CONDENSED AND LOCALLY COMPACT ABELIAN GROUPS

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Abstract. These are notes for a workshop on Peter Scholze's lectures on Clausen-Scholze's condensed mathematics. This is merely a huge expansion (in particular concerning results from algebraic topology, see Section 3) of the first four of these lectures. Many details have been added to help the attendees of the workshop and might be useful for students interested in the marvelous results obtained by Dustin Clausen and Peter Scholze. Needless to mention that all results and idea of proofs are solely due to them.

Contents

Notations and conventions. We will simply say *space* for topological space.

Given a cardinal κ, a κ-sets (resp. κ-space) will be a set (resp. space whose underlying set is) of cardinal less than κ.

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Given an abelian category $\mathscr{A}$, we let $\mathrm{C}(\mathscr{A})$, $\mathrm{D}(\mathscr{A})$ and $\mathscr{D}(\mathscr{A})$ be respectively the associated abelian category of complexes, derived category and derived ∞-category. Note $\mathscr{D}(\mathscr{A})$ is a stable ∞-category, which is also a $\mathbb{Z}$-dg-category.

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1. Condensed sets

1.1. **Recall on topology.** Here is a (non-exhaustive) list of separation axioms on topological spaces.

Definition 1.1. Let X be a topological space. One says that X is:

- (T0) *Kolmogorov*: if $\forall x_0, x_1 \in X$, there exists i and a neighborhood V_i of x_i which does not contain x_{1-i}.
- (T1) if $\forall x_0, x_1 \in X$, for all i, there exists neighborhoods V_i of x_i such that $x_{1-i} \notin V_i$.
- (T2) *Hausdorff*: (*séparé* for Bourbaki) if $\forall x_0, x_1 \in X$, for all i, there exists neighborhoods V_i of x_i such that $V_0 \cap V_1 = \emptyset$.
- (T3) *regular*: if for all closed subset $Z \subset X$ and all point $x \in X$, there exists a neighborhood V of Z and W of x in X such that $V \cap W = \emptyset$.

Obviously: (T0) $\Rightarrow$ (T1) $\Rightarrow$ (T2) $\Rightarrow$ (T3). A T1-space is a space such that for every point $x \in X$, $\{x\}$ is closed in X.

Remarque 1.2. Some authors say *Fréchet* for T1. However, we think it is not a good idea as the terminology Fréchet spaces is used in functional analysis[1] and the latter concept is much more relevant than being T1.

Now, some classical variants of compactness.

Definition 1.3. Let X be a topological space. Recall:

1. X is *compact* (sometime "quasi-compact") if all open cover of X admits a subcover which is finite.
2. X is *locally compact* if any point $x \in X$ admits a compact neighborhood.
3. X is *compactly generated* (also "k-space") if it has the final topology for all continuous maps $K \to X$ from a compact Hausdorff space K.

We will adopt the convention of Barwick-Haine ([BH19]) and call *compacta* a compact Hausdorff space.

Remarque 1.4.
1. One easily checks that: compact $\Rightarrow$ locally compact $\Rightarrow$ compactly generated.
2. Hausdorff (resp. compactly generated) spaces are obviously stable under projective limits.

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[1] *i.e.* metrizable, complete locally convex vector space;

(3) Tychonov theorem says that compact spaces are stable under projective limits (originally arbitrary products).[2] This implies that locally compact spaces are stable under projective limits.

1.5. In algebraic topology, the last two notions of the previous definition have been chosen for their good categorical properties.

Locally compact spaces are (almost exactly[3]) those spaces X such that the functor $Y \times X \times Y$ admits a right adjoint, $Y \mapsto \mathrm{Map}(X, Y)$. In other words, the category of locally compact spaces is cartesian closed. Note that the space $\mathrm{Map}(X, Y)$ is given as follows:

- Its underlying set is that of continuous application $X \to Y$, $\mathrm{Hom}_{\mathscr{T}op}(X, Y)$.
- The topology on $\mathrm{Hom}_{\mathscr{T}op}(X, Y)$ is the so-called *compact-open topology.* It is the topology generated[4] by the open subsets of the form

$$V(X, U) = \{f : X \to Y \mid f(K) \subset U\},$$

for any compact subspace $K \subset X$ and any open $U \subset Y$.

The category $\mathscr{T}op^{\mathrm{lc}}$ of locally compact spaces is stable under limits (see preceding remark) and coproducts. However, it is not stable under cokernel (quotient space) so it does not admit all colimits. This is the reason to introduce compactly generated spaces.

