted by  $C_*^\delta(X, \mathbb{E})$ , as the complex of abelian groups located at the line  $q = 0$  of the  $E_1$ -term of the  $\delta$ -niveau spectral sequence. The  $p$ -th homology of this complex will be called the *Gersten*  $\delta$ -homology of  $\mathbb{E}$  **Should we use the terminology Gersten or Rost here ? Or nothing ?** and be denoted by:

$$A_p^\delta(X, \mathbb{E}) = H_p(C_*^\delta(X, \mathbb{E})).$$

Note that the Gersten complex is concentrated in degrees  $[\delta_-(X), \delta_+(X)]$ . Moreover, using Definition 3.1.2 for fiber  $\delta$ -homology, we get the following form of this complex:

$$C_*^\delta(X, \mathbb{E}) \bigoplus_{x \in X_{(p)}} H_0^\delta \mathbb{E}_p(x/S).$$

*Remark 3.2.5.* The complex defined above is deeply linked with Rost's theory of cycle modules. We have used a simplification here. Indeed recall that cycle modules, as well as their complexes, are  $\mathbf{Z}$ -graded. We can recover this  $\mathbf{Z}$ -grading by applying the above definition to the  $\mathbf{Z}$ -graded spectrum:

$$\mathbb{E}\{*\} = (\mathbb{E}(n)[n])_{n \in \mathbf{Z}}.$$

It is usual in  $\mathbb{A}^1$ -homotopy theory to call this the  $\mathbf{G}_m$ -grading.

Then the complex  $C_*^\delta(X, \mathbb{E}\{*\})$  is  $\mathbf{Z}$ -graded, and has the following form:

$$C_*^\delta(X, \mathbb{E}\{*\}) = \bigoplus_{x \in X_{(p)}} H_0^\delta \mathbb{E}_{p-*}(x/S).$$

The main reason to use this grading is that the differentials of the Gersten complex are then homogeneous of degree  $-1$ .

**3.2.6.** Consider the notations of the above definition. The differentials of the Gersten complex associated with  $\mathbb{E}$  and  $X/S$  have the following form:

$$C_p^\delta(X, \mathbb{E}) = \bigoplus_{x \in X_{(p)}} \mathbb{E}_p(x/S) \xrightarrow{d_p} \bigoplus_{s \in X_{(p-1)}} \mathbb{E}_p(s/S) = C_{p-1}^\delta(X, \mathbb{E})$$

The differential  $d_p$  is obtained as the inductive limit of differentials  $d_p^Z$  associated with a  $\delta$ -flag  $Z_*$  and is described as follows:

$$\begin{array}{ccccc} \mathbb{E}_p(Z_p/S) & \longrightarrow & E_p(Z_p - Z_{p-1}/S) & \xrightarrow{c} & E_{p-1}(Z_{p-1}/S) \\ & & & \searrow d_p & \parallel \\ & & E_{p-1}(Z_{p-2}/S) & \longrightarrow & E_{p-1}(Z_{p-1}/S) \xrightarrow{b} E_{p-1}(Z_{p-1} - Z_{p-3}/S) \end{array}$$

where the horizontal lines are part of the localization long exact sequences.

Using the functoriality of the localization long exact sequences with respect to pullback along open immersions, it is therefore possible to explicitly describe the differentials of this complex as follows. Consider a pair  $(x, s) \in X_{(p)} \times X_{(p-1)}$  and denote by  $(d_p)_y^x : \mathbb{E}_p(x/S)_{r-p} \rightarrow \mathbb{E}_{p-1}(y/S)_{r-p}$  the corresponding component of the above differential. Let us write  $Z(x)$  the reduced closure of  $x$  in  $X$ . Then, from the construction of the Gersten complex given above, one deduces that:

- if  $s \in Z(x)$ : we let  $Z(x)_{(s)}$  be the localization of  $Z(x)$  at  $s$ ; this is a 1-dimensional scheme so that  $Z(x)_{(s)} = \{x, s\}$ . Then  $(d_p)_s^x$  is the boundary map of the localization long exact sequence associated with the closed immersion  $i : \{s\} \rightarrow Z(x)_{(s)}$ . Explicitly, it is the middle map in the following exact sequence:

$$\mathbb{E}_p(Z(x)_{(s)}/S) \xrightarrow{j^*} \mathbb{E}_p(x/S) \xrightarrow{(d_p)_s^x} \mathbb{E}_{p-1}(s/S) \xrightarrow{i_*} \mathbb{E}_{p-1}(Z(x)_{(s)}/S)$$

- Otherwise,  $(d_p)_s^x = 0$ .

Here are the formal properties of Gersten complexes.

**Proposition 3.2.7.** *Consider the notations of the above definition.*

- (1) The complex  $C_*^\delta(X, \mathbb{E})$  is covariantly functorial in  $\mathbb{E}$ . Given any integer  $p$ , the induced maps

$$C_*^\delta(X, \mathbb{E}) \rightarrow C_*^\delta(X, \tau_{\leq p}\mathbb{E}), \quad C_*^\delta(X, \tau_{\geq -p}\mathbb{E}) \rightarrow C_*^\delta(X, \mathbb{E})$$

are isomorphisms provided  $p \geq 0$ . In particular, one has a canonical isomorphism:

$$C_*^\delta(X, \mathbb{E}) \simeq C_*^\delta(X, H_0^\delta \mathbb{E}).$$

- (2) The complex  $C_*^\delta(X, \mathbb{E})$  is functorial in  $X/S$ , covariantly with respect to proper maps and contravariantly with respect to étale maps.  
 (3) If the  $\mathcal{T}$ -spectrum  $\mathbb{E}$  is homologically non- $t_\delta$ -negative (resp. non- $t_\delta$ -positive), there exists a canonical epimorphism (resp. monomorphism):

$$\begin{aligned} \mathbb{E}_p(X/S) &\rightarrow A_p^\delta(X, \mathbb{E}) \\ \text{resp. } A_p^\delta(X, \mathbb{E}) &\rightarrow \mathbb{E}_p(X/S). \end{aligned}$$

These two maps are functorial in  $X/S$ , covariantly with respect to proper maps and contravariantly with respect to étale maps.

If  $\mathbb{E}$  is in the  $\delta$ -homotopy heart, or more generally is concentrated in one degree for the  $\delta$ -homotopy  $t$ -structure, these two maps are inverse isomorphisms giving a functorial identification:

$$\mathbb{E}_p(X/S) \simeq A_p^\delta(X, \mathbb{E}).$$

*Proof.* Point (1) follows from the  $\delta$ -niveau spectral sequence and its obvious functoriality in  $\mathbb{E}$  (subsection 3.2.2).

Point (2) follows from the classical functoriality of the ( $\delta$ -)niveau spectral sequences. We refer the reader to [Jin16], proof of Proposition 3.11 for proper functoriality, and proof of Proposition 3.12 for étale contravariance. Note an important technical point here: we deal with the case of a general triangulated category  $\mathcal{T}$ , in contrast with the special case  $\mathcal{T} = \text{DM}$  which is oriented. In the case of proper functoriality, this does not come into play. The case of étale contravariance works as well as the tangent bundle of an étale map is trivial.

Point (3) is a consequence of the convergence of the  $\delta$ -niveau spectral sequence and of the computation of its  $E_1$ -term.  $\square$

*Remark 3.2.8.* (1) According to the isomorphism of Point (3) above, one obtains using the main result of [DJK18] that, given a  $\delta$ -homotopy module  $\mathbb{E}$ , the Gersten homology  $A_*^\delta(X, \mathbb{E})$  is contravariant in  $X$  with respect to any lci smoothifiable morphism  $f : Y \rightarrow X$ . More precisely, one gets a canonical morphism:

$$A_*^\delta(X, \mathbb{E}) \simeq \mathbb{E}_p(X/S) \rightarrow \mathbb{E}_p(Y/S, \langle L_{Y/X} \rangle) \simeq A_*^\delta(Y, \mathbb{E} \langle L_{Y/X} \rangle)$$

where the last identification uses the fact  $\mathbb{E} \langle L_{Y/X} \rangle$  is concentrated in one degree over any connected component of  $Y$  — which is the virtual rank of  $L_{Y/X, \eta}$  where  $\eta$  is any generic point of the chosen connected component.

- (2) Gersten complexes are closely connected to Rost's theory of cycle complexes. For example, if  $S$  is the spectrum over a perfect field  $k$ ,  $\delta$  is the obvious dimension function, and we are in the motivic case  $\mathcal{T} = \text{DM}(k, R)$ . Then any object  $\mathbb{E}$  in the heart of  $\text{DM}(k, R)$  is a homotopy module with transfers in the sense of [Dégl11] which canonically corresponds to a cycle modules  $\hat{\mathbb{E}}_* = \hat{\mathbb{E}}_*^\delta$  — this notation corresponds to the one of Definition 3.1.2. Moreover, according to [Dégl12, 2.7(ii)], there exists a canonical isomorphism of complexes

$$C_*^\delta(X, \mathbb{E}) \simeq C_*(X, \hat{\mathbb{E}}_*)$$

where the right hand-side is Rost cycle complex associated with the cycle module  $\hat{\mathbb{E}}_*$ .

### 3.3. Products.

#### 3.3.1. Consider a $\mathcal{T}$ -spectrum $\mathbb{E}$ over $S$ .

In order to describe products, we will use cohomological notations. We put:

$$C_\delta^p(S, \mathbb{E}) = C_{-p}^\delta(S, \mathbb{E}), \quad \text{resp. } A_\delta^p(S, \mathbb{E}) := A_{-p}^\delta(S, \mathbb{E})$$

and call it the *Gersten  $\delta$ -cohomological complex* (resp.  *$\delta$ -cohomology*) of  $S$  with coefficients in  $\mathbb{E}$ . Note in particular that, when  $\mathbb{E}$  is in the heart, we get from Proposition 3.2.7 a canonical isomorphism:

$$A_\delta^p(S, \mathbb{E}) \simeq H^p(S, \mathbb{E}).$$

Consider now a morphism of  $\mathcal{T}$ -spectra over  $S$ :

$$\mu : \mathbb{E} \otimes \mathbb{F} \rightarrow \mathbb{G}.$$

We deduce as usual a morphism at the level of cohomologies:

$$\mathbb{E}^p(S) \otimes_{\mathbf{Z}} \mathbb{F}^q(S) \rightarrow \mathbb{G}^{p+q}(S)$$

by sending a pair of maps  $(a : \mathbb{1}_S \rightarrow \mathbb{E}[p], b : \mathbb{1}_S \rightarrow \mathbb{F}[q])$  to the following map:

$$\mathbb{1}_S \xrightarrow{a \otimes b} \mathbb{E} \otimes \mathbb{F}[p+q] \xrightarrow{\mu} \mathbb{G}[p+q].$$

Using the preceding isomorphism, we deduce a canonical pairing:

$$A_\delta^p(S, \mathbb{E}) \otimes_{\mathbf{Z}} A_\delta^q(S, \mathbb{F}) \rightarrow A_\delta^{p+q}(S, \mathbb{G}).$$

We deduce the following results.

**Proposition 3.3.2.** *Let  $\mathbb{E}$  be a ring  $\mathcal{T}$ -spectrum over  $S$ . Then  $A_\delta^*(S, \mathbb{E})$  has the structure of a graded-commutative ring.*

*If  $\mathbb{F}$  is a  $\mathcal{T}$ -spectrum with a structure of an  $\mathbb{E}$ -module, then  $A^*(S, \mathbb{F})$  has a structure of module over  $A_\delta^*(S, \mathbb{E})$ .*

*Example 3.3.3.* Let  $\mathbb{E}$  be a ring  $\mathcal{T}$ -spectrum over  $S$ . Then according to Example 2.1.14, the graded  $\delta$ -homotopy module  $H_*^\delta(\mathbb{E})$  has the structure of a graded commutative ring spectrum. According to the preceding proposition,  $A_\delta^*(S, H_*^\delta(\mathbb{E}))$  has the structure of a bigraded commutative ring.

