Notes: The \mathbb{A}^1 -homotopy category

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1 Introduction

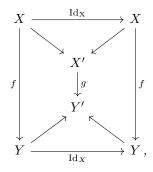
The aim of the present exposé is to give a quick overview of the construction of the unstable \mathbb{A}^1 -homotopy category of smooth schemes over a base S. Roughly, the construction goes as follow : starting from spaces Sm_S , one adds colimits by embedding in $\mathrm{Pre}(\mathrm{Sm}_S)$, the category of presheaves on Sm_S . Then, one adds homotopy colimits by embedding in $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_S)$, the category of simplicial presheaves on Sm_S . Then, one localises first with respect to the Nisnevich hypercoverings on $\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_S)$, and localises again with respect to the projections $X \times \mathbb{A}^1 \to X$. This gives a model category $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\mathbf{\Delta}^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}_S)$. The unstable \mathbb{A}^1 -homotopy category \mathbb{H} is defined as the homotopy category of this model category.

The content of this exposé has been inspired by [AE16].

2 Model categories

Definition 1 Let M be a category with small colimits and small limits. Given three classes W, F and C of morphism, which we call weak equivalences, fibrations and cofibrations respectively. One says the de 4-uple (M, W, F, C) is a Model Category if the following axioms are satisfied :

- (M1) Given two composable morphism $X \xrightarrow{f} Y \xrightarrow{g}$, if two elements of $\{f, g, g \circ f\}$ are in W (resp. F, resp. C), then so is the third.
- (M2) The three classes are stable by retraction, that is to say, given two morphisms f, g fitting in the following commutative diagram



if g is in one of the class W, F or C, then so is f.

(M3) Trivial fibrations (elements of $W \cap F$) have the right lifting properties with respect to cofibrations, and trivial cofibrations (elements of $W \cap C$) have the left lifting property with respect to fibration. In other words, given a diagram¹



if the left arrow or the right arrow is a weak equivalence, there is a solid arrow filling the dotted arrow.

For all morphisms $f : X \to Y$, f can be factored as an acyclic cofibration $X \to Z$ followed by a fibration $h : Z \to X$, or as a cofibration $X \to Z'$ followed by a trivial fibration $Z' \to X$.

Example Let A be a ring, then $\operatorname{Ch}_{\geq 0}(A)$, the category of (homological) chain complexes concentrated on positive degrees have the structure of a model category by taking W to be the quasiisomorphisms, F to be the maps that are degreewise surjective on strictly positive degrees, and taking $C = (W \cap F)_{\perp}^2$.

Definition 2 Let M be a model category. Denote by \emptyset its initial element and * its terminal. We say Y is fibrant if the unique morphism $Y \to *$ is a fibration, and we say that Y is cofibrant if the unique morphism $\emptyset \to Y$ is a cofibration.

Given any object X, we can factor the morphism $X \to *$ as $X \hookrightarrow RX \to *$ where the cofibration is trivial. In such case, we say that RX is a fibrant replacement of X. Similarly, we can factor $\emptyset \to X$ as $\emptyset \hookrightarrow QX \to X$ where the fibration is trivial. In such case, we say QX is a cofibrant replacement of X.

Example Let us give an important example of model category. The category of simplicial sets Δ^{op} Sets has a model structure with the following data : weak equivalences are the maps of simplicial sets $f: X_{\bullet} \to Y_{\bullet}$ such that the induced map $|f|: |X_{\bullet}| \to |Y_{\bullet}|$ between geometric realization is a weak equivalence in the sense of classical homotopy theory. We define the cofibrations to be the maps that are levelwise monomorphisms. Fibrations are then defined by $_{\perp}(C \cap W)$, in this case, this turns out to be the Kan fibrations of simplicial sets.

Definition 3 Given a category M and W any set of morphism, a localization of M at W, is a category $M[W^{-1}]$ endowed with a functor $\gamma: M \to M[W^{-1}]$ such that for all $w \in W$, $\gamma(w)$ is an isomorphism and that is universal with respect to this property. Localizations are then unique up to unique isomorphism and we often say that $M[W^{-1}]$ is the localization of M at W.

Remark The localization of an arbitrary category at an arbitrary set of morphism always exist, but might not be small, i.e said category may not be in the same universe \mathscr{U} as the category we started with, and the set of morphisms between two objects might not be a small \mathscr{U} -set. Often, we say that the localization "exists" to mean that it is a \mathscr{U} -category.

^{1.} It is customary to denote fibrations as two-headed arrows, and cofibration as arrow with a hooked tail.

^{2.} Given a class A of morphism, we denote A_{\perp} to be the class of morphisms that have the left lifting property with respect to all morphisms in A and $_{\perp}A$ to be the class of morphisms that have the right lifting property with respect to all morphisms in A.

