Notes: Milnor-Witt K-theory

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Definition 1 *Let F be a field, we define $K_*^{\mathrm{MW}}(F)$, the* Milnor-Witt K-theory of F*, to be the associative graded ring defined by the following data :*

- *Generators are given by symbols $[u]$ for u in $F^\times$ of degree 1, and a symbol η of degree -1.*
- *The generator must satisfy the following relations :*
  1 *For all $a \in F^\times$, $a \neq 1$, $[a].[1-a] = 0$.*
  2 *For all $a, b \in (F^\times)^2$, $[ab] = [a] + [b] + \eta.[a].[b]$.*
  3 *For all $u \in F^\times$, $\eta.[u] = [u].\eta$.*
  4 *Let $h = \eta[-1] + 2$. Then $\eta.h = 0$.*

The quotient $K_*^{\mathrm{MW}}(F)/\eta$ is naturally isomorphic to $K_*^M(F)$, the usual Milnor K-theory of F. Similarly, one can define the Witt K-theory of F $K_*^W(F)$ as $K_*^{MW}(F)/h$.

Proposition 1.1 *Let F be a field, and n be an integer. Let $\tilde{K}_n^{MW}(F)$ be the group presented in the following way :*

- *Generators are given by symbols $[\eta^m, u_1, \ldots, u_r]$ where η is a symbol, r is an integer, $m = r-n$, and $u_1, \ldots, u_r$ are elements of $F^\times$.*
- *The generators are subject to the following relations :*
  - *$[\eta^m, u_1, \ldots, u_r] = 0$ if there exists i such that $0 \leq i < r$ and $u_i + u_{i+1} = 0$.*
  - *For every r, for every pair $a, b \in F^\times$, and for every $1 < i < r$, one has*
  $$[\eta^m, \ldots, u_{i-1}, ab, u_{i+1}, \ldots, u_r] = [\eta^m, \ldots, u_{i-1}, a, u_{i+1}, \ldots, u_r] + [\eta^m, \ldots, u_{i-1}, b, u_{i+1}, \ldots, u_r] + [\eta^{m+1}, \ldots, u_{i-1}, a, b, u_{i+1}, \ldots, u_r]\,.$$
  - *For every i,*
  $$[\eta^m, u_1, \ldots, u_{i-1}, -1, u_{i+1}, \ldots, u_r] + 2[\eta^{m-1}, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_r] = 0\,.$$

Then, the natural morphism $\tilde{K}_n^{MW}(F) \to K_n^{MW}(F)$ that sends the generator $[\eta^m, u_1, \ldots, u_r]$ to $\eta^m[u_1]\ldots[u_r]$ is an isomorphism.

Proof Let R be the associative graded ring with unit generated by the elements of $F^\times$ with degree 1 and a symbol η in degree -1, which is central. We define a map $\phi : R_n \to \tilde{K}_n^{MW}(F)$ that sends $\eta^m[u_1]\ldots[u_r]$ to $[\eta^m, u_1, \ldots, u_r]$. It will be enough to show that this map sends I_n, the $n-th$ graded part of the ideal defining the relations of $K_n^{MW}(F)$, to zero. But I_n is generated by elements of the form $x.r.y$ where r is an element of the following from :

— $[a][1-a]$ for $a \neq 1$.

— $[ab] = [a] + [b] + \eta.[a].[b]$ for some $a, b \in F^{\times}$.

— $\eta^2[-1] + 2\eta = 0$.

and x and y are elements of the form $[\eta^m][u_1] \cdots [u_r]$, such that the degrees of x, r and y add to m. Those elements are mapped to zero in $\tilde{K}_n^{MW}(F)$ since they correspond exactly to the relations defining $\tilde{K}_n^{MW}(F)$. Thus, the map ϕ factors to a map $K_n^{MW}(F) \to \tilde{K}_n^{MW}(F)$, and this map is inverse to the map in the proposition. $\square$

For an element $a \in F^{\times}$, we set $\langle a \rangle = 1 + \eta[a] \in K_0^{MW}(F)$. We now give a few relations that hold in the ring $K_*^{MW}(F)$:

Lemma 1.2 *Let $a, b \in F^{\times}$. We have the following relations :*

i) $[ab] = [a] + \langle a \rangle.[b]$ *and* $[ab] = [a].\langle b \rangle + [b]$.

ii) $\langle ab \rangle = \langle a \rangle.\langle b \rangle$, *and* $K_0^{MW}(F)$ *is contained in the center of* $K_*^{MW}(F)$.

iii) $\langle 1_F \rangle = 1_{K_*^{MW}(F)}$ *and* $[1] = 0$.

iv) $\langle a \rangle$ *is a unit of* $K_*^{MW}(F)$ *whose inverse is* $\langle a^{-1} \rangle$.

v) $[\frac{a}{b}] = [a] - \langle \frac{a}{b} \rangle [b]$, *in particular* $[a^{-1}] = -\langle a^{-1} \rangle [a]$.