*Remark 3.3.4.* (1) Products are thus easy to obtain on our Gersten  $\delta$ -homology. It is possible to follow Rost's approach in [Ros96] and to get a definition of the product on the level of complexes. Using classical techniques due to Levine, it is even possible to find a dg-algebra underlying our Gersten  $\delta$ -homology (see [Lev06, Lev08, BY18]). An advantage of our approach is that it circumvents these technicalities.

(2) One can also extend the preceding considerations to the case of smooth  $S$ -schemes. Indeed, in that case, one has:

$$A_\delta^p(X, \mathbb{E}) = A_{-p}^\delta(X, \mathbb{E}) \simeq \mathbb{E}_{-p}(X/S) = H^p(X, \mathbb{E}\langle L_{X/S} \rangle)$$

according to Proposition 2.2.4. The preceding proposition obviously extends to that case. Note that the product obtained for smooth  $S$ -schemes is now compatible with étale pullbacks as defined in Proposition 3.2.7, and even with respect to smooth pullbacks.

## 4. THE $\delta$ -HOMOTOPY LERAY SPECTRAL SEQUENCES

### 4.1. The homological version.

**4.1.1.** We are now ready to build the hoped-for spectral sequence.

Let us fix a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$  and consider the following geometric situation:

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ & \searrow \pi & \\ & & S \end{array}$$

where  $f$  is any morphism of schemes, which plays the role of the fibration. We assume  $\pi$  is a separated morphism essentially of finite type and put  $\mathbb{E}_X = \pi^! \mathbb{E}$ . Given a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $B$ , it will be convenient to use the notation:

$$H_p(B, \mathbb{E}) = \mathrm{Hom}_{\mathcal{T}(B)}(\mathbb{1}_B[p], \mathbb{E}).$$

[To be added in the introduction ?](#)

Then one can look at the tower of homological truncation of  $f_*(\mathbb{E}_X)$  for the  $t_\delta$ -homotopy  $t$ -structures, which is the analogue of the Postnikov tower in our situation:

$$\dots \rightarrow \tau_{\geq q+1}^\delta(f_* \mathbb{E}_X) \rightarrow \tau_{\geq q}^\delta(f_* \mathbb{E}_X) \rightarrow \tau_{\geq q-1}^\delta(f_* \mathbb{E}_X) \rightarrow \dots$$

It is standard to deduce from this filtration a spectral sequence. Let us be more precise. We first consider the canonical distinguished triangle:

$$\tau_{\geq q+1}^\delta(f_* \mathbb{E}_X) \rightarrow \tau_{\geq q}^\delta(f_* \mathbb{E}_X) \rightarrow \tau_{=q}^\delta(f_* \mathbb{E}_X) \rightarrow \tau_{\geq q+1}^\delta(f_* \mathbb{E}_X)[1]$$

where:

$$\tau_{=q}^\delta(f_* \mathbb{E}_X) = \tau_{\leq q}^\delta \tau_{\geq q}^\delta(f_* \mathbb{E}_X) = H_q^\delta(f_* \mathbb{E}_X)[q].$$

Applying the functor  $H_{p+q}(B, -)$  to the preceding distinguished triangle, we get a long exact sequence:

$$H_{p+q}(B, \tau_{\geq q+1}^\delta(f_*\mathbb{E}_X)) \xrightarrow{a} H_{p+q}(B, \tau_{\geq q}^\delta(f_*\mathbb{E}_X)) \xrightarrow{b} H_p(B, H_q^\delta(f_*\mathbb{E}_X)) \xrightarrow{c} H_{p+q-1}(B, \tau_{\geq q+1}^\delta(f_*\mathbb{E}_X)).$$

Thus, using the conventions of definition 3.2.1, we get a homological exact couple of degree 2 such that:

$$\begin{aligned} D_{p,q} &= H_{p+q}(B, \tau_{\geq q}^\delta(f_*\mathbb{E}_X)), \\ E_{p,q} &= H_p(B, H_q^\delta(f_*\mathbb{E}_X)). \end{aligned}$$

We deduce from that exact couple our main construction.

**Theorem 4.1.2.** *Consider the above assumptions and notations. The exact couple defined above gives a converging spectral sequence of the form:*

$$E_{p,q}^2(f, \mathbb{E}) = H_p(B, H_q^\delta(f_*\mathbb{E}_X)) \simeq A_p^\delta(B, H_q^\delta(f_*\mathbb{E})) \Rightarrow \mathbb{E}_{p+q}(X/S).$$

The  $\mathbb{E}_2$ -term is concentrated in the range  $p \in [\delta_-(X'), \delta_+(X')]$  and is the homology in degree  $p$  of the  $\delta$ -homological Gersten complex  $C_*^\delta(B, H_q^\delta(f_*\mathbb{E}))$  (see Definition 3.2.4) which takes the form:

$$\dots \rightarrow \bigoplus_{x \in B^{(p)}} \mathbb{E}_{p+q}(X_x/S) \xrightarrow{d_p^G} \bigoplus_{s \in B^{(p-1)}} \mathbb{E}_{p+q-1}(X_s/S) \rightarrow \dots$$

where  $X_x$  is the fiber of  $f$  above the point  $x \in B$ .

Note in particular that the differentials  $d_r$  are trivial for  $r > (\delta_+(X) - \delta_-(X))$ .

*Proof.* The spectral sequence follows from the theory of (derived) exact couples. One can compute the  $E_2$ -term as in the above statement by using Proposition 3.2.7:

$$H_p(B, H_q^\delta(f_*\mathbb{E}_X)) \simeq A_p^\delta(B, H_q^\delta(f_*\mathbb{E}_X)).$$

Going back to the definition, this is the  $p$ -th homology of the following Gersten  $\delta$ -homological complex:

$$\dots \rightarrow \bigoplus_{x \in B^{(p)}} \mathbb{E}_{p+q}(x/B, f_*\mathbb{E}_X) \xrightarrow{d_p^G} \bigoplus_{s \in B^{(p-1)}} \mathbb{E}_{p+q-1}(s/B, f_*\mathbb{E}_X) \rightarrow \dots$$

Given any point  $x \in B^{(p)}$  one considers the cartesian square:

$$\begin{array}{ccc} X_x & \xrightarrow{i_x} & X \\ f_x \downarrow & & \downarrow f \\ \text{Spec}(\kappa_x) & \xrightarrow{x} & B \end{array}$$

The following computation

$$\mathbb{E}_*(x/B, f_*\mathbb{E}_X) = [x! \mathbb{1}[*], f_*\mathbb{E}_X] = [\mathbb{1}[*], x^! f_*\mathbb{E}_X] = [\mathbb{1}[*], f_{x*} i_x^! \mathbb{E}_X] = [\mathbb{1}[*], i_x^! \pi^! \mathbb{E}] = \mathbb{E}_*(X_x/S)$$

Shouldn't we put  $\mathbb{1}_x[*]$  in the second term from the right hand side ? gives the  $E_2$ -term stated in the above proposition.

This implies the spectral sequence converges.  $\square$

**Definition 4.1.3.** The above spectral sequence will be called the (homological)  $\delta$ -homotopy Leray spectral sequence associated with  $f$  and with coefficients in  $\mathbb{E}$ .

The spectral sequence is obviously fully functorial in the  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$ . If we want to take care of Tate twists, we can look at the spectral sequence with coefficients in  $\mathbb{E}(n)$ , which has the following form:

$$E_{p,q}^2 = A_p^\delta(B, H_q^\delta(f_*\mathbb{E})(n)) \Rightarrow \mathbb{E}_{p+q,n}(X/S).$$

Similarly, twists by Thom spaces can be considered.

The spectral sequence moreover enjoys good geometrical functorialities:

**Proposition 4.1.4.** *We consider the following diagram of schemes:*

$$\begin{array}{ccccc} & & Y & & \\ & \swarrow \pi' & \downarrow p & \searrow f' & \\ S & & X & & B \\ & \swarrow \pi & & \searrow f & \end{array}$$

and a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$ . We put  $\mathbb{E}_X = \pi^! \mathbb{E}$  and  $\mathbb{E}_Y = \pi'^! \mathbb{E}$ . Given any point  $x \in B$ , we let  $p_x : Y_x \rightarrow X_x$  the induced morphism on the fibers over a point  $x \in B$ .



- Assume  $p$  is proper. Then there is an adjunction map:

$$\phi_p : f'_* \mathbb{E}_Y = f'_* \pi^! (\mathbb{E}) = f_* p_! p^! \pi^! (\mathbb{E}) \xrightarrow{ad(p_!, p^!)} f_* \pi^! (\mathbb{E}) = f_* (\mathbb{E}_X)$$

which induces a morphism of spectral sequences converging to the indicated map on the abutment:

$$\begin{array}{ccc} A_p^\delta(B, H_q^\delta(f'_* \mathbb{E}_Y)) & \Rightarrow & \mathbb{E}_{p+q}(Y/S) \\ \phi_{p*} \downarrow & & \downarrow p_* \\ A_p^\delta(B, H_q^\delta(f_* \mathbb{E}_X)) & \Rightarrow & \mathbb{E}_{p+q}(X/S). \end{array}$$

Moreover, the map  $\phi_{p*}$  is induced by the following morphism of Gersten complexes:

$$\begin{array}{ccccc} \dots & \longrightarrow & \bigoplus_{x \in B(p)} \mathbb{E}_{p+q}(Y_x/S) & \xrightarrow{d_{p,Y}^G} & \bigoplus_{s \in B(p-1)} \mathbb{E}_{p+q-1}(Y_s/S) & \longrightarrow & \dots \\ & & \downarrow \sum_x p_{x*} & & \downarrow \sum_s p_{s*} & & \\ \dots & \longrightarrow & \bigoplus_{x \in B(p)} \mathbb{E}_{p+q}(X_x/S) & \xrightarrow{d_{p,X}^G} & \bigoplus_{s \in B(p-1)} \mathbb{E}_{p+q-1}(X_s/S) & \longrightarrow & \dots \end{array}$$

- Assume  $p$  is étale. Then there is an adjunction map:

$$\psi_p : f_* (\mathbb{E}_X) = f_* \pi^! (\mathbb{E}) \xrightarrow{ad(p^*, p_*)} f_* p_* p^* \pi^! (\mathbb{E}) = f'_* \pi^! (\mathbb{E}) = f'_* \mathbb{E}_Y$$

which induces a morphism of spectral sequences, converging to the indicated map on the abutment:

$$\begin{array}{ccc} A_p^\delta(B, H_q^\delta(f_* \mathbb{E}_X)) & \Rightarrow & \mathbb{E}_{p+q}(X/S) \\ \psi_{p*} \downarrow & & \downarrow p_* \\ A_p^\delta(B, H_q^\delta(f'_* \mathbb{E}_Y)) & \Rightarrow & \mathbb{E}_{p+q}(Y/S). \end{array}$$

Moreover, the map  $\psi_{p*}$  is induced by the following morphism of Gersten complexes:

$$\begin{array}{ccccc} \dots & \longrightarrow & \bigoplus_{x \in B(p)} \mathbb{E}_{p+q}(X_x/S) & \xrightarrow{d_{p,X}^G} & \bigoplus_{s \in B(p-1)} \mathbb{E}_{p+q-1}(X_s/S) & \longrightarrow & \dots \\ & & \downarrow \sum_x p_{x*} & & \downarrow \sum_s p_{s*} & & \\ \dots & \longrightarrow & \bigoplus_{x \in B(p)} \mathbb{E}_{p+q}(Y_x/S) & \xrightarrow{d_{p,Y}^G} & \bigoplus_{s \in B(p-1)} \mathbb{E}_{p+q-1}(Y_s/S) & \longrightarrow & \dots \end{array}$$

*Proof.* Each point is obtained by using the functoriality of the  $\delta$ -homotopy Leray spectral sequence with respect to the  $\mathcal{T}$ -spectrum  $\mathbb{E}$ . The computation of the map on the abutment follows from the definition of the functoriality of the bivariant theory. The map on the  $E_2$ -terms follows from point (2) of Proposition 3.2.7.  $\square$

*Remark 4.1.5.* Using fundamental classes as defined in [DJK18], and the induced functoriality on bivariant theory, one can extend the contravariant étale functoriality to lci smoothifiable morphisms  $f : Y \rightarrow X$ , up to considering twists by the Thom space of the cotangent complex of  $f$ . We leave the formulation to the reader.