1.6. Let $\mathscr{T}op^{\mathrm{cg}}$ be the category compactly generated spaces. As this subcategory of $\mathscr{T}op$ is table under projective limit, we formally get an adjunction :

$$i : \mathscr{T}op^{\mathrm{cg}} \leftrightarrows \mathscr{T}op : k$$

where i is the canonical inclusion. The right adjoint k can be described as follows: for X a topological space, $k(X)$ is the topological space whose underlying set is X and whose closed subsets are those F such that: for all compacta $K \subset X$, $K \cap F$ is closed in X.

The existence of k immediately implies that $\mathscr{T}op^{\mathrm{cg}}$ admits all colimits which can be computed in $\mathscr{T}op$ (*i.e.* a colimit of compactly generated spaces is compactly generated). Moreover, $\mathscr{T}op^{\mathrm{cg}}$ admits all limits which are obtained by computing in $\mathscr{T}op$ and then applying k.

Note also that a product of compactly generated spaces is compactly generated. As compactly generated spaces are in particular locally compact, is follows from the previous paragraph that $\mathscr{T}op^{\mathrm{cg}}$ is cartesian closed, using the formula:

$$\mathrm{Map}_{\mathscr{T}op^{\mathrm{cg}}}(X, Y) = \mathrm{Map}_{\mathscr{T}op^{\mathrm{lc}}}(X, Y).$$

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[2]Recall Tychonov theorem is actually equivalent to the axiom of choice.

[3]In full generality, one has to use the notion of *core-compactness*: see [EH02, Th. 5.3]

[4]*i.e.* the set of open subsets is the smallest one containing object of the base and stable under unions and finite intersections;

In particular, the underlying set to $\mathrm{Map}_{\mathscr{T}op^{\mathrm{cg}}}(X,Y)$ is $\mathrm{Hom}_{\mathscr{T}op}(X,Y)$ but its topology is a variant of the compact-open topology. Remark nevertheless it coincides with the compact-open topology whenever Y is Hausdorff.

All in all, $\mathscr{T}op^{\mathrm{cg}}$ is what Steenrod called a *convenient category of topological spaces* (see Remark 1.30 for more discussion).

Remarque 1.7. Of course, a similar construction exists for compactas. In fact, the inclusion

$$\mathscr{T}op^{\mathrm{K}} \to \mathscr{T}op$$

commutes with projective limits (Remark 1.4(3) for products). One can therefore show the existence of a left adjoint $\beta : \mathscr{T}op \to \mathscr{T}op^{\mathrm{K}}$ by the usual limit formula, suitably controlled. Alternatively, there is a nice model of the Stone-Čech compactification $\beta(X)$ of a space X: one defines $\beta(X)$ as the set of ultrafilters of X equipped with the topology generated by open subsets of the form $\{\mathcal{U} \in \beta(X) \mid Y \in \mathcal{U}\}$ for any subset $Y \subset X$.

The following terminology is classical in topology, but we recall it for completeness.

Definition 1.8. A surjective continuous map $p : T \to S$ between topological epimorphisms is a *quotient map* if the following equivalent conditions are satisfied:

(i) for any subset $U \subset S$, $p^{-1}(U)$ is open if and only if U is open.
(ii) for any map $f : S \to X$ between spaces, f is continuous if and only if $f \circ p$ is continuous.

The equivalence of the two conditions is an easy exercise in topology.

Example 1.9. Let $p : T \to S$ be a continuous map between compacta. Then p is a quotient map if and only if it is surjective. This follows from the facts that the subsets of S (resp. T) are compactas if and only if they are closed and that the image of a compacta by a continuous map is a compacta.

Definition 1.10. Let S be a topological space. One says that:

- S is *totally disconnected* if for any $s \in S$, $\{s\}$ is a connected component of S.
- S is a *Stone space* if S is compact, Hausdorff and totally disconnected.

Note that totally disconnected spaces, as well as Stone spaces are stable under projective limits.

1.11. Let $\mathscr{S}et^{\mathrm{f}}$ be the category of finite sets. A pro-object $(S_i)_{i\in I}$ of $\mathscr{S}et^{\mathrm{f}}$ is simply called a *profinite set* and the corresponding category is denoted by $\mathscr{S}et^{\mathrm{pf}}$. To such a pro-object, one can associate a topological space

$$S_\infty := \varprojlim_{i\in I} S_i$$

by considering the discrete topology on each S_i and taking the projective limit. For short, we call such a space a *profinite space*.