Proof Let's prove i) :

$$\begin{aligned} [ab] &= [a] + [b] + \eta[a][b] && \text{by relation } \mathbf{2} \\ &= [a] + (1 + \eta[a])[b] \\ &= [a] + \langle a \rangle [b] && \text{by definition of } \langle a \rangle. \end{aligned}$$

Since $[ab] = [ba]$, we also get that $[ab] = [a] + [b].\langle a \rangle$ and $[ab] = [b] + \langle b \rangle.[a]$. Let's now prove ii) :

$$\begin{aligned} \langle ab \rangle &= 1 + \eta[ab] \\ &= 1 + \eta\left([a] + [b] + \eta[a][b]\right) && \text{by relation } \mathbf{2} \\ &= 1 + \eta[a] + \eta[b] + \eta^2[a][b] \\ &= (1 + \eta[a])(1 + \eta[b]) \\ &= \langle a \rangle.\langle b \rangle\,. \end{aligned}$$

Using i), one has that $\langle a \rangle [b] = [b] \langle a \rangle$ for all b, thus $\langle a \rangle$ is central as claimed.
Let's prove iii), notice that $h = \langle -1 \rangle + 1$, and that $\eta[1] = \langle 1 \rangle - 1$. Thus, relation **4** implies

$$(\langle 1 \rangle - 1)(\langle -1 \rangle + 1) = 0\,.$$

But

$$(\langle 1 \rangle - 1)(\langle -1 \rangle + 1) = \langle 1 \rangle \langle -1 \rangle + \langle 1 \rangle - \langle -1 \rangle - 1$$

and $\langle 1 \rangle \langle -1 \rangle = \langle -1 \rangle$ by ii), so that

$$(\langle 1 \rangle - 1)(\langle -1 \rangle + 1) = \langle 1 \rangle - 1\,.$$

Thus it being 0 implies the desired relation. Now by i),

$$\begin{aligned}[1] &= [1] + \langle 1 \rangle [1] \\ &= [1] + [1]\end{aligned}$$

so that $[1] = 0$, as desired.
Let us prove iv).

$$\begin{aligned}1 &= \langle 1 \rangle && \text{by iii)} \\ &= \langle aa^{-1} \rangle \\ &= \langle a \rangle \langle a^{-1} \rangle && \text{by ii).}\end{aligned}$$

Similarly, $\langle a^{-1} \rangle \langle a \rangle = 1$, hence iv). Finally, let's prove v), one has

$$\begin{aligned}[a] &= [ab^{-1}b] \\ &= \left[\frac{a}{b}\right] + \left\langle \frac{a}{b} \right\rangle [b] && \text{by i).}\end{aligned}$$

Which proves that $[\frac{a}{b}] = [a] - \langle \frac{a}{b} \rangle [b]$. Applying to $a = 1$, and using that $[1] = 0$, one has the expected particular case. $\square$

Lemma 1.3 *1. For $n \geq 1$, $K_n^{MW}(F)$ is generated by products of the form $[u_i] \dots [u_n]$.*

2. For $n \leq 0$, $K_n^{MW}(F)$ is generated by products of the form $\eta^n \langle u \rangle$.

Proof Assume first that $n \geq 1$. We already know that $K_n^{MW}(F)$ is generated by elements of the form $\eta^m [u_1] \dots [u_r]$ with $n = r - m$. Relation **2** gives that $\eta[a][b] = [ab] - [a] - [b]$, thus, we can inductively diminish m until $m = 0$, leaving only terms of the desired form.
In the case $n < 0$. The group $K_n^{MW}(F)$ is again generated by elements of the form $\eta^m [u_1] \dots [u_r]$ with $n = r - m$. Since

$$\eta^m [u_1] \dots [u_r] = \eta^{-n} \eta^r [u_1] \dots [u_r] \,.$$

But

$$\begin{aligned}\eta[a] &= \langle a \rangle - 1 \\ &= \langle a \rangle - \langle 1 \rangle && \text{by iii) of the previous lemma.}\end{aligned}$$

This relation as well as relation ii) of the previous lemma ensure that $\eta^r [u_1] \dots [u_r]$ reduces to a sum of elements of the form $\langle u \rangle$ as desired. $\square$

Let ϵ denote the element $-\langle -1 \rangle$ of $K_0^{MW}(F)$. Relation **4** can be rephrased as $\epsilon.\eta = \eta$.