It is possible to describe the filtration on the abutment of the  $\delta$ -homotopy Leray spectral sequence in geometrical terms. The following definition is the obvious generalization of the classical definition of Grothendieck (see [BO74]).

**Definition 4.1.6.** Consider the setting of Paragraph 4.1.1. We define the  $\delta$ -niveau filtration on  $\mathbb{E}_*(X/S)$  relative to  $f$  as:

$${}^\delta N_p^f \mathbb{E}_*(X/S) = \bigcup_{i: Z \rightarrow X, \delta(Z) \leq p} \text{Im}(i_* : \mathbb{E}_*(X \times_B Z/S) \rightarrow \mathbb{E}_*(X/S)).$$

where  $i$  runs over the closed immersions with target  $X$ .

*Remark 4.1.7.* One can also describe this filtration by the following formula:

$${}^\delta N_p^f \mathbb{E}_*(X/S) = \bigcup_{p: Y \rightarrow X, \delta(Y) \leq p} \text{Im}(p_* : \mathbb{E}_*(X \times_B Y/S) \rightarrow \mathbb{E}_*(X/S)).$$

where  $p : Y \rightarrow X$  runs over the proper morphism with target  $X$ . Indeed, such a morphism always factors as

$$Y \xrightarrow{\bar{p}} p(Y) \xrightarrow{i} X$$

where  $p(Y)$  denotes the image of  $Y$ , with its canonical structure of a closed subscheme of  $X$ . As the natural map  $\bar{p}$  is surjective, one has  $\delta(p(Y)) \leq p$ . This concludes.

**4.1.8.** Before stating the computation of the filtration on the  $\delta$ -homotopy spectral sequence, we will introduce notations in order to simplify the proof. We consider again the assumptions and notations of Paragraph 4.1.1.

First, note that by definition, the filtration induced by the  $\delta$ -homotopy spectral sequence on  $\mathbb{E}_*(X/S)$  is defined as:

$${}^\delta F_q^f \mathbb{E}_*(X/S) = \text{Im}(H_*(B, \tau_{\geq q} f_* \mathbb{E}) \rightarrow \mathbb{E}_*(X/S)).$$

Second, we can define the  $\delta$ -niveau filtration relative to  $f$  at the level of schemes, by considering the following ind-schemes:

$$B_{\leq p} = \text{''}\varinjlim\text{''}_{Z_* \in \mathcal{F}(B)} Z_p.$$

Therefore we get a closed ind-immersion  $B_{\leq p} \xrightarrow{i_p} B$ . We can define a kind of complementary immersion by considering the following pro-objects:

$$B_{> p} = \text{''}\varprojlim\text{''}_{Z_* \in \mathcal{F}(B)} (B - Z_p)$$

together with the pro-open immersion  $B_{> p} \xrightarrow{j_p} B$ .

Using these notations, we can consider the localization long exact sequence

$$\mathbb{E}_*(X \times_B B_{\leq p}/S) \xrightarrow{i_{p*}} \mathbb{E}_*(X/S) \xrightarrow{j_p^*} \mathbb{E}_*(X \times_B B_{> p}/S) \xrightarrow{\partial_p} \mathbb{E}_*(X \times_B B_{\leq p}/S)$$

where the third (resp. first, fourth) member(s) is the obvious colimit, using the contravariance of  $\mathbb{E}_*(-/S)$  with respect to open immersion (resp. contravariance with respect to closed immersions). This long exact sequence is nothing else than the filtered colimit of the localisation sequences with respect to the closed immersions  $X \times_B Z_p \rightarrow X$  for  $\delta$ -flags  $Z_*$  of  $B$ .

With these notations, the  $\delta$ -niveau filtration relative to  $f$  simply equals the image of  $i_{p*}$ .

**Proposition 4.1.9.** *Consider the above assumptions 4.1.1. Then for any pair of integer  $(p, n) \in \mathbf{Z}^2$ , one has the following relation:*

$${}^\delta F_p^f \mathbb{E}_n(X/S) = {}^\delta N_{n-p}^f \mathbb{E}_i(X/S),$$

where the left hand-side is the filtration on the abutment of the  $\delta$ -homotopy Leray spectral sequence associated with  $f$  and the right hand side is the  $\delta$ -niveau filtration relative to  $f$ .

*Proof.* Let us put  $\mathbb{E}' = f_* \mathbb{E}_X = f_* \pi^! \mathbb{E}$ . We want to compare the following filtrations:

$$\begin{aligned} {}^\delta F_q^f \mathbb{E}_*(X/S) &= \text{Im}(H_*(B, \tau_{\geq q} \mathbb{E}') \rightarrow H_*(B, \mathbb{E}')), \\ {}^\delta N_p^f \mathbb{E}_*(X/S) &= \text{Im}(H_*(B_{\leq p}, \mathbb{E}') \rightarrow H_*(B, \mathbb{E}')). \end{aligned}$$

So, reasoning with  $\mathbb{E}'$  instead of  $\mathbb{E}$ , we reduce to the case  $X = B = S$ ,  $f = Id_X$ ,  $\pi = Id_X$ .

We will start with the following lemma:

**Lemma 4.1.10.** *Let  $\mathbb{E}$  be a  $\mathcal{T}$ -spectrum over a scheme  $X$ . Then one has the following vanishing:*

$$\begin{aligned} H_n(X_{\leq p}, \tau_{< q}^\delta \mathbb{E}) &= 0 \text{ if } n \geq p + q, \\ H_n(X_{> p}, \tau_{\geq q}^\delta \mathbb{E}) &= 0 \text{ if } n \leq p + q. \end{aligned}$$

This follows from the case  $X = S$ ,  $m = q$ ,  $i = 0$  of Proposition 2.1.10.

We can now consider the following commutative diagram, whose rows and columns are exact sequences:

$$\begin{array}{ccccccc} & & & & H_{n+1}(X_{\leq p}, \tau_{< q}^\delta \mathbb{E}) & & \\ & & & & \downarrow & & \\ H_{n+1}(X_{> p}, \tau_{\geq q}^\delta \mathbb{E}) & \longrightarrow & H_n(X_{\leq p}, \tau_{\geq q}^\delta \mathbb{E}) & \xrightarrow{b} & H_n(X, \tau_{\geq q}^\delta \mathbb{E}) & \longrightarrow & H_n(X_{> p}, \tau_{\geq q}^\delta \mathbb{E}) \\ & & \downarrow a & & \downarrow c & & \\ & & H_n(X_{\leq p}, \mathbb{E}) & \xrightarrow{d} & H_n(X, \mathbb{E}) & & \\ & & \downarrow & & & & \\ & & H_n(X_{\leq p}, \tau_{< q}^\delta \mathbb{E}) & & & & \end{array}$$

According to the preceding lemma, one gets that:

- $a$  is an isomorphism if  $n \geq p + q$ .
- $b$  is an isomorphism if  $n \leq p + q - 1$ , and an epimorphism if  $n = p + q$ .

Therefore, we obtain that  $\text{Im}(c) = \text{Im}(d)$  if  $n = p + q$ . This concludes.  $\square$

*Example 4.1.11.* The preceding proposition recovers the conjecture of Washnitzer proved by Bloch and Ogus in [BO74, 6.9]. This is obtained as follows:

- $S$  is the spectrum of a perfect field  $k$ ,  $X$  is smooth over  $k$ ;
- $\mathbb{E} = \mathbb{E}_{dR}$  is the (ring) spectrum representing De Rham cohomology as in [CD12, §3.1] while  $\mathcal{T} = \text{DM}(-, \mathbf{Q})$ .

The fact the truncation of  $\mathbb{E}_{dR}$  for the homotopy  $t$ -structure agrees with the truncation of the De Rham complex follows from the construction of  $\mathbb{E}_{dR}$  ([CD12, 3.1.5]).

*Remark 4.1.12.* The proof of Bloch and Ogus is less precise and more theoretical. It consists in proving that the niveau spectral sequence for De Rham cohomology agrees with the hypercohomology spectral sequence associated with the De Rham complex.

We do not need such a comparison to prove our result, and our proof is more direct. But however, let us indicate that there is also an underlying comparison of spectral sequences. In fact, it is possible to prove that the  $\delta$ -homotopy Leray spectral sequence defined above agrees from  $E_2$ -on with the  $\delta$ -niveau spectral sequence 3.2.3 of  $X/S$  with coefficients in the spectrum  $f_*(\mathbb{E})$ . The case where  $X = B$ ,  $f = \text{Id}_B$ ,  $S$  is the spectrum of a perfect field was proved in [Bon10] and [Dég14b]. The general case will be treated in future work.

## 4.2. The cohomological version.

**4.2.1.** Let us consider again the assumptions of Paragraph 4.1.1, but with  $X = S$ . So we fix a morphism of schemes:

$$f : X \rightarrow B$$

and a  $\mathcal{T}$ -spectrum  $\mathbb{E}$ .

To get products on the  $\delta$ -homotopy spectral sequence of  $(f, \mathbb{E})$ , we will use a classical construction of Douady (see [Dou59, Section II]) which uses the theory of *spectral diagrams* from [CE99, XV.7] rather than that of exact couples.

Here is how one gets such a spectral diagram underlying the spectral sequence (4.2.6.a). We first define the following tautological functors in a  $\mathcal{T}$ -spectrum  $\mathbb{F}$  over  $B$  (in the end  $\mathbb{F}$  plays the role of  $f_*\mathbb{E}$ ):

$$\begin{aligned} \tau_{\geq -\infty}^{\delta}(\mathbb{F}) &= \mathbb{F}, \quad \tau_{\geq +\infty}^{\delta}(\mathbb{F}) = 0, \\ \tau_{< -\infty}^{\delta}(\mathbb{F}) &= 0, \quad \tau_{< +\infty}^{\delta}(\mathbb{F}) = \mathbb{F}. \end{aligned}$$

Consider the following poset:

$$\mathcal{P} = \{(p, q) \mid p, q \in \mathbf{Z} \cup \{\pm\infty\}, p \leq q\}.$$

such that  $(p, q) \leq (p', q')$  when  $p \leq p'$  and  $q \leq q'$ . Then for any  $(p, q) \in \mathcal{P}$ , we put:

$$\tau_{[p, q]}^{\delta}(\mathbb{F}) := \tau_{\geq p}^{\delta} \tau_{< q}^{\delta}(\mathbb{F}).$$

This defines a contravariant functor  $\tau^{\delta} : \mathcal{P} \rightarrow \mathcal{T}(X)$ .