There is an alternate description of profinite spaces. As each S_i is a Stone space, one deduces that S_∞ is a Stone space. Moreover, one has the following results, a particular case of the general "Stone duality" for Boolean algebras (see [Joh82]).

Theorem 1.12. *Consider the above notations. Then the functor from profinite sets to Stone spaces*

$$L : \mathscr{S}et^{\mathrm{pf}} \to \mathscr{T}op^{\mathrm{Stone}}, (S_i)_{i\in I} \mapsto S_\infty$$

is an equivalence of categories.

Indeed, an inverse functor π to the projective limit functor L can be defined as follows: given a Stone space S, we let $\mathcal{P}^f(S)$ be the set of finite subsets of S, ordered by inclusion. Obviously, $\mathcal{P}^f(S)$ is cofiltered. Then we define $\pi(S)$ as the profinite set $(F)_{F\in\mathcal{P}^f(S)}$.

Remarque 1.13. Given this theorem, and following Scholze, we will say *profinite spaces* for *Stone topological spaces.*

Definition 1.14. A topological space is *extremally disconnected* if the closure of every open subset is open.

Note that extremally disconnected implies totally disconnected but the converse is not true: the topological group $\mathbb{Q}$ is totally disconnected but not extremally disconnected.

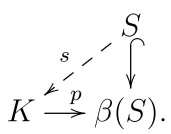
Here is a classy theorem of Gleason (see Th. 2.5 of [Gle58]).

Theorem 1.15. *In the category of compactas, the projective objects are the compacta which are extremally disconnected spaces.*

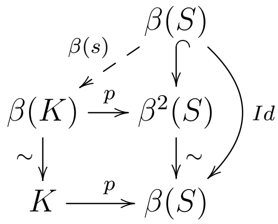
In other words, for a compacta X, the following conditions are equivalent:

(i) *X is extremally disconnected;*
(ii) *for any compacta K, any surjection $K \to X$ splits.*

Example 1.16. The Stone-Čech compactification $\beta(S)$ of any discrete topological space S is extremally disconnected. Indeed, let $K \xrightarrow{p} \beta(S)$ be a surjection. As S is a subset of $\beta(S)$, there exists a lift s:



As the Stone-Čech compactification functor is the left adjoint to the inclusion functor from compacta to topological spaces, applying β to the previous diagram gives:



so that $\beta(s)$ gives the desired splitting.

1.2. **Clausen-Scholze's condensed set.**

Definition 1.17. Consider the notations of 1.11. We define a Grothendieck topology on $\mathscr{S}et^{\mathrm{pf}}$ by choosing for coverings the finite families $(S'_i \to S)_{i \in I}$ of profinite spaces such that $\sqcup_i S'_i \to S$ is surjective.

We denote by $\mathscr{S}et^{\mathrm{cds}}$ the subcategory of accessible objects of the topos associated with $\mathscr{S}et^{\mathrm{pf}}$ (see [BH19, §0.3, 1.4]). An object (resp. group, abelian group) of $\mathscr{S}et^{\mathrm{cds}}$ is called a *condensed* set (resp. *group, abelian group*). We also denote by $\mathscr{A}b^{\mathrm{cds}}$ the abelian category of condensed abelian groups.

The accessibility condition can be translated into the fact that a condensed set is a sheaf on $\mathscr{S}et^{\mathrm{pf}}$ which is determined by[5] its restriction to the full subcategory $\kappa - \mathscr{S}et^{\mathrm{pf}}$ of profinite spaces whose underlying set is of cardinal less than some cardinal κ. Then one says that T is *determined by* κ.

Remarque 1.18. A condensed set T is therefore a presheaf on profinite κ-sets for some cardinal κ such that:

(C1) $T(\emptyset) = \emptyset$.
(C2) T commutes with finite sums: $T(S \sqcup S') \simeq T(S) \times T(S')$.
(C3) Given any surjection $p : S' \to S$ of profinite sets, the following map is a bijection:
$$T(S) \xrightarrow{p^*} \operatorname{coKer}\left(T(S') \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} T(S' \times_S S')\right)$$
where p_1, p_2 are the obvious projection maps.

Proposition 1.19. *We consider the inclusion of sites:*
$$\mathscr{T}op^{\mathrm{edK}} \subset \mathscr{T}op^{\mathrm{Stone}} \subset \mathscr{T}op^{\mathrm{K}}$$
respectively of extremally disconnected compacta (ed-compacta for short), profinite spaces and compacta, each of which is equipped with the (proper) surjective topology.[6]

Then the associated toposes (resp. accessible toposes) are all equivalent.