Lemma 1.4 *Let $a, b \in F^\times$, We have the following relations :*

i) $[a].[-a] = 0$.

ii)

$$\begin{aligned}[a][a] &= [a][-1] \\ &= \epsilon[a][-1] \\ &= [-1][a] \\ &= \epsilon[-1][a]\,.\end{aligned}$$

iii) $\langle a^2 \ \rangle = 1$.

iv) $\langle a \rangle + \langle -a \rangle = h$.

v) $[a][b] = \epsilon[b][a]$.

Proof Let us prove i). If $a = 1$, this is clear since $[1] = 0$. Assume $a \neq 1$. Then, we have the equality $-a = \frac{1-a}{1-a^{-1}}$. We can use relation v) of 1.2 to get

$$[-a] = [1-a] - \langle -a \rangle [1-a^{-1}]\,.$$

Multiplying by $[a]$, we get

$$\begin{aligned}[a][-a] &= [a][1-a] - \langle -a \rangle [a][1-a^{-1}] \\ &= -\langle -a \rangle [a][1-a^{-1}] && \text{by relation } \mathbf{1} \\ &= \langle -a \rangle \langle a \rangle \ [a^{-1}][1-a^{-1}] && \text{by relation v) of 1.2} \\ &= 0 && \text{by } \mathbf{1} \text{ again.}\end{aligned}$$

Let's prove ii). By i) of 1.2, we have that $[-a] = [-1] + \langle -1 \rangle [a]$. Multiplying by $[a]$ on the left and using the previously proved relation $[a].[-a] = 0$, we have $[a][-1] = -\langle -1 \rangle [a][a]$, which is, $[a][-1] = \epsilon[a][a]$. Similarly, by multiplying $[-a] = [-1] + \langle -1 \rangle [a]$ by α on the right, we get that $[-1][a] = \epsilon[a][a]$. We notice that $\epsilon^2 = 1$, so that we have indeed proved that $[a][a] = \epsilon[a][-1]$ and that $[a][a] = \epsilon[-1][-a]$.
Since we have

$$\begin{aligned}0 &= [1] \\ &= [(-1)^2] \\ &= [-1] + \langle -1 \rangle [-1]\,.\end{aligned}$$

We conclude that $\epsilon[-1] = [-1]$, so that $\epsilon[a][-1] = [a][-1]$ and $\epsilon[-1][-a] = [-1][a]$, which finishes proving ii). Let us prove iii). It is enough to prove that $\eta[a^2] = 0$. We have that $[a^2] = 2[a] + \eta[a][a]$. By ii), $[a][a] = [-1][a]$ so that $[a^2] = 2[a] + \eta[-1][a]$, that is to say, $[a^2] = (2 + \eta[-1])[a]$. Multiplying by η and using relation **4**, we conclude that $\eta[a^2] = 0$.
Let use prove iv). One starts with $[a][-a] = 0$ and multiplies by η^2, to get $\eta[a]\eta[-a] = 0$. Since $\eta[a] = \langle a \rangle - 1$ and $\eta[-a] = \langle -a \rangle - 1$, we can develop $(\langle a \rangle - 1)(\langle -a \rangle - 1)$ and get $\langle a \rangle + \langle -a \rangle = 1 + \langle -a^2 \rangle$. By iii) along with ii) of 1.2, $\langle -a^2 \rangle = \langle -1 \rangle$, thus $\langle a \rangle + \langle -a \rangle = h$.
Let us prove v). We start with the relation $[ab][-ab] = 0$, which was proved in i). Expanding using relation i) of 1.2, this relation becomes

$$0 = ([a] + \langle a \rangle [b])([-a] + \langle -a \rangle [b])\,.$$

Expanding, we get that

$$0 = [a][-a] + \langle -a\rangle [a][b] + \langle a\rangle [b][-a] + \langle -a^2\rangle [b][b]\,.$$

By i), $[a][-a] = 0$. By ii), $[b][b] = [-1][b]$. As noted before, $\langle -a^2\rangle = \langle -1\rangle$. Thus the relation becomes

$$0 = \langle a\rangle \left([b][-a] + \langle -1\rangle [a][b]\right) + \langle -1\rangle [-1][b]\,.$$

We can use the fact that $[-a] = [a] + \langle a\rangle [-1]$, and we have

$$0 = \langle a\rangle \left([b][a] + \langle -1\rangle [a][b]\right) + \langle a^2\rangle [b][-1] + \langle -1\rangle [-1][b]\,.$$

Once again, $\langle a^2\rangle = 1$, and by ii), $[b][-1] + \langle -1\rangle [-1][b] = 0$ (recall that $\epsilon = -\langle -1\rangle$). thus we have that

$$\langle a\rangle \left([b][a] + \langle -1\rangle [a][b]\right) = 0\,.$$

Since $\langle a\rangle$ is a unit by iv) of 1.2, we get $[b][a] = \epsilon[a][b]$, which is the desired relation since $\epsilon^2 = 1$. □