Using the truncation triangles associated with the  $\delta$ -homotopy  $t$ -structure, we get distinguished triangles for  $p \leq q \leq r$ :

$$\tau_{[q, r]}^{\delta}(\mathbb{F}) \rightarrow \tau_{[p, r]}^{\delta}(\mathbb{F}) \rightarrow \tau_{[p, q]}^{\delta}(\mathbb{F}) \xrightarrow{\partial} \tau_{[q, r]}^{\delta}(\mathbb{F})[1]$$

where the first two maps are given by the functoriality of  $\tau^{\delta}$ . These formulas imply that the contravariant functor:

$$\mathcal{P} \mapsto \mathcal{A}b, (p, q) \mapsto H_*(B, \tau_{[p, q]}^{\delta}(f_*\mathbb{E}))$$

defines a  $\mathbf{Z}$ -graded spectral diagram in the sense of *loc. cit.*<sup>10</sup> Moreover, this construction is obviously functorial in  $\mathbb{E}$ .

Note that for  $\mathbb{F} = f_*(\mathbb{E})$ , the exact couple of Paragraph 4.1.1 is contained in the preceding spectral diagram — it corresponds to couples  $(p, p + 1)$ . Therefore, the spectral sequence associated with the

<sup>10</sup>Property (SP.5) of *loc. cit.* is not immediate. It follows from the convergence of the  $\delta$ -homotopy Leray spectral sequence — see Theorem 4.1.2.

preceding spectral diagram in [CE99] coincides with the  $\delta$ -homotopy Leray spectral sequence of  $(f, \mathbb{E})$ . In particular, we get the following formula:

$$E_{p,q}^r(f, \mathbb{E}) = \text{Im}\left(H_{p+q}(B, \tau_{[p-r, p]}^\delta(f_*\mathbb{E})) \rightarrow H_{p+q}(B, \tau_{[p-1, p+r-1]}^\delta(f_*\mathbb{E}))\right).$$

*Remark 4.2.2.* After the seminal work of Cartan and Eilenberg, spectral diagrams have appeared in other forms in the literature. In the triangulated context, one can refer, mainly for historical purposes, to [Ver96]. The  $\infty$ -categorical context is much more recent, but also much more powerful and satisfactory. The notion of spectral diagram in the  $\infty$ -categorical context is introduced by Lurie in [Lur18, Section 1.2.2], under the name of  $\mathbf{Z}$ -complex. The advantage of using  $\infty$ -category with pushouts (eg: stable  $\infty$ -categories) is that a  $\mathbf{Z}$ -complex is essentially equivalent to a tower of objects: see [Lur18, Lemma 1.2.2.4]. The fact that we can stay in the old-fashioned world of triangulated categories comes from the good behavior of  $t$ -structures.

To get products on the  $\delta$ -homotopy Leray spectral sequence following Douady, we need to refine Proposition 2.1.13 as follows.

**Proposition 4.2.3.** *We consider the preceding notations. Suppose we are given a morphism of  $\mathcal{T}$ -spectra over a scheme  $B$ :*

$$\mu : \mathbb{F} \otimes \mathbb{F}' \rightarrow \mathbb{F}''.$$

*Then for any triple of integers  $(p, q, r)$  such that  $r \geq 0$ , there exists a canonical map:*

$$\tau_{[p, p+r]}^\delta(\mathbb{F}) \otimes \tau_{[q, q+r]}^\delta(\mathbb{F}') \xrightarrow{\varphi_{p,q,r}} \tau_{[p+q, p+q+r]}^\delta(\mathbb{F}'').$$

*Moreover, this pairing satisfies the formulas (SPP 1) and (SPP 2) of [Dou59, II.A, Définition, p. 19-06].<sup>11</sup>*

*Proof.* It is sufficient to treat the case where  $\mathbb{F}'' = \mathbb{F} \otimes \mathbb{F}'$  and  $\mu$  is the identity. As explained in the proof of Proposition 2.1.13, we can also assume that  $\delta \geq 0$  so that the tensor product will respect homologically positive objects (see the end of Paragraph 2.1.11).

According to Proposition 2.1.13, we get a canonical pairing:

$$\tau_{\geq p}^\delta(\mathbb{E}) \otimes \tau_{\geq q}^\delta(\mathbb{F}) \rightarrow \tau_{\geq p+q}^\delta(\mathbb{E} \otimes \mathbb{F}).$$

Let us consider the following diagram:

$$\begin{array}{ccc} \tau_{\geq p}^\delta(\mathbb{E}) \otimes \tau_{\geq q+r}^\delta(\mathbb{F}) & \longrightarrow & \tau_{\geq p+q+r}^\delta(\mathbb{E} \otimes \mathbb{F}) \\ \downarrow & \searrow \text{dashed} & \downarrow \\ \tau_{\geq p}^\delta(\mathbb{E}) \otimes \tau_{\geq q}^\delta(\mathbb{F}) & \longrightarrow & \tau_{\geq p+q}^\delta(\mathbb{E} \otimes \mathbb{F}) \\ \downarrow & \searrow \text{dashed} & \downarrow \\ \tau_{\geq p}^\delta(\mathbb{F}) \otimes \tau_{[q, q+r]}^\delta(\mathbb{F}') & \dashrightarrow & \tau_{[p+q, p+q+r]}^\delta(\mathbb{F} \otimes \mathbb{F}') \end{array}$$

where the solid arrows form a commutative diagram. The two columns of this diagram come from distinguished triangles. First this implies that the slanted dotted arrow is zero. Second, it implies that there exists a unique dotted horizontal arrow making the bottom square commutative.

Then we consider the following diagram:

$$\begin{array}{ccc} \tau_{\geq p+r}^\delta(\mathbb{E}) \otimes \tau_{[q, q+r]}^\delta(\mathbb{F}) & & \\ \downarrow & \searrow \text{dashed (1)} & \\ \tau_{\geq p}^\delta(\mathbb{E}) \otimes \tau_{[q, q+r]}^\delta(\mathbb{F}) & \longrightarrow & \tau_{[p+q, p+q+r]}^\delta(\mathbb{E} \otimes \mathbb{F}) \\ \downarrow & \searrow \text{dashed (2)} & \\ \tau_{[p, p+r]}^\delta(\mathbb{F}) \otimes \tau_{[q, q+r]}^\delta(\mathbb{F}') & & \end{array}$$

As the tensor product is right  $t_\delta$ -exact, the object  $\tau_{\geq p+r}^\delta(\mathbb{E}) \otimes \tau_{[q, q+r]}^\delta(\mathbb{F})$  is in homological degree  $\geq p + q + r$ . By definition, the object  $\tau_{[p+q, p+q+r]}^\delta(\mathbb{E} \otimes \mathbb{F})$  is in homological degree  $< p + q + r$ . So the map labeled (1) must be 0. Therefore, the map (2) must exist and gives the existence of the pairing

<sup>11</sup>(SPP1) is the functoriality of this pairing with respect to  $(p, p+r)$  (resp.  $(q, q+r)$ ) considered as an object of  $\mathcal{P}$ . (SPP2) is the Leibniz rule for the boundary map of type  $\partial$ .

$\varphi_{p,q,r}$ . The uniqueness of the construction then guarantees Douady's coherence properties (SPP1) and (SPP2).  $\square$

**4.2.4.** Therefore, one can now apply the construction of [Dou59, Th. II]. Going back to the setting of Paragraph 4.2.1, and to our morphism  $f : X \rightarrow B$ , we consider a pairing of  $\mathcal{T}$ -spectra over  $X$ :

$$\mu : \mathbb{E} \otimes \mathbb{E}' \rightarrow \mathbb{E}''.$$

As  $f_*$  is weakly monoidal (left adjoint of a monoidal functor), we get a pairing:

$$\mu : f_*(\mathbb{E}) \otimes f_*(\mathbb{E}') \rightarrow f_*(\mathbb{E}'').$$

Applying the preceding proposition and Douady's construction, we get a pairing of spectral sequences:

$$E_r^{p,q}(f, \mathbb{E}) \otimes_{\mathbf{Z}} E_r^{s,t}(f, \mathbb{E}') \rightarrow E_r^{p+s, q+t}(f, \mathbb{E}'')$$

such that the differentials  $d_r$  satisfy the usual Leibniz rule.

To comply with usage, when considering products, we renumber the  $\delta$ -homological spectral sequence cohomologically, and use our cohomological conventions (see in particular Paragraph 3.3.1). In the end, we get the following result.

**Theorem 4.2.5.** *Suppose  $f : X \rightarrow B$  is a morphism of schemes, and let  $\mathbb{E}_X$  be a  $\mathcal{T}$ -spectrum over  $X$ . The constructions of Paragraph 4.2.1 and 4.2.4 yield a convergent spectral sequence of the form:*

$$E_2^{p,q}(f, \mathbb{E}_X) = A_\delta^p(B, H_\delta^q(f_*\mathbb{E}_X)) \Rightarrow H^{p+q}(X, \mathbb{E}_X).$$

*If  $\mathbb{E}_X$  admits a ring structure, then the spectral sequence is equipped with a convergent multiplicative structure; the product on the  $E_2$ -term is induced by the construction of Example 3.3.3.*

**4.2.6.** An important point for the applications is the ability to compute the  $E_2$ -term of the spectral sequence. Recall from Theorem 4.1.2 (with  $X = S$  and with cohomological conventions) that  $E_2^{p,q}(f, \mathbb{E})$  is the  $p$ -cohomology of the Gersten  $\delta$ -cohomological complex  $C_\delta^*(B, H_\delta^q f_*\mathbb{E})$ . Moreover, for  $p \in \mathbf{Z}$ , one has:

$$(4.2.6.a) \quad C_\delta^p(B, H_\delta^q f_*\mathbb{E}) = \bigoplus_{x \in B^{(p)}} H^{p+q}(X_x/X, \mathbb{E}_X).$$

To formulate the next result, let us introduce a classical notation. Consider a point  $x \in B$ , identified with the map:  $x : \text{Spec}(\kappa_x) \rightarrow B$ . Let  $Z$  be the reduced closure of its image in  $B$ . Then one can factor the morphism  $x$  as the following composite:

$$\text{Spec}(K) \xrightarrow{j} Z \xrightarrow{i} X$$

where  $i$  is a closed immersion and  $j$  a pro-open immersion. One puts:  $x^! = j^*i^!$ .

**Proposition 4.2.7.** *Consider the assumptions of the above proposition, and the above notation. Then one has the following computations.*

- Assume that  $f$  is smooth and  $\mathbb{E}_X = f^*\mathbb{E}$  for a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $B$ . Then for any  $p \in \mathbf{Z}$ , one has:

$$C_\delta^p(B, H_\delta^q f_*\mathbb{E}) = \bigoplus_{x \in B^{(p)}} H^{p+q}(X_x, \mathbb{E}_x)$$

where  $\mathbb{E}_x = x^!\mathbb{E}$ . *This form of the  $E_1$ -term of the spectral sequence is much more useful for applications.*

- Assume that  $X$  is regular, the fibers of  $f$  are regular and  $\mathbb{E}_X$  is  $X$ -pure (Definition 2.2.2). For any  $p \in \mathbf{Z}$ , one has:

$$C_\delta^p(B, H_\delta^q f_*\mathbb{E}) = \bigoplus_{x \in B^{(p)}} H^{p+q}(X_x, \mathbb{E}(-N(X_x/X)))$$

where  $N(X_x/X)$  denote the normal bundle of the regular closed immersion  $\nu_x : X_x \rightarrow X$ .