Theorem 2.1 If (M, W, C, F) is a model category. then $M[W^{-1}]$ exists. Moreover, it is equivalent to the category Ho(M), where objects are objects of M that are both fibrant and cofibrant, and morphisms are homotopy classes of morphisms between two such objects³.

Let X, Y be elements of Δ^{op} Sets, recall that their mapping space $\operatorname{map}_{\Delta}(X, Y)$ is the simplicial set defined by $(\operatorname{map}_{\Delta}(X, Y))_n = \operatorname{Hom}_{\Delta^{\text{op}} \text{Sets}}(X \times \Delta^n, Y).$

Definition 4 Let C be a category. One says that the category is a Simplicial category if there is a simplicial mapping space bifunctor $\operatorname{map}_{\mathcal{C}}(-,-)$ from $\mathcal{C}^{\operatorname{op}} \times \mathcal{C}$ to the category $\Delta^{\operatorname{op}}$ Sets, with the following properties :

- (1) One has $\operatorname{map}_{\mathcal{C}}(A, B)_0 = \operatorname{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathcal{C}$.
- (2) For all A, the functor $\operatorname{map}_{\mathcal{C}}(A, -)$ has a left adjoint $A \otimes -: \Delta^{\operatorname{op}} \operatorname{Sets} \to \mathcal{C}$, called the action.
- (3) For all B, the functor $\operatorname{map}_{\mathcal{C}}(-, B)$ has a left adjoint $B^- : \Delta^{\operatorname{op}} \operatorname{Sets} \to \mathcal{C}^{\operatorname{op}}$ called the exponential.

Remark This definition is equivalent to the fact that C is enriched over Δ^{op} Sets and is both tensored and cotensored over it, we refer to [Lur09] A.1.3 for explanation of this terminology.

One of the most important consequence of being a simplicial category is that there is an adjunction $- \otimes X \dashv \operatorname{map}_{\mathcal{C}}(X, -)$ for all X.

Definition 5 Let M be a model category. We say that it is a simplicial model category if it is a simplicial category, and that moreover, the following property, usually called "SM7", is satisfied : For all X in C and $i : A \hookrightarrow X$, for all $p : E \twoheadrightarrow B$, the map

$$\operatorname{map}_{M}(X, E) \to \operatorname{map}_{M}(A, E) \times_{\operatorname{map}_{M}(A, B)} \operatorname{map}_{M}(X, B)$$

is a fibration of simplicial sets, i.e a Kan fibration. Moreover, it is trivial if either i or p is trivial.

Definition 6 Let M and N be model categories. and let $F : M \cong N : G$ be an adjoint pair. We say that it is a Quillen adjunction if either F respects cofibrations and trivial cofibrations or G respects fibrations and trivial fibrations⁴.

Proposition 2.2 Given two model categories M and N, and a Quillen adjunction $F : M \leftrightarrows N : G$, there is an adjunction $\mathbf{L}F : \operatorname{Ho}(M) \leftrightarrows \operatorname{Ho}(N) : \mathbf{R}G$.

Definition 7 In the situation above, if $\mathbf{L}F \dashv \mathbf{R}G$ is an equivalence, we say that $F \dashv G$ is a Quillen equivalence.

3 Bousfield Localization

Definition 8 Let M be a simplicial model category, and I be a set of maps in M. Let $X \in M$. We say that X is I-local if X is fibrant and for any map $i : A \to B$ in I, the induced map $i^* : \operatorname{map}_M(B, X) \to \operatorname{map}_M(A, M)$ is a weak equivalence of simplicial sets.

We say that a morphism $f : A \to B$ is an I-local weak equivalence if for all I-local object X, the map $f^* : \operatorname{map}_M(B, X) \to \operatorname{map}_M(A, X)$ is a weak equivalence.

^{3.} We will not give the definition of the relation of homotopy between two objects in a model category, we refer the reader to [Lur09] A.2.2

^{4.} This turns out to be equivalent.

Definition 9 Given a model category (M, W, F, C) and a class of morphism I. We define W_I to be the set of I-local weak equivalences, $C_I = C$, and $F_I = {}_{\perp}(C_I \cap W_I)$. We say that (M, W_I, C_I, F_I) is the left Bousfield localization of M at I. We denote it $L_I M$.

Theorem 3.1 If M is a simplicial model category that is left proper and combinatorial⁵, then for any class I of morphism, L_IM is a simplicial model category, that is left proper and combinatorial.

We refer the reader to [Lur09] A.3.7.3 for a proof of this theorem.

4 Simplicial presheaves and hypercoverings

Let \mathcal{C} be an essentially small category. Let $\Delta^{\mathrm{op}}\mathrm{Pre}(\mathcal{C})$ be the category of simplicial presheaves on \mathcal{C} .