Corollary 1.5 *The ring $K_*^{MW}(F)$ is ϵ-graded commutative.*

Proof We have to show that for $\alpha \in K_m^{MW}(F)$ and $\beta \in K_n^{MW}(F)$, one has $\alpha\beta = \epsilon^{mn}\beta\alpha$.
It is sufficient to show that it holds for the multiplicative generators of $K_*^{MW}(F)$, that is to say for elements of the form $[u]$ or η.
The relation $[a][b] = \epsilon[b][a]$ is known from v) of the previous lemma. η is central, of degree -1 and $\eta[a] = [a]\eta$, but $\eta\epsilon = \eta$, so the relation hold as well, the same thing happens with the product η^2. □

Let us recall some facts about about the Grothendieck-Witt ring of a field F.

Definition 2 *Let F be a field, we denote by $GW(F)$ the free group generated by the set of isomorphism classes of symmetric inner product spaces over F, modulo the relation that $[x \oplus y] = [x] + [y]$, where $\oplus$ denotes the sum orthogonal sum of inner product spaces.*
The Grothendieck-Witt group inherits a ring structure via the tensor product of inner product space.

For a unit element u of F, we denote by $\langle u\rangle$ the inner product space of dimension one, with one basis element e, such that $(e|e = u)$. If we denote by h again the element $1 + \langle -1\rangle$ of $GW(F)$, then GW2) shows that (h), the subgroup generated by h, is actually an ideal of $GW(F)$. The Witt ring $W(F)$ of F is then the quotient $GW(F)/(h)$. Our next goal is to establish the following theorem.

Theorem 1.6 *The Grohendieck-Witt Ring of F, denoted as $GW(F)$, is, as a group, generated by the $\langle u\rangle$, $u \in F^\times$, subject to the relations :*

GW1) $\langle uv^2\rangle = \langle u\rangle$.

GW2) $\langle u\rangle + \langle -u\rangle = 1 + \langle -1\rangle$.

GW3) $\langle u\rangle + \langle v\rangle = \langle u+v\rangle + \langle (u+v)uv\rangle$ *if* $u + v \neq 0$.

Proof In characteristic different from 2, any symmetric inner product space admits an orthogonal basis. In characteristic 2, any symmetric inner product space can be written as an orthogonal sum of dimension 1 spaces and some symplectic space N. The symplectic space must admits a

symplectic basis, such that the matrix of its product is $H_n = \begin{pmatrix} 0 & \mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix}$. Then, it suffices to show that $H_n = 2n\langle 1\rangle$ in the Grothendieck-Witt group. An easy reordering of the basis elements shows that H_n is the orthogonal sum of n hyperbolic planes H_1. It suffices to show $H_1 = 2$ in $GW(F)$. We will show that $3\langle 1 \cong \langle 1\rangle + H$. Let V be a symmetric inner product space over a field of characteristic 2 with an orthogonal basis (e_1, e_2, e_3) such that $(e_i|e_i) = 1$ for all i. Then the basis $(e_1 + e_2 + e_3, e_1 + e_3, e_2 + e_3)$ exhibits V as isomorphic to $\langle 1\rangle + H_1$.
For the relations, we will instead prove them for $W(F)$, it will be equivalent since a class of space is 0 in $W(F)$ if and only if the space is split, and because the relations are between spaces of the same rank. All three relations are easily seen to be satisfied in $W(F)$. To show that any relation stems from those, we will need the following lemma, that appears as lemma 5.6 in Milnor-Husemoller :

Lemma 1.7 *Let $\alpha_1, \dots, \alpha_n$ and $\beta_1, \dots, \beta_n$ be elements of $F^\times$ such that $\sum_{i=1}^{n}\langle\alpha_i\rangle = \sum_{i=1}^{n}\langle\beta_i\rangle$ in $W(F)$, then, it is possible to pass from $(\alpha_1, \cdots, \alpha_n)$ to $(\beta_1, \cdots, \beta_n)$ by changing only two entries at a time, using only the relations of the theorem, and thus preserving the Witt class.*