*Proof.* Consider the first property. Given a point  $x \in B$ , we introduce, as in the proof of Theorem 4.1.2, the following cartesian diagram:

$$\begin{array}{ccc} X_x & \xrightarrow{i_x} & X \\ f_x \downarrow & & \downarrow f \\ \text{Spec}(\kappa_x) & \xrightarrow{x} & B \end{array}$$

Note that, as  $f$  is smooth, one gets a canonical isomorphism of functors:  $i_x^! f^* = f_x^* x^!$  (using the notation introduced just before the statement of the proposition for the shriek functors). Then the following computation concludes the proof:

$$H^{p+q}(X_x/X, \mathbb{E}_X) = [\mathbb{1}_{X_x}, i_x^! f^*(\mathbb{E})[p+q]] = [\mathbb{1}_{X_x}, f_x^* x^!(\mathbb{E})[p+q]] = [\mathbb{1}_x, \mathbb{E}_x[p+q]].$$

The second property is a direct consequence of the form (4.2.6.a) of the Gersten complex and of Proposition 2.2.4.  $\square$

Again, one has good functorial properties of this spectral sequence, for example in the (ring)  $\mathcal{T}$ -spectrum  $\mathbb{E}$ . This can be applied to get the following geometrical functorial property.

**Proposition 4.2.8.** *Consider a commutative diagram:*

$$\begin{array}{ccc} Y & \xrightarrow{q} & X \\ & \searrow g & \swarrow f \\ & & B \end{array}$$

and a ring  $\mathcal{T}$ -spectrum  $\mathbb{E}$ . Let us put  $\mathbb{E}_Y = q^* \mathbb{E}$ . Then there exists a morphism of converging spectral sequences:

$$\begin{array}{ccc} E_2^{p,q}(f, \mathbb{E}) = A_\delta^p(B, H_\delta^q(f_* \mathbb{E})) & \Longrightarrow & H^{p+q}(X, \mathbb{E}) \\ p^* \downarrow & & \downarrow q^* \\ E_2^{p,q}(f', \mathbb{E}') = A_\delta^p(B, H_\delta^q(g_* \mathbb{E}')) & \Longrightarrow & H^{p+q}(Y, \mathbb{E}_Y) \end{array}$$

where  $q^*$  is the usual pullback on cohomology.

The morphism of spectral sequences is simply obtained using the functoriality in  $\mathbb{E}$  with respect to the following map:

$$\mathbb{E} \rightarrow q_* q^*(\mathbb{E}) = q_* \mathbb{E}_Y.$$

### 4.3. Remarkable properties of homotopy modules.

**4.3.1.  $\delta$ -effectivity.**– Consider the situation of Paragraph 4.1.1:  $\mathbb{E}$  is a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$  and one looks at morphisms:

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ & \searrow \pi & \\ & & S \end{array}$$

where  $\pi$  is separated essentially of finite type, and put  $\mathbb{E}_X = \pi^! \mathbb{E}$ . We assume in addition:

- (1)  $\mathbb{E}_X$  and  $\mathbb{1}_B$  are  $\delta$ -effective (Definition 3.1.6).
- (2)  $f$  is proper.

Then the first and second assumptions imply  $f_*(\mathbb{E}_X) = f_!(\mathbb{E}_X)$  is  $\delta$ -effective (see Paragraph 3.1.9). Moreover, one can compute the  $E_2$ -term of the  $\delta$ -homotopy Leray spectral sequence as follows:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}(B)}(\mathbb{1}_B[p], H_q^\delta f_* \mathbb{E}) &= \mathrm{Hom}_{\mathcal{T}(B)}(s \mathbb{1}_B[p], H_q^\delta s f_* \mathbb{E}) \\ &= \mathrm{Hom}_{\mathcal{T}^{\delta\text{-eff}}(B)}(\mathbb{1}_B[p], w H_q^\delta s f_* \mathbb{E}) \\ &= \mathrm{Hom}_{\mathcal{T}^{\delta\text{-eff}}(B)}(\mathbb{1}_B[p], H_q^{\delta\text{-eff}} w s f_* \mathbb{E}) \\ &= \mathrm{Hom}_{\mathcal{T}^{\delta\text{-eff}}(B)}(\mathbb{1}_B[p], H_q^{\delta\text{-eff}} f_* \mathbb{E}). \end{aligned}$$

The first identification uses the assumption (1), the second the adjunction  $(s, w)$ , the third the fact  $w$  is  $t_\delta$ -exact and the last one the fact  $s$  is fully faithful.

We have obtained the following remarkable result.

**Proposition 4.3.2.** *Under the preceding assumptions, the  $\delta$ -homotopy Leray spectral sequence can be written:*

$$E_{p,q}^2 = A_p^\delta(B, H_q^{\delta\text{-eff}} f_* \mathbb{E}_X) \Rightarrow \mathbb{E}_{p+q}(X/S).$$

If  $X = S$ , we can also consider the cohomological form of the  $\delta$ -homotopy Leray spectral sequence:

$$E_2^{p,q} = A_\delta^p(B, H_{\delta\text{-eff}}^q f_* \mathbb{E}) \Rightarrow \mathbb{E}^{p+q}(X).$$

*Example 4.3.3.* The interest of the preceding proposition is that it is easier to get bounded objects with respect to the  $\delta$ -effective category: see Example 3.1.8.

Let us consider either the homotopical or motivic case. We consider a proper morphism  $f : X \rightarrow B$  of schemes essentially of finite type over  $k$  such that  $X$  is  $k$ -smooth. We let our dimension functions be induced by that computed relative to  $k$ . Then  $\mathbb{1}_X(n)$  for  $n \geq 0$  and  $\mathbb{1}_B$  are both  $\delta$ -positive.

Moreover, if we assume  $X$  is of pure dimension  $d$ , so that  $\delta(X) = d$  according to the preceding choice, then the constant object  $\mathbb{1}_X$  is concentrated in homological degree  $d$ . Therefore the preceding spectral sequence, in its homological form, is concentrated in degrees  $q \in [d, d + \dim(f)]$ .

The following definition will serve as an analogue of the property of being locally constant **Shouldn't we write constant instead of locally constant ?** for sheaves, or that of admitting trivial monodromy for local systems.

**Definition 4.3.4.** Consider a separated morphism  $p : B \rightarrow S$  of finite type. We will say that a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $B$  is  $S$ -simple if there exists a  $\mathcal{T}$ -spectrum  $\mathbb{E}_0$  over  $S$  and an isomorphism  $\mathbb{E} \simeq p^!\mathbb{E}_0$ .

*Remark 4.3.5.* This notion will come into play mainly when  $S$  is the spectrum of a base field  $k$ , or, in the motivic case, of a base Dedekind ring  $A$ . For us, the interest comes in the study of the  $\delta$ -homotopy Leray spectral sequence, from Definition 4.1.3. Usually, we will start with a  $k$ -simple  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $S$  – thus  $\mathbb{E}_X$  is  $k$ -simple. If we know that the  $\delta$ -homotopy module  $H_q^\delta f_* \mathbb{E}$  over  $B$  is  $k$ -simple, then the  $E_2$ -term depends only on the homotopy type (resp. motive) of  $B$  over  $k$ .

We will give examples in Section 5.

*Remark 4.3.6.* (1) Obviously,  $S$ -simple  $\mathcal{T}$ -spectra over  $B$  are stable under suspensions, twists and even tensor products by Thom spaces of virtual bundles over  $B$  that come from  $S$ . They are not stable under extensions or even direct factors in general.

(2) Let  $f : B \rightarrow S$  be smooth and consider the motivic case. Then  $f^! = f^*(d)[2d]$  where  $d$  is the relative dimension of  $f$ . One deduces that  $S$ -simple motives over  $B$  are stable under tensor products. The same remark applies to oriented spectra over  $B$ , but not to arbitrary spectra.

(3) Note that if  $k$  is a field,  $k$ -simple over a scheme  $B$  implies  $B$ -pure (Definition 2.2.2).

(4) One could say that a  $\delta$ -homotopy module over  $B$  is  $S$ -constructible if it is obtained by a finite number of extensions and direct factors of **simple  $S$ -modules ?** within the abelian category of  $\delta$ -homotopy modules over  $B$ .

This notion is not so well-behaved, compared to its model for torsion étale sheaves, as it lacks some notion of finiteness. It would be desirable to have some good finiteness condition on  $S$ -simple  $\delta$ -homotopy modules. But even when  $S$  is a base field, it is not obvious to find such a finiteness condition; see [Dég11, Rem. 6.7] for further discussion.

Note the following fact.

**Lemma 4.3.7.** Consider a separated morphism  $f : B \rightarrow S$  of finite type, and a  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $B$ . Recall we assume that  $S$  and  $B$  each admit a dimension function. Let us fix a dimension function  $\delta$  on  $B$ .

If  $\mathbb{E}$  is  $S$ -simple, then for any  $p \in \mathbf{Z}$ , the  $\delta$ -homotopy module  $H_q^\delta(\mathbb{E})$  over  $B$  is  $S$ -simple.

*Proof.* We just need to be precise about dimension functions. By additivity, we can assume that  $B$  is connected. Let us fix  $\delta_0$  an arbitrary dimension function on  $S$ . Then the dimension function on  $B$  induced by  $\delta_0$  satisfies the relation  $\delta_0 = \delta + n$  for a fixed integer  $n \in \mathbf{Z}$  (see Remark 2.1.6). In particular,  $H_q^{\delta_0}(\mathbb{E}) = H_{q+n}^\delta(\mathbb{E})$  according to *loc. cit.*

By assumption,  $\mathbb{E} = f^!\mathbb{E}_0$ . it remains to apply the fact  $f^!$  is  $t_{\delta_0}$ -exact to conclude:

$$H_q^\delta(\mathbb{E}) = H_{q-n}^{\delta_0}(\mathbb{E}) \simeq H_{q-n}^{\delta_0}(f^!\mathbb{E}_0) = f^!H_{q-n}^{\delta_0}(\mathbb{E}_0).$$

□

*Example 4.3.8.* In the motivic case, if  $f : X \rightarrow S$  is smooth, the constant object  $\mathbb{1}_X$  is  $S$ -simple.

Similarly, the classical oriented ring spectra  $\mathbf{HR}_X$ ,  $\mathbf{KGL}_X$  ( $K$ -theory),  $\mathbf{MGL}_X$  (algebraic cobordism),  $\hat{\mathbf{H}}\mathbf{Q}_{\ell,X}$  (representing continuous  $\ell$ -adic cohomology) are all  $S$ -simple.

On the contrary, spectra representing non orientable theories such as  $\mathbf{HR}_X$ ,  $\mathbf{KQ}_X$  (hermitian  $K$ -theory), or the sphere spectrum  $\mathbf{S}_X^0$ , are not  $S$ -simple except when the tangent bundle of  $f$  is trivial (or is the pullback of a vector bundle over  $S$ ).

Note finally that when  $f : X \rightarrow S$  is arbitrary separated of finite type, the main result of [Jin18], Theorem 1.3, tells us that the spectrum  $\mathbf{GGL}_X$  representing algebraic  $G$ -theory in  $\mathrm{SH}(X)$  is  $S$ -simple.

As expected, here is the generic case where the homotopy modules appearing in the  $E_2$ -term of the  $\delta$ -homotopy Leray spectral sequence are simple.