Proposition and definition 4.1 Define W to be the set of morphisms that are pointwise weak equivalences of simplicial sets, F to be the set of morphism that are are pointwise fibrations, and C to be $(W \cap F)_{\perp}$. The 4-uple $(\Delta^{\text{op}}\text{Pre}(\mathcal{C}), W, F, C)$ satisfies the axioms of a model category. Moreover, this model category is simplicial, combinatorial and left proper. We say that it is the projective model structure on $\Delta^{\text{op}}\text{Pre}(\mathcal{C})$ or the projective model category on \mathcal{C} .

This fact is proved in [Lur09] A.2.8.2 and A.2.8.4.

One has a Yoneda embedding $y : \mathcal{C} \to \Delta^{\text{op}} \text{Pre}(\mathcal{C})$ and the projective model category is initial with respect to the model categories M with an embedding $\mathcal{C} \to M$.

Now assume that a Grothendieck topology τ is given on \mathcal{C} . Let U_{\bullet} be an object of $\Delta^{\text{op}}\text{Pre}(\mathcal{C})$, in other words, U_n is a presheaf of sets on \mathcal{C} . Then, let V be a representable object in $\Delta^{\text{op}}\text{Pre}(\mathcal{C})$. We say that a map $U \to V$ is an hypercovering if every U_n is a coproduct of representable, $U_0 \to V$ is a τ -cover and, for each n. The map $U^{\Delta^n} \to U^{\partial\Delta^n}$ induced by applying the exponential functor to the inclusion $\partial\Delta^n \to \Delta^n$ induces τ -covers on degree zero. The standard example of hypercovers are the ones arising as the Čech complex associated to a τ -cover $U \to V$.

Theorem 4.2 The left Bousfield Localization with respect to hypercovers exist. It is denoted by $L_{\tau} \Delta^{op} Pre(\mathcal{C})$.

The existence here comes from the fact that hypercovers form a set since we have assumed that our base category C is essentially small.

Remark that if τ is subcanonical, *i.e* if all representable presheaves are sheaves for τ , then, \mathcal{C} embeds in Ho(L_{τ} Δ^{op} Pre(\mathcal{C})).

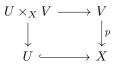
5 The unstable \mathbb{A}^1 -homotopy category of smooth schemes

We begin by applying the preceding construction with $C = Sm_S$ and τ is the Nisnevich topology. We recall its definition for convenience.

^{5.} We will note define these terms. We refer to [Lur09] A.2.4.1 for the definition of a left proper model category, and to [Lur09] A.2.6.1 for the definition of a combinatorial model category. It will be enough to say that all the model categories we care about satisfy these condition

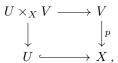
Definition 10 A family of morphisms $\{U_i \to U\}_{i \in I}$ in Sm_S is said to be a Nisnevich covering if the $U_i \to U$ are étale and of finite type, and there exists a finite decreasing chain of closed subschemes $\emptyset \subset Z_n \subset \cdots \subset Z_0 = X$ of X such that the map $\coprod_i p_i^{-1}(Z_j \setminus Z_{j+1}) \to Z_j \setminus Z_{j+1}$ has a section for all $0 \leq j \leq n-1$.

Definition 11 Consider a commutative diagram



where p is étale, $U \hookrightarrow X$ is an open immersion. If $Z = (X \setminus U)_{\text{red}}$ is such that $p^{-1}(Z) \cong Z$, we say that this square is a Nisnevich Distinguished square.

Proposition 5.1 If S is Noetherian of finite Krull dimension, then $F \in \Delta^{\text{op}}\text{Pre}(\text{Sm}_S)$ is fibrant in $L_{\text{Nis}}\Delta^{\text{op}}\text{Pre}(\text{Sm}_S)$ if for any Nisnevich distinguished square



then the canonical map $F(X) \to F(U) \times_{F(U \times_X V)} F(V)$ is a weak equivalence of simplicial sets.

Definition 12 Let $I = \{\mathbb{A}_S^1 \times_S X \to X, X \in \mathrm{Sm}_S\}$. Then, the left Bousfield localization of $\mathrm{L}_{\mathrm{Nis}} \Delta^{\mathrm{op}} \mathrm{Pre}(\mathrm{Sm}_S)$ with respect to I exists, we denote the resulting model category by $\mathrm{L}_{\mathbb{A}^1} \mathrm{L}_{\mathrm{Nis}} \mathbb{A}^1$. It is a left proper combinatorial simplicial model category. We call it the (unstable) \mathbb{A}^1 -homotopy category

Example One can check that if $f, g : X \to Y$ are \mathbb{A}^1 -homotopic, *i.e* if there is a map $H : \mathbb{A}^1 \times_S X \to Y$ such that $H(i_0 \times_S \operatorname{Id}_X) = f$ and $H(i_1 \times_S \operatorname{Id}_X) = g$, then f and g induce the same morphism in the homotopy category of $L_{\mathbb{A}^1} L_{\operatorname{Nis}} \mathbb{A}^1$.

One can also check that if $p: E \to X$ is a vector bundle, then p is a weak \mathbb{A}^1 -local equivalence.

Références

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