Proof We proceed by induction on n. If $n = 2$, there are two cases. Either $\langle\alpha_1\rangle + \langle\alpha_2\rangle$ is isotropic, then, there are some $x_1, x_2 \in F^2$ such that $\alpha x_1^2 + \alpha x_2^2 = 0$. then, this means that $\langle\alpha_2\rangle = \langle-\alpha_1\rangle$, so we may replace the couple by $(\alpha_1, -\alpha_1)$, then by $(1, -1)$, and do the same with β_1, β_2. If the space are anisotropic, then they are isomorphic, then $\beta_1 = \alpha_1 x_1^2 + \alpha_2 x_2^2$ for some x_1, x_2, so that we may replace (α_1, α_2), by $(\alpha_1 x_1^2, \alpha_2 x_2^2)$ by GW1), then by $(\beta_1, \beta_1\alpha_1\alpha_2)$ by GW3). But the spaces are isomorphic, so their determinant modulo $(F^\times)^2$ must be the same, so that $\beta_2 = \beta_1\alpha_1\alpha_2 c^2$ for some $c \in F^\times$. So that by GW1) again, we can replace $(\beta_1, \beta_1\alpha_1\alpha_2)$ by (β_1, β_2).
By induction on n, assume the result is proved for all integers less that $n - 1$. Once again, we can argue wether the spaces are anisotropic or not. If the spaces are isotropic, then there is a non-trivial n-uple $(x_1, \dots, x_n)$ such that $\sum_{i=1}^{n} \alpha_i x_i^2 = 0$, we can argue by induction on the number of non-zero x_is. If this number is 2, we can do as before and replace α_2 by $-\alpha_1$ to get $\langle\alpha_1\rangle + \langle\alpha_2\rangle = 0$ in $W(F)$. If $k > 2$, we can assume $\alpha_1 x_1^2 + \alpha_2 x_2^2 = \gamma \neq 0$, so that $\langle\alpha_1\rangle + \langle\alpha_2\rangle = \langle\gamma\rangle + \langle\delta\rangle$ for some δ in $W(F)$ by GW3). This diminishes k, so by induction we can find a sequence such that in the end, $\langle\alpha_1\rangle + \langle\alpha_2\rangle = 0$ in $W(F)$. We can do the same with β_1 and β_2, so that by induction on n, we are done.
If the spaces are anisotropic, they are isomorphic, just as before, we can find an equation $\beta_1 = \sum_{i=1}^{n} \alpha_i x_i^2$, once again, by induction on the number of non-zeros x_is, if it is two, we already treated the case, otherwise, by the same trick as before, we can find $\gamma \neq 0$ which is $\alpha_1 x_1^2 + \alpha_2 x_2^2$, and replace $\langle\alpha_1\rangle + \langle\alpha_2\rangle$ by $\langle\gamma\rangle + \langle\delta\rangle$, and conclude by induction. □

□

The Witt ring $W(F)$ has a unique prime ideal $I(F)$ such that $W(F)/I(F) = \mathbb{Z}/2\mathbb{Z}$. This ideal is the kernel of the rank modulo 2 morphism. We will make use of the following result, due to Pfister :

Proposition 1.8 *Let F be a field, then, we have an exact sequence of groups* $1 \to I^2(F) \to I(F) \to F^\times/(F^\times)^2 \to 1$.

Proof Let X be an inner product space of rank r, let $d(X)$ be its discriminant, it is the class mod $(F^\times)^2$ of $(-1)^{\frac{r(r-1)}{2}} \det(X)$.

One checks that split spaces have trivial discriminant, and that the discriminant of an orthogonal sum of spaces is the product of the discriminant, so that this defines a group morphism $W(F) \to F^\times/(F^\times)^2$. Its restriction to $I(F)$ is surjective as $d(\langle u\rangle + \langle -1\rangle) = x(F^\times)^2$. Any element of I^2 must of course have trivial discriminant since I^2 is generated by products $(\langle\alpha\rangle + 1)(\langle\beta\rangle + 1) = \langle\alpha\beta\rangle + \langle\alpha\rangle + \langle\beta\rangle + 1$, which we can check have trivial discriminant. Let $\sum_{i=1}^{2r}\langle\alpha_i\rangle$ be an element of I. By using the following relations

$$\langle\alpha\rangle + \langle\beta\rangle \equiv \langle -\alpha\beta\rangle + \langle -1\rangle \mod I^2(F)$$
$$3\langle -1\rangle \equiv \langle 1\rangle \mod I^2(F)\,.$$

We can reduce the class $\sum_{i=1}^{2r}\langle\alpha_i\rangle$ to $\langle\beta\rangle + \langle -1\rangle$ modulo I^2. If $d(\langle\beta\rangle + \langle -1\rangle) = F^2$, then β must be a square so that $\langle\beta\rangle + \langle -1\rangle = 0$, i.e $\sum_{i=1}^{2r}\langle\alpha_i\rangle \in I^2(F)$. $\square$