**Proposition 4.3.9.** *Let  $B$  and  $F$  be  $S$ -schemes separated essentially of finite type. We consider the trivial fibration  $f : X = F \times_S B \rightarrow B$ . Then for any  $S$ -simple  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $X$ , the  $\mathcal{T}$ -spectrum  $f_*\mathbb{E}$  is  $S$ -simple.*

*Proof.* This is a trivial exercise on the six functors formalism. We consider the cartesian square:

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ q \downarrow & \Delta & \downarrow p \\ F & \xrightarrow{f_0} & S. \end{array}$$

Then we get an associated exchange isomorphism:  $Ex(\Delta_*^!) : p^!f_{0*} \xrightarrow{\sim} f_*q^!$ . By assumption, there exists a  $\mathcal{T}$ -spectrum  $\mathbb{E}_0$  over  $S$  such that  $\mathbb{E} = h^!\mathbb{E}_0$ ,  $h = pf = f_0q$ . Thus we can do the computation:

$$f_*\mathbb{E} = f_*h^!\mathbb{E}_0 = f_*q^!f_0^!\mathbb{E}_0 \xrightarrow{Ex(\Delta_*^!)^{-1}} p^!f_{0*}f_0^!\mathbb{E}_0.$$

This concludes the proof.  $\square$

## 5. APPLICATIONS

In this section, we study the  $\delta$ -homotopy Leray spectral sequence in various simple cases and include some applications. We first study morphisms with  $\mathbf{A}^1$ -contractible fibers, and then morphisms with fibers that are “ $\mathbf{A}^1$ -homology spheres”. We conclude with some applications to relative cellular spaces.

**5.1. Morphisms with  $\mathbf{A}^1$ -contractible fibers.** We first analyze the case where the fibers are  $\mathbf{A}^1$ -homotopically “as simple as possible”, i.e.,  $\mathbf{A}^1$ -contractible.

**Proposition 5.1.1.** *Suppose  $f : X \rightarrow B$  be a smooth morphism with  $\mathbf{A}^1$ -contractible fibers. Let  $M$  be a spectrum (resp. a motive) that is a  $\delta$ -homotopy module over  $B$  and set  $M_X := f^*M$ . Then,*

$$H_\delta^q f_* M_X = \begin{cases} M & q = 0, \\ 0 & q \neq 0. \end{cases}$$

*In particular, the  $\delta$ -homotopy Leray spectral sequence is concentrated on the line  $q = 0$ , and thus degenerates; pullback along  $f$  yields identifications (in cohomological notation) of the form:*

$$H^p(X, M_X) = H^p(B, H_\delta^0 f_*(M_X)) = H^p(B, M).$$

*Proof.* Consider the canonical map

$$H_\delta^q(M) \rightarrow H_\delta^q(f_* M_X).$$

We need only to prove it is an isomorphism on fiber  $\delta$ -homology (Remark 3.1.3). Evaluated at a point  $x : \text{Spec } K \rightarrow B$  and in  $\mathbf{G}_m$ -degree  $r \in \mathbf{Z}$ ,  $p = \delta(x)$ , the above map takes the form:

$$H^{q+r-2p, r-p}(x/B, M) \rightarrow H^{q+r-2p, r-p}(x/B, f_* f^* M) = H^{q+r-2p, r-p}(X_x/X, f^* M)$$

According to the computation of Proposition 4.2.7(1), which we can apply as  $f$  is smooth, one obtains:

$$H^{q+r-2p, r-p}(X_x/X, f^* M) = H^{q+r-2p, r-p}(X_x, M_x).$$

In particular, if we set  $M_x = i_x^! M$  (using the notation of Paragraph 4.2.6), the above map is isomorphic to the pullback map:

$$f_x^* : H^{q+r-2p, r-p}(\text{Spec}(\kappa_x), M_x) \rightarrow H^{q+r-2p, r-p}(X_x, M_x),$$

which is itself an isomorphism as  $f_x$  is assumed to be an  $\mathbf{A}^1$ -weak equivalence. Note also that, because the spectral sequence is functorial with respect to pullbacks (Proposition 4.2.8), we know that the identification of the statement arises from the pullback map along  $f$ . The remaining assertions are straightforward.  $\square$

*Example 5.1.2.* There exist many examples of morphisms satisfying the hypotheses of Proposition 5.1.1. For concreteness, assume  $k$  is a field having characteristic 0. In [AD07, Theorem 1.3], it is shown that there exist connected  $k$ -schemes  $B$  of arbitrary dimension and smooth morphisms  $f : X \rightarrow B$  of relative dimension  $\geq 6$  whose fibers are  $\mathbf{A}^1$ -contractible and such that fibers over distinct  $k$ -points of  $B$  are pairwise non-isomorphic. Because of the last point, such morphisms  $f$  are *not* Zariski locally trivial. These results were improved in [DF18, DPØ18], where it was shown that one could build  $f$  as above that



are smooth of relative dimension  $\geq 3$ . On the other hand, it is expected that the only  $\mathbf{A}^1$ -contractible smooth  $k$ -scheme of dimension 2 is  $\mathbf{A}^2$ , and a long-standing conjecture of Dolgachev–Weisfeiler [VD74, 3.8.5] states that every flat morphism of (say) smooth schemes with all fibers isomorphic to affine space is Zariski locally trivial.

Suppose  $f$  is a smooth morphism  $f$  with  $\mathbf{A}^1$ -contractible fibers. It is not clear to the authors whether such an  $f$  is *unstably* an  $\mathbf{A}^1$ -weak equivalence without imposing further hypotheses (e.g., that  $f$  is Nisnevich locally trivial). Nevertheless, the following remark demonstrates that such  $f$  are *stable*  $\mathbf{A}^1$ -weak equivalences in a strong sense, which makes Proposition 5.1.1 somewhat unsurprising.

*Remark 5.1.3.* Given a map  $f : X \rightarrow B$  as in Proposition 5.1.1, one can directly show that the adjunction map:

$$Id \rightarrow f_* f^*$$

is an isomorphism of functors. Indeed, we can use the same argument as above and the fact the family of functors  $x^! : \mathrm{SH}(B) \rightarrow \mathrm{SH}(\mathrm{Spec}(\kappa(x)))$ , indexed by schematic points  $x \in B$ , is conservative.<sup>12</sup> In particular, one does not need to work over a base field; nor does one need to invert any integers.

As  $F$  is smooth, the natural transformation  $f_{\#} f^* \rightarrow Id$  is an isomorphism, and that tells us, in particular, that the map

$$\Sigma^\infty X_+ = f_{\#} f^*(\mathbb{1}_B) \rightarrow \mathbb{1}_B = \Sigma^\infty B_+$$

is an isomorphism in  $\mathrm{SH}(B)$ . In fact,  $f$  is a *universal* stable  $\mathbf{A}^1$ -weak equivalence since  $f$  is a stable  $\mathbf{A}^1$ -weak equivalence and the property mentioned above remains true after base change.

**5.2. Gysin and Wang sequences.** The relative Atiyah–Hirzebruch spectral sequence takes a particularly simple form when the Serre fibration  $F \rightarrow X \rightarrow B$  under consideration has either the property that  $B$  is a sphere, or  $F$  is a homology sphere and the associated local system on  $B$  is trivial. In those cases, the spectral sequences yields the so-called Wang or Gysin long exact sequences. The fact that the differentials in the cohomological form of the spectral sequence are derivations yields additional structure in these long exact sequences that is frequently useful in computations.

*$\mathbf{A}^1$ -homology spheres.* In motivic homotopy theory, there are many smooth schemes over a base  $S$  that have the stable  $\mathbf{A}^1$ -homotopy type of a motivic sphere  $\Sigma^i \mathbf{G}_m^{\wedge j}$ . For example, Morel–Voevodsky showed that  $\mathbf{A}^n \setminus 0_S$  has the  $\mathbf{A}^1$ -homotopy type of  $\Sigma^{n-1} \mathbf{G}_m^{\wedge n}$ . Likewise, the split smooth affine quadric  $Q_{2n-1}$  defined by the hypersurface  $\sum_i x_i x_{n+i} = 1$  in  $\mathbf{A}_S^{2n}$  is  $\mathbf{A}^1$ -weakly equivalent to  $\mathbf{A}^n \setminus 0$  and [ADF17, Theorem 2] demonstrates that the smooth affine quadric  $Q_{2n}$  defined by the equation  $\sum_i x_i x_{n+i} = x_{2n+1}(1 - x_{2n+1})$  in  $\mathbf{A}_S^{2n+1}$  is a model of  $\Sigma^n \mathbf{G}_m^{\wedge n}$ . On the other hand, it is known that  $\Sigma^i \mathbf{G}_m^{\wedge j}$  has no smooth model if  $i > j$  and conjecturally has no smooth model if  $i < j - 1$ . We now formulate a definition of “homology sphere” in  $\mathbf{A}^1$ -homotopy theory.

**Definition 5.2.1.** We say that a (pointed) smooth  $S$ -scheme  $X$  is an  $\mathbf{A}^1$ -homology sphere if there exist integers  $p, q, r \geq 0$  and an  $\mathbf{A}^1$ -weak equivalence  $\Sigma^r X \sim \Sigma^p \mathbf{G}_m^{\wedge q}$ .

The next proposition gives a construction of many  $\mathbf{A}^1$ -homology spheres, at least over a field.

**Proposition 5.2.2.** *Fix a smooth base scheme  $S$ , and an  $\mathbf{A}^1$ -contractible smooth  $S$ -scheme  $X$ . Assume there exists a closed immersion of  $S$ -schemes  $x : S \rightarrow X$  with trivial normal bundle  $\nu_{x/X}$ .*

- (1) *A choice of trivialization of  $\nu_{x/X}$  determines an  $\mathbf{A}^1$ -weak equivalence  $\Sigma X \setminus x \cong \Sigma \mathbf{A}^d \setminus 0$ , i.e.,  $X \setminus x$  is an  $\mathbf{A}^1$ -homology sphere.*
- (2) *If  $S = \mathrm{Spec}(k)$  for a perfect field  $k$ ,  $X$  has dimension  $d \geq 3$  and is  $\mathbf{A}^1$ -connected, then  $X \setminus x$  is  $\mathbf{A}^1$ -simply connected as well.*

*Proof.* For the first point, note that there is a cofiber sequence of the form

$$X \setminus x \longrightarrow X \longrightarrow X/(X \setminus x) \longrightarrow \Sigma X \setminus x \longrightarrow \cdots$$

Because the  $\mathbf{A}^1$ -local model structure is left proper, the fact that  $X$  is  $\mathbf{A}^1$ -contractible implies that the map  $Th(\nu_{x/X}) \rightarrow \Sigma X \setminus x$  is an  $\mathbf{A}^1$ -weak equivalence. Under the assumptions on  $S$ , there is a purity isomorphism  $X/(X \setminus x) \cong Th(\nu_{x/X})$  and a choice of trivialization of  $\nu_{x/X}$  determines an  $\mathbf{A}^1$ -weak equivalence  $Th(\nu_{x/X}) \xrightarrow{\sim} \Sigma^d \mathbf{G}_m^{\wedge d}$ , which can be written  $\Sigma \mathbf{A}^d \setminus 0$ .

<sup>12</sup>To define  $x^!$ , one uses the factorization:

$$\mathrm{Spec} \kappa(x) \xrightarrow{j} Z \xrightarrow{i} X,$$

where  $Z$  is the closure of  $x$  in  $X$  with its reduced induced scheme structure, and set  $x^! = j^* i^!$ . The conservativity property is obtained using the continuity property of  $\mathrm{SH}$  and the localization property.