Let us recall that we have a canonical morphism $W(F) \to \mathbb{Z}/2\mathbb{Z}$ which is unique. That sends the the class of a symmetric inner product space (X, β) to its dimension mod 2. We have a cartesian square of commutative rings :

$$\begin{array}{ccc} GW(F) & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ W(F) & \longrightarrow & \mathbb{Z}/2\mathbb{Z}\,. \end{array}$$

By what precedes, the subgroup of $K_0^{MW}(F)$ generated by the elements $\langle u\rangle$ satisfies the relation of $GW(F)$, indeed, the symbol $\langle u\rangle$ is multiplicative, by lemma 1.4 iii), GW1) is satisfied. By 1.4 iv), along with the fact that $\langle 1\rangle = 1$, GW2) is satisfied. Let us show that GW3) is satisfied. Since the symbol $\langle u\rangle$ is multiplicative, we can assume that $u + v = 1$, then, applying **1**, one has $[u][v] = 0$, multiplying by η^2 and using that $\eta[u] = \langle u\rangle - 1$ and that $\eta[v] = \langle v\rangle - 1$, developing, we get $\langle u\rangle + \langle v\rangle = 1 + \langle uv\rangle$ as expected.
This gives a surjective morphism $\phi_0 : GW(F) \to K_0^{MW}(F)$ (surjectivity comes from 1.3).
By relation **4**, multiplication by η kills h. Thus, the composition of surjective morphisms

$$GW(F) \longrightarrow K_0^{MW}(F) \xrightarrow{\times\eta^n} K_{-n}^{MW}(F)$$

factors to a surjective map $\phi_{-n} : W(F) \to K_{-n}^{MW}(F)$

Proposition 1.9 *For all $n \geq 0$, the map ϕ_{-n} is an isomorphism.*

Proof We will use Milnor's epimorphism $s_n : K_n^M(F) \to i^n(F)$ where $i^n(F) = I^n(F)/I^{n+1}(F)$ is the n-th graded piece of the graded ring associated to the filtration $I^n(F)$ of $W(F)$. For $n \leq 0$, we set $I^n(F) = W(F)$. Let us recall the construction of this epimorphism. It sends the symbol $(a_1, \dots, a_m) \in (F^\times)^m$ to the class $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$.
We define $J^n(F)$ by the following cartesian square :

$$\begin{array}{ccc} J^n(F) & \longrightarrow & I^n(F) \\ \downarrow & & \downarrow \\ K_n^M(F) & \longrightarrow & i^n(F)\,. \end{array}$$

in which the rightmost vertical map is the quotient map $I^n(F) \to I^n(F)/I^{n+1}(F)$. The collection of modules $J^*(F)$ form a graded ring, indeed we have such a square for all $n \in \mathbb{Z}$ and the maps forming the pullback respect multiplication.
Notice that for $n = -1$, the pullback square is actually the square

$$\begin{array}{ccc} W(F) & \longrightarrow & W(F) \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 1\,. \end{array}$$

So that $J^{-1}(F) = W(F)$. Indeed we know that $K_n^M(F) = K_n^{MW}(F)/(\eta)$ and by 1.3, η is a factor of all ements of $K_n^{MW}(F)$ when $n \leq -1$.
So there is an element $\eta \in J^{-1}(F)$ that corresponds to the element $1 \in W(F)$. Let $u \in F^\times$ and let $[u]$ be the element of $J^1(F)$ corresponding to the pair $([u], \langle u\rangle - 1) \in K_1^M(F) \times I(F)$. This element is well-defined, as u is sent to $\langle -1, u\rangle \in i^1(F)$ and since $\langle u\rangle - 1 = \langle u\rangle + \langle -1\rangle$ in $W(F)$ and both classes represent a symmetric inner product space with an orthogonal basis e_1, e_2 such that $e_1.e_1 = -1$ and $e_2.e_2 = \beta$, they indeed map to the same element of $i^1(F)$.
Let us check that the four relations defining $K_n^{MW}(F)$ holds in the graded ring $J^*(F)$ with the given symbols η and $[u]$, so that we thus have a morphism $\psi : K_*^{MW}(F) \to J^*(F)$.

— Relation **1** holds by definition for the symbol $[u]$ of $K_*^M(F)$. In $W(F)$, we have

$$\begin{aligned}
(\langle u\rangle + \langle -1\rangle) \otimes (\langle 1-u\rangle + \langle -1\ \rangle) &= \langle u(1-u)\rangle + \langle -u\rangle + \langle u-1\rangle + 1 \\
&= \langle u(1-u)\rangle + \langle -1\rangle + \langle u(u-1)\rangle + 1 && \text{by GW3)} \\
&= 0 && \text{by GW2).}
\end{aligned}$$

So relation **1** holds in $J^*(F)$ for the symbol $[u]$.