For the second point, since  $X \setminus x$  is  $\mathbf{A}^1$ -connected,  $X \setminus x$  has a non-empty set of  $k$ -points by the unstable 0-connectivity theorem [MV99, §2 Corollary 3.22]. Fix a base  $k$ -point in  $X \setminus x$ , and point  $X$  by its composite with the open immersion  $X \setminus x \rightarrow X$ . Finally, since  $d \geq 3$ , we may appeal to [AD09, Theorem 4.1] to conclude that the morphism  $\pi_1^{\mathbf{A}^1}(X \setminus x) \rightarrow \pi_1^{\mathbf{A}^1}(X)$  is an isomorphism. Since  $X$  is  $\mathbf{A}^1$ -contractible, the latter sheaf is trivial and  $X \setminus x$  is thus  $\mathbf{A}^1$ -simply connected.  $\square$

*Remark 5.2.3.* If  $k$  is a field, and  $X$  is furthermore affine, then  $X \setminus x$  is isomorphic to  $\mathbf{A}^d \setminus 0$  if and only if  $X$  is isomorphic to  $\mathbf{A}^d$ . Indeed, if there exists an isomorphism from  $X \setminus x$  to  $\mathbf{A}^d \setminus 0$ , then normality of  $X$  allows one to extend this isomorphism to an isomorphism of  $X$  with  $\mathbf{A}^d$ ; the other implication is immediate.

*Example 5.2.4.* If  $X$  is the smooth affine threefold defined by  $x + x^2y + z^2 + t^3 = 0$ , then the main result of [DF18] implies that  $X$  is  $\mathbf{A}^1$ -contractible, at least if  $k$  is an infinite field. In that case, for any extension  $L/k$ , B. Antieau observed that  $X$  is connected by chains of affine lines (see [DPØ18, Example 2.28] for a proof). If  $L$  is an infinite field, we may always assume our chains avoid a codimension  $\geq 2$  subset, and in particular it follows that  $X \setminus x$  is connected by chains of affine lines; it follows that,  $X \setminus x$  is  $\mathbf{A}^1$ -connected. Thus, if  $k$  is infinite and perfect,  $X \setminus x$  satisfies the hypotheses of the proposition and yields an “exotic motivic sphere”. In dimension  $d \geq 4$ , the examples in [AD07] or [ADF17] also satisfy the hypotheses of the theorem, at least over an infinite base field.

*Gysin and Wang sequences for homotopy modules.* Our goal now is to analyze the relative coniveau spectral sequence for  $f : X \rightarrow B$  a smooth morphism where  $B$  is an  $\mathbf{A}^1$ -homology sphere in the sense above; the outcome will be a version of the Gysin sequence. We begin by observing that from the appropriate cohomological standpoint,  $\mathbf{A}^1$ -homology spheres behave in a fashion analogous to spheres in classical homotopy theory, i.e., cohomology with “locally constant coefficients” is concentrated in precisely 2 degrees. To make this precise, assume we work over a field, and let  $\delta$  be the usual dimension function relative to  $k$ . We consider homotopy modules  $M$  over  $k$  such that  $f_*M$  is  $k$ -simple in the sense of Definition 4.3.4.

**Lemma 5.2.5.** *Assume  $k$  is a field,  $X$  is a (pointed) smooth  $k$ -scheme that is an  $\mathbf{A}^1$ -homology sphere (with  $p, q \geq 1$ ). For any homotopy module  $M$  over  $k$ ,*

$$H_\delta^i(f_*M_X) = \begin{cases} M & \text{if } i = 0 \\ M_{-q} & \text{if } i = p - r \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* This result follows essentially immediately from [AF14, Lemma 4.5]. However, to keep the presentation self-contained, we sketch a proof in the spirit of this paper: one proceeds along the same lines as the proof of Proposition 5.1.1 and appeals to Proposition 4.2.7(1). The statement for  $i = 0$  follows from the fact that  $X$  is pointed. Since  $f$  is smooth, and  $M_X = f^*M$ , the Gersten complex for  $H_\delta^i(f_*M_X)$  on  $\text{Spec}(k)$  takes the form

$$C_\delta^j(\text{Spec}(k), H_\delta^i(f_*M_X)) = \bigoplus_{x \in B^{(j)}} H^{i+j}(X, M).$$

By the suspension isomorphism we have identifications in *reduced* cohomology of the form:

$$\tilde{H}^{i+j}(X, M) \cong \tilde{H}^{i+j+r}(\Sigma^r X, M) \cong \tilde{H}^{i+j+r}(\Sigma^p \mathbf{G}_m^{\wedge q}, M) \cong H^{i+j+r-p}(\mathbf{G}_m^{\wedge q}, M).$$

Since  $M$  is a homotopy module, the last group vanishes if  $i + j + r - p$  is not equal to 0 and is precisely  $M_{-q}(\text{Spec}(k))$  if  $i + j = p - r$ . The result then follows by unwinding the definitions.  $\square$

**Proposition 5.2.6.** *Assume  $k$  is a field and suppose  $f : X \rightarrow B$  is a Zariski locally trivial smooth morphism of  $k$ -varieties with fibers  $F$  that are  $\mathbf{A}^1$ -homology sphere (where  $\Sigma^r X \sim \Sigma^p \mathbf{G}_m^{\wedge q}$  with  $p, q \geq 1$ ). If  $M$  is a homotopy module, then*

$$H_\delta^j(f_*M_X) = \begin{cases} M & \text{if } j = 0 \\ M_{-q} & \text{if } j = p - r, \\ 0 & \text{otherwise.} \end{cases}$$

(in particular,  $H_\delta^j(f_*M_X)$  is  $k$ -simple) and there is a long exact “Gysin” sequence of the form

$$\dots \rightarrow H^i(B, M) \rightarrow H^i(X, M) \rightarrow H^{i+r-p}(B, M_{-q}) \rightarrow H^{i+1+r-p}(B, M) \rightarrow \dots$$

*Proof.* By Theorem 4.2.5, there is a spectral sequence with

$$E_2^{i,j} = A^i(B, H_\delta^j(f_*M_X)) \implies H^{i+j}(X, M).$$

Granted the first statement, the spectral sequence is concentrated in two rows, and the existence of the resulting long exact sequence is immediate. Thus, it remains to prove the vanishing statement. If  $B$  is a field, the computation of  $H_\delta^j(f_*M_X)$  is simply Lemma 5.2.5. The general case reduces to that statement as follows.  $\square$

**Proposition 5.2.7.** *Assume  $k$  is a field, and  $f : X \rightarrow B$  is a smooth morphism where  $B$  is an  $\mathbf{A}^1$ -homology sphere (where  $\Sigma^r X \sim \Sigma^p \mathbf{G}_m^{\wedge q}$  with  $p, q \geq 1$ ). If  $M$  is a homotopy module such that  $H_\delta^j(f_*M)$  is  $k$ -simple for each  $j \geq 0$ , then there is a long exact “Wang” sequence of the form:*

$$\cdots \rightarrow H^i(X, M) \rightarrow H_\delta^i(f_*M)(k) \xrightarrow{\theta} H_\delta^{i+1+r-p}(f_*M)_{-p+r}(k) \rightarrow H^{i+1}(X, M) \rightarrow \cdots,$$

where the map  $\theta$  is a graded derivation.

*Proof.* This result follows immediately by combining Theorem 4.2.5 and Lemma 5.2.5, which shows that the resulting spectral sequence is concentrated in two columns.  $\square$

*Remark 5.2.8.* To guarantee that  $H_\delta^j(f_*M)$  is  $k$ -simple for each  $j \geq 0$ , one may assume that  $f$  is Zariski locally trivial.

**5.3. Relative cellular spaces.** The notion of “algebraic cell decomposition” and the related notion of “relative algebraic cell decomposition” has a long history. E.g., if a split torus acts on a smooth projective scheme  $X$  over a field, then Bialynicki–Birula [ByB73] showed that  $X$  may be decomposed as a disjoint union of smooth varieties that are total spaces of vector bundles over connected components of the fixed point loci (which are necessarily smooth). The cohomological consequences of the existence of such filtrations were observed almost immediately (e.g., one immediately computes Chow groups for smooth projective varieties equipped with a torus action with isolated fixed points).

Karpenko was one of the first to exploit the existence of such algebro-geometric cell decompositions to produce motivic decompositions of varieties ([Kar00]), in his case absolute Chow motives (over a base field). Such results have been developed in numerous directions, but we will mainly be concerned with the relative version studied in [MNP13, 8.4.2]. We begin by introducing our own version of “cellularity”.

**Definition 5.3.1.** Suppose  $f : X \rightarrow B$  be a morphism of schemes. Say that  $f$  (or  $X/B$ ) admits a flat (resp. lci) cellular structure if there exists a filtration:

$$\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_n \subset X_{n+1} = X$$

of closed subschemes of  $X$  such that for all  $\alpha$ , the following conditions are satisfied:

- (a)  $f_\alpha = f|_{X_\alpha} : X_\alpha \rightarrow B$  is flat (resp. lci);
- (b) setting  $U_\alpha := X_\alpha - X_{\alpha-1}$ , the restriction  $p_\alpha = f|_{U_\alpha} : U_\alpha \rightarrow B$  is a smooth morphism and stable  $\mathbf{A}^1$ -weak equivalence (eg: see Remark 5.1.3).

*Remark 5.3.2.* In Karpenko’s definition, the morphisms  $p_\alpha$  are assumed to be vector bundles, which is sufficient for his purposes since he essentially uses the method of Bialynicki–Birula, where the geometric decomposition arises from the action of a split torus on a smooth projective variety. In contrast, our definition allows, e.g., cellular spaces where the “cells” are merely  $\mathbf{A}^1$ -contractible and we allow ourselves to consider varieties that are not necessarily projective. Examples show that this additional generality is natural. For example, the smooth affine quadrics  $Q_{2n}$  of dimension  $2n$  (the hypersurface in affine space defined by the equation  $\sum_i x_i y_i = z(1-z)$ ) admit a decomposition as an affine space ( $x_1 = \cdots = x_n = z = 0$ ) of dimension  $n$ , and an open complement that is a strictly quasi-affine  $\mathbf{A}^1$ -contractible scheme [ADF17] (in particular the complement is an  $\mathbf{A}^1$ -contractible variety that is not isomorphic to affine space). Likewise, the Panin–Walter model of quaternionic projective space  $\mathbf{HP}^n$  admits a “cell decomposition” where the cells are strictly quasi-affine  $\mathbf{A}^1$ -contractible schemes.

*$p_\alpha$  is assumed smooth, so isn’t it better than lci?*

Note that  $p_\alpha$  is lci, as it is the restriction of  $f_\alpha$  to an open subscheme. In particular, its relative dimension is a Zariski locally constant function on  $U_\alpha$  that we denote by  $d_\alpha$ . In other words,  $d_\alpha$  consists simply of integers for each connected component of  $U_\alpha$ . To state the next results, we will use the following notation:

$$(5.3.2.a) \quad \mathbb{1}_B(d_\alpha)[2d_\alpha] = \bigoplus_{x \in U_\alpha^{(0)}} \mathbb{1}_B(d_\alpha(x))[2d_\alpha(x)].$$

**5.3.3.** Let us first recall the computation obtained in [MNP13, 8.4.3], for Chow motives. We work over a quasi-projective base  $B$  over a perfect field  $k$ . Taking into account the comparison result of Jin [Jin16, Th. 3.17], the category of relative Chow motives defined by Corti-Hanamura corresponds to the weight 0 part of the triangulated category of rational mixed motives  $\mathrm{DM}(B, \mathbf{Q})$ .

Given a projective morphism  $f : X \rightarrow B$  such that  $X$  is smooth over  $k$ , one defines the Chow motive (*i.e.* cohomological motive) of  $X/B$  as:

$$h_B(X) = f_*(\mathbb{1}_X).$$

With this notation, [MNP13, Prop. 8.4.3] can be stated as follows.