— For relation **2** : by definition, $[ab] = [a] + [b]$ in $K_1^M(F)$, and $\eta.([a][b])$ corresponds in $K_1^M(F) \times I(F)$ to the tuple $(0, (\langle a\rangle - 1)(\langle b\rangle - 1))$. In $I(F)$, we have

$$(\langle a\ \rangle - 1) \otimes (\langle b\rangle - 1) = \langle ab\rangle - \langle a\rangle - \langle b\rangle + 1\,.$$

So that, in $J^1(F)$:

$$\begin{aligned}
[a] + [b] + \eta[ab] &= ([a] + [b], \langle a\rangle - 1 + \langle b\rangle - 1 + \langle ab\rangle - \langle a\rangle - \langle b\rangle + 1) \\
&= ([a] + [b], \langle ab\rangle - 1) \\
&= [ab]\,.
\end{aligned}$$

So relation **2** holds.

— Relation **3** is automatic due to our choice of the symbol η as the identity of $W(F)$.

— Relation **4** holds since

$$\begin{aligned}
h &= \eta[-1] + 2 \\
&= (0, \langle -1\rangle - 1) + 2 \\
&= (1, \langle -1\rangle + 1) \\
&= (1, 0) && \text{By GW2).}
\end{aligned}$$

So that $\eta.h = (0, 0)$.

We have the following claim :

Lemma 1.10 *The map $\psi : K_*^{MW}(F) \to J^*(F)$ is an epimorphism.*

Proof First, notice that given any class $v \in I$, there exists some $u \in F^\times$ such that $([u], v) \in J^1$. Indeed, the congruence relations

$$\begin{aligned} \langle\alpha\rangle + \langle\beta\rangle &\equiv \langle -\alpha\beta\rangle + \langle -1\rangle \mod I^2 \\ \langle -1\rangle + \langle -1\rangle + \langle -1\rangle &\equiv \langle 1\rangle \mod I^2 \end{aligned}$$

in $W(F)$ imply that any class $v = \langle\alpha_1\rangle + \cdots \langle\alpha_{2r}\rangle$ is congruent modulo I^2 to some $\langle u\rangle + \langle -1\rangle$. This precisely means that $([u], v) \in J^1$. Moreover, any two u_1, u_2 such that $([u_1], v) \in J^1(F)$ and $([u_2], v) \in J^1(F)$ are such that u_1 and u_2 only differ by a square in $F^\times$. Indeed it is known that the epimorphism $s_1 : K_1^M(F) \to I/I^2$ has a kernel precisely equal to $2K_1^M(F)$, i.e s_1 induces an isomorphism $F^\times/(F^\times)^2 \to I/I^2$. That $([u_1], v)$ and $([u_2], v)$ are in $J^1(F)$ means that $s_1(u_1) = s_1(u_2)$, i.e $u_1 u_2^{-1} \in (F^\times)^2$.
Now, notice that if $([u], v)$ is in the image of ψ, then so is $([u'], v)$ for any u' such that $([u'], v) \in J^1(F)$. Indeed, we can write $u = u'w^2$, if $([u], v) = \sum_{i=1}^{r}([\alpha_i], \langle\alpha_i\rangle + \langle -1\rangle)$, then $u = \prod_{i=1}^{r} \alpha_i$. Replacing α_1 by $\alpha_1 w^{-2}$ does not affect the Witt class v, and shows that

$$([u'], v) = ([\alpha_1 v^{-2}], \langle\alpha_1\rangle + \langle -1\rangle) + \sum_{i=2}^{r}([\alpha_i], \langle\alpha_i\rangle + \langle -1\rangle)\,.$$

Thus, to show surjectivity on $J^1(F)$, it suffices to show that for every $v \in I$, there are some $u \in F^\times$ such that $([u], v) \in J^1(F)$ and $([u], v)$ is in the image of $K_1^{MW}(F) \to J^1(F)$. This certainly is true if $v = \langle\alpha\rangle - 1$ for some α. It is known that such classes generate additively I. So $K_1^{MW}(F) \to J^1(F)$ is an epimorphism.
To show surjectivity on $J^n(F)$ for $n \geq 1$, it is enough to show it for J^1. Indeed, let $(u, v) \in J^n(F)$. We recall that $K_*^M(F)$ is generated by $K_1^M(F)$, so that $u = \sum_{i=1}^{r}\prod_{j=1}^{n}[a_{ij}]$. By definition, we have that $v \equiv s_n(u) \mod I^{n+1}$. So that we have

$$v = \sum_{i=1}^{r}\bigotimes_{j=1}^{n}\langle 1, -a_{ij}\rangle + c$$

where $c \in I^{n+1}$. So that

$$(u, v) = \sum_{i=1}^{r}\prod_{j=1}^{n}([a_{ij}], \langle a_{ij}\rangle + \langle -1\rangle) + (0, c)\,.$$

Notice that since $c \in I^{n+1}$, $(0, c) \in J^1(F) = \mathrm{Im}(\psi_1)$. So that ψ_n is surjective if $n \geq 1$.
It remains to show surjectivity if $n \leq 0$. For $n = 0$. The pullback is the following one :

$$\begin{array}{ccc} GW(F) & \longrightarrow & W(F) \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \end{array}$$

and the symbol $\langle u\rangle$ of $K_0^{MW}(F)$ is sent to $\langle u\rangle$ of $GW(F)=J^1(F)$. Indeed $\langle u\rangle = 1+\eta[u]$, so that $\langle u\rangle$ corresponds to the element $\eta.([u],\langle u\rangle -1)+(1,1)$. Which is the element $(1,\langle u\rangle)$ since multiplication by η sends a class $(a,b)\in J^k(F)$ to $(0,b)$ where b is viewed as an element of $J^{k-1}(F)$. The element $(1,\langle u\rangle)$ corresponds to $\langle u\rangle$ with the identification $J^0(F)=GW(F)$. For $n<0$, the pullback square is again the square

$$\begin{array}{ccc} W(F) & \longrightarrow & W(F) \\ \downarrow & & \downarrow \\ * & \longrightarrow & *\,. \end{array}$$

The element $\langle u\rangle$ of $J^n(F)=W(F)$ corresponds to $(*,\langle u\rangle)$ with this identification. This element is exactly $\eta^{-n}(1,\langle u\rangle)$, wich is $\eta^{-n}\psi(\langle u\rangle)$. So the map ψ is indeed surjective. □

Back to the proof of proposition 1.9. We know that ψ is an epimorphism, and for $n\geq 0$, the composite $\phi_n\circ\psi_n$ is the identity. Indeed, the proof of the preceding lemma shows that this is the case on generators. But since both maps are epimorphism, this means that ϕ_n is an isomorphism. □

Corollary 1.11 *The canonical morphism $K_*^{MW}(F)\to W(F)[\eta,\eta^{-1}]$ defined by $[u]\to\eta^{-1}(\langle u\rangle -1)$ induces an isomorphism $K_*^{MW}(F)[\eta^{-1}]\to W(F)[\eta,\eta^{-1}]$.*

Lemma 1.12 *Let $a\in F^\times$ and $n\in\mathbb{Z}$. Let $n_\epsilon = \sum_{i=1}^{n}\langle(-1)^{i-1}\rangle$ if $n\geq 0$ and $-\langle -1\rangle(-n)_\epsilon$ if $n<0$. Then, we have $[a^n]=n_\epsilon[a]$ in $K_1^{MW}(F)$.*

Proof We proceed by induction : if $n=0$, this is clear : both expressions are 0. For $n\geq 1$,

$$\begin{aligned}
[a^n] &= [a^{n-1}]+[a]+\eta[a^{n-1}][a] \\
&= (n-1)_\epsilon[a]+[a]+(n-1)_\epsilon\eta[a][a] \\
&= (n-1)_\epsilon[a]+[a]+(n-1)_\epsilon\eta[-1][a] && \text{ii) of 1.4} \\
&= ((n-1)_\epsilon+1+(n-1)_\epsilon\eta[-1])\,[a] \\
&= ((n-1)_\epsilon\,(1+\eta[-1])+1)\,[a] \\
&= (\langle -1\rangle(n-1)_\epsilon+1)\,[a] \\
&= n_\epsilon[a]\,.
\end{aligned}$$

□

Proposition 1.13 *Let F be a field in which any unit is a square. Then the epismorphism $K_*^{MW}(F)\to K_*^M(F)$ is an epimorphism in degree ≥ 0, and $K_*^{MW}(F)\to K_*^W(F)$ is an isomorphism in degree <0.*

Proof Since -1 is a square. We have that $\langle -1\rangle = 1$, so that $h=2$ and relation **4** becomes $2\eta=0$. By the previous lemma, $\eta[a^2]=2\eta[a]$, so $\eta[a^2]=0$. Since any unit is a square, $\eta[b]=0$ for all b, so that relation **2** becomes $[ab]=[a]+[b]$. Thus in degree ≥ 0, K_*^M and K_*^{MW} have the same generator and relations. In degree <0, we know that $K_n^{MW}(F)\cong W(F)$ by proposition 1.9. Since $h=2$, $K_n^W(F)\cong W(F)/2W(F)$. But it is known that for a field in which every unit is a square, $W(F)\cong\mathbb{Z}/2\mathbb{Z}$. So that taking the quotient by 2 will be an isomorphism. □