**Proposition 5.3.4.** *Assume  $f : X \rightarrow B$  is a projective morphism between  $k$ -varieties, with  $X$  smooth and admitting a flat relative cellular structure. One has a canonical isomorphism (using notation (5.3.2.a)):*

$$h_B(X) = \bigoplus_{\alpha} \mathbb{1}_B(-d_{\alpha})[-2d_{\alpha}].$$

Note [MNP13, 8.4.3] is stated for  $k = \mathbb{C}$  but this assumption is not used in the proof, which works over an arbitrary base field  $k$ . We appeal to the assumption  $k$  perfect implicitly via the comparison result (*i.e.*, [Jin16]).

**5.3.5.** We can actually extend the previous computation to  $\mathbf{A}^1$ -homotopy, if one restricts to *oriented spectra*. Let us adopt the following definition. We will say that the abstract triangulated motivic category  $\mathcal{T}$  is *oriented* if:

- There exists a premotivic adjunction  $\varphi^* : \mathrm{SH} \rightleftarrows \mathcal{T} : \varphi_*$ ;
- The ring spectrum  $\varphi_*(\mathbb{1}_{\mathcal{T}})$  is oriented.

These assumptions in place, we will use the theory of Borel-Moore objects as described in [BD17, section 1.3], together with the theory of fundamental classes as developed in [Dég18] (see also [DJK18] for a more general account).

For the convenience of the reader, we recall the definitions and properties we use. Given any separated  $B$ -scheme of finite type  $\pi : Y \rightarrow B$ , we define its *Borel-Moore  $\mathcal{T}$ -motive*<sup>13</sup> by the formula:

$$\mathrm{M}^{\mathrm{BM}}(Y/B) := \pi_!(\mathbb{1}_Y).$$

We will need the following properties:

- (P1) According to the localization property of triangulated motivic categories, given any closed immersion  $i : Z \rightarrow X$  of  $B$ -schemes with complementary open immersion  $j : U \rightarrow X$ , one obtains a distinguished triangle:

$$\mathrm{M}^{\mathrm{BM}}(U/B) \xrightarrow{j_*} \mathrm{M}^{\mathrm{BM}}(X/B) \xrightarrow{i^*} \mathrm{M}^{\mathrm{BM}}(Z/B) \rightarrow \mathrm{M}^{\mathrm{BM}}(U/B)[1].$$

- (P2) Using the main construction of [Dég18, Th. 2.5.3], one associates to any quasi-projective morphism  $f : Y \rightarrow X$  of relative dimension  $d$ , a morphism (also called a fundamental class in *loc. cit.*):

$$f_! : \mathrm{M}^{\mathrm{BM}}(Y/X) = f_!(\mathbb{1}_Y) \rightarrow \mathbb{1}_X(-d)[-2d].$$

- (P3) If in addition to the assumptions of the previous point, one suppose that  $f$  is smooth, then the following diagram is commutative:

$$\begin{array}{ccc} f_!(\mathbb{1}_Y) & \xrightarrow{f_!} & \mathbb{1}_X(-d)[-2d] \\ \mathrm{p}_f \downarrow \sim & & \uparrow \\ f_{\#}(\mathbb{1}_Y)(-d)[-2d] & \xrightarrow{f_*(-d)[-2d]} & \mathbb{1}_X(-d)[-2d] \end{array}$$

where  $\mathrm{p}_f$  is the purity isomorphism associated with the smooth morphism  $f$  in the six functors formalism,  $f_{\#}(\mathbb{1}_Y)$  is the  $\mathcal{T}$ -motive represented by  $Y/B$ , and  $f_*$  refers to the classical functoriality. Note in particular that  $f_*$  is the image of the canonical map  $\Sigma^{\infty} Y_+ \rightarrow \Sigma^{\infty} B_+$  of spectra over  $B$  under the map  $\varphi^* : \mathrm{SH}(B) \rightarrow \mathcal{T}(B)$ .

<sup>13</sup>Another appropriate terminology would be cohomological  $\mathcal{T}$ -motive with compact support.

(P4) If under the assumptions of point (2),  $X$  is separated of finite type over a scheme  $B$ , with structural morphism  $p$ , we obtain by applying  $p_!$  a Gysin morphism:

$$f_! : M^{\text{BM}}(Y/B) \rightarrow M^{\text{BM}}(X/B)(-d)[-2d].$$

It follows from the compatibility of fundamental classes with composition [Dég18, 2.5.3] that these Gysin morphism are compatible with composition.

**Proposition 5.3.6.** *We consider a morphism  $f : X \rightarrow B$  with an lci cellular structure, and  $B$  an arbitrary scheme.*

*Then, using notation (5.3.2.a), there exists a canonical isomorphism in  $\mathbf{MGL}\text{-mod}(B)$ :*

$$f_!(\mathbb{1}_X) = \bigoplus_{\alpha} \mathbb{1}_B(-d_{\alpha})[-2d_{\alpha}]$$

The principle of proof is well-known...

*Proof.* We look at the localization triangle of property (P1) associated with the closed immersion  $i_{\alpha} : X_{\alpha} : X_{\alpha-1} \rightarrow X_{\alpha}$  with complementary open immersion  $j_{\alpha} : U_{\alpha} \rightarrow X_{\alpha}$ :

$$M^{\text{BM}}(U_{\alpha}/B) \xrightarrow{j_{\alpha*}} M^{\text{BM}}(X_{\alpha}/B) \xrightarrow{i_{\alpha}^*} M^{\text{BM}}(X_{\alpha-1}/B) \rightarrow M^{\text{BM}}(U_{\alpha}/B)[1].$$

Next we apply property (P4) to get a commutative diagram:

$$\begin{array}{ccc} M^{\text{BM}}(U_{\alpha}/B) & \xrightarrow{j_{\alpha*}} & M^{\text{BM}}(X_{\alpha}/B) \\ & \searrow p_{\alpha*} & \swarrow f_{\alpha*} \\ & M^{\text{BM}}(B/B)(-d_{\alpha})[-2d_{\alpha}] & \end{array}$$

Point (P3) applied to the smooth morphism  $p_{\alpha}$ , which by assumption is a stable weak  $\mathbf{A}^1$ -equivalence, implies the map  $p_{\alpha*}$  is an isomorphism. In particular,  $j_{\alpha*}$  is a split monomorphism (with splitting  $p_{\alpha*}^{-1}f_{\alpha*}$ ) and one gets an isomorphism  $M^{\text{BM}}(U_{\alpha}/B) = \mathbb{1}_B(-d_{\alpha})[-2d_{\alpha}]$ .

This completes the proof.  $\square$

Changed the indices in part (1) of the statement below

**Corollary 5.3.7.** *Assume that  $f : X \rightarrow B$  is a projective morphism with lci cellular structure.*

(1) *Then one gets an isomorphism of the following cohomological  $\mathcal{T}$ -motives:*

$$f_*(\mathbb{1}_X) = \bigoplus_{\alpha} \mathbb{1}_B(-d_{\alpha})[-2d_{\alpha}].$$

*In particular, one gets an isomorphism of  $\mathcal{T}$ -cohomology:*

$$H^{**}(X, \mathbb{1}_X) \simeq \bigoplus_{\alpha} H^{*-2d_{\alpha}, *-d_{\alpha}}(B, \mathbb{1}_B)$$

(2) *For any  $\mathcal{T}$ -spectrum  $\mathbb{E}$  over  $B$ , there exists a canonical isomorphism:*

$$f_*f^*(\mathbb{E}) \simeq \bigoplus_{\alpha} \mathbb{1}_B(-d_{\alpha})[-2d_{\alpha}].$$

Only the second point needs a proof. One simply uses the preceding proposition and the projection formula from the six functors formalism:

$$f_*f^*(\mathbb{E}) \simeq f_!(\mathbb{1}_X \otimes f^*(\mathbb{E})) \simeq f_!(\mathbb{1}_X) \otimes \mathbb{E}.$$

*Example 5.3.8.* (1) In the motivic case, one gets back the result of Proposition 5.3.4. In fact, with rational coefficients, we have shown that one can get rid of a base field, if one replaces the flatness by lci. (See also next Remark.)

(2) We can apply the preceding result to classical oriented ring spectra, such as  $\mathbf{MGL}$  and  $\mathbf{KGL}$ , by using the theory of modules over ring spectra (see [CD09, §7]). The corollary extends a previously known result in [NZ06], from characteristic 0 to the absolute case (without the need of a base field).

*Remark 5.3.9.* (1) The above corollary applies to strict **MGL**-modules  $M$  over  $B$ . That is ring spectra, the **MGL**-module structure on  $B$  must be defined at the model category level. This assumption is too strong in many situations.

One can avoid this assumption, assuming only that  $M$  is a spectrum in  $\mathrm{SH}(B)$  with a module structure over  $\mathbf{MGL}_B$ . Indeed, it is possible to use the proof of Proposition 5.3.6 after replacing  $f_!(\mathbb{1}_X) = f_!f^*(\mathbb{1}_X)$  with the spectrum  $f_!f^*(M)$ . One uses the theory of fundamental classes as developed in [DJK18] and the fact that one has Thom isomorphisms for  $M$ , using the  $\mathbf{MGL}_B$ -module structure.

We leave the details to the reader.

(2) The orientation assumptions in the preceding proposition and corollary is essential. Indeed, recall that the Chow-Witt ring does not satisfy the projective bundle formula though it is representable in  $\mathrm{SH}$  by the Milnor-Witt homotopy module.

Nevertheless, one can extend the validity of the above result by considering weaker orientability conditions. For example, we can consider a symplectically oriented (ring) spectrum  $M$  (see [?]), provided that we give for each index  $\alpha$  a symplectic structure on the tangent bundle of  $U_\alpha/B$ .

(3) At least in the rational motivic case (which is equivalent to the rational orientable case), it is possible in principle to generalize both the flat and lci case. Indeed, it is visible in the proof that we only need a good theory of fundamental classes for the projections  $X_\alpha \rightarrow B$ . The recent work [Jim18] opens the way to define these fundamental classes for arbitrary *finite tor-dimension morphisms*, which contains both flat and lci cases. The main tool to do that is to extend Jin's work to a suitable representability theorem for relative K-theory (as defined in [BG17, IV, 3.3]).

The relative cellular space provides us with a situation where the  $\delta$ -homotopy Leray spectral sequence is particularly simple.

**Proposition 5.3.10.** *Let  $f : X \rightarrow B$  be a proper morphism with an lci relative cellular structure. We consider a  $\delta$ -homotopy module  $M$  over  $B$  satisfying one of the following conditions:*

- *Homotopical case.*–  $M$  is an oriented  $\delta$ -homotopy module in  $\mathrm{SH}(B)$ .
- *Motivic case.*–  $M$  is a  $\delta$ -homotopy module in  $\mathrm{DM}(B, R)$ .

*We consider the  $\delta$ -homotopy module  $M_X = f^!M$ . Then there exists an isomorphism:*

$$f_*(M_X) \simeq \sum_{i=0}^d H_\delta^i(f_*M_X)[-i]$$

where  $d$  is the maximum of the dimension of the fibers of  $f$ .

Moreover, the  $\delta$ -homotopy spectral sequence:

$$E_2^{p,q} = A^p(B, H_\delta^q(f_*M_X)) \Rightarrow A^{p+q}(X, M_X)$$

degenerates at  $E_2$ , and the abutting filtration splits giving an isomorphism:

$$A^n(X, M_X) \simeq \bigoplus_{p=0}^n A^p(B, H_\delta^{n-p}(f_*M_X)).$$

Not sure about the indices.

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