[5] more precisely, is the left Kan extension of

[6] *i.e.* covers are generated by finite families of surjective maps (which are necessarily proper as spaces are compacta).

Proof. This follows from the fact each object of one of the two bigger site admits a cover with objects from the immediately smaller site, as obtained by the following lemma.

Lemma 1.20. *Any compacta K admits a continuous surjection $K' \to K$ whose source is an extremally disconnected compacta.*

Let us prove the lemma. We let K_d be the associated discrete topological space to K. We have a continuous bijection $K_d \to K$. As the Stone-Čech compactification functor β is left adjoint to the forgetful functor from compacta to topological spaces, we get a continuous surjection $K' := \beta(K_d) \to \beta(K) \simeq K$. But then K' is extremally disconnected (see Ex. 1.16). □

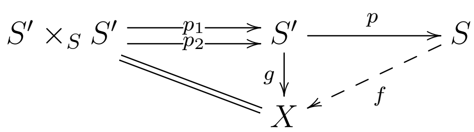
Corollary 1.21. *A condensed set (resp. abelian group) T is equivalent to an accessible presheaf (resp. abelian presheaf) on the category of ed-compacta which commutes with finite products;*
i.e. *for any ed-compactas E, E', $T(E \times E') = T(E) \sqcup T(E')$.*

1.3. Embeddings in condensed sets.

1.22. Let X be a topological space. Through the Yoneda embedding, it can be considered as a presheaf on $\mathscr{T}\!op$. We will denote by $\underline{X}$ its restriction to the category of profinite spaces $\mathscr{S}\!et^{\mathrm{pf}}$.

Lemma 1.23. *The presheaf $\underline{X}$ is a sheaf on the condensed site. Moreover, if X is Fréchet, then $\underline{X}$ is accessible (Definition 1.17).*

Proof. We check the conditions of Remark 1.18 for the presheaf $\underline{X}$. As the functor $\mathrm{Hom}_{\mathscr{T}\!op}(-, X)$ commutes with colimits, conditions (C1) and (C2) are clear. Let us check condition (C3). In other words, we have to show the existence and uniqueness of the dotted arrow satisfying the following problem:



where p is any surjection of profinite spaces of some bounded cardinality κ. The existence of f as an application is an easy exercise in set theory[7]. We have to prove that f is continuous. That simply follows as S' and S are compacta, so that any surjection $p : S' \to S$ is a quotient map (see Example 1.9). □

In particular, one gets a canonical functor: $u : \mathscr{T}\!op^{\mathrm{T1}} \to \mathscr{S}\!et^{\mathrm{cds}}$ from Fréchet spaces to condensed sets.

[7]It says that the effective topology on the category of sets is subcanonical *i.e.* any representable presheaf is a sheaf.

Remarque 1.24. Note that u restricted to the category of profinite spaces is just the Yoneda embedding. So u restricted to $\mathscr{T}op^{\text{Stone}} \simeq \mathscr{S}et^{\text{pf}}$ is obviously fully faithful.

We are going to build a left adjoint to u. For this we need a supplementary condition on condensed sets which ensures the condensed structure will be "sufficiently compatible with topology".

Definition 1.25. Let T be a condensed set.

(1) We define the *canonical topology* on the underlying set $T(*)$ as the final topology for the set of maps:
$$\alpha_* : S \to T(*)$$
indexed by (S, α) for S a profinite space of cardinal less than κ and $\alpha \in T(S)$, identified with a map $\eta : \underline{S} \to T$.
From now on, if we refer to a topology on $T(*)$, it will always be the canonical topology.

(2) One says that T is *pointwise quasi-compact* if for any $x \in T(*)$, the induced map of topological spaces
$$\{x\} \to T(*)$$
is quasi-compact.

We will denote by $\mathscr{S}et^{\text{cdsqc}}$ the full subcategory of pointwise quasi-compact condensed set.

Note in particular that the last property of T implies that $T(*)$ is Fréchet.

Remarque 1.26. Beware that pointwise quasi-compact condensed sets are not stable under finite products.

Proposition 1.27. *For any Fréchet space X, the condensed set $\underline{X}$ is pointwise quasi-compact.*

Moreover, the induced functor $u : \mathscr{T}op^{\text{T1}} \to \mathscr{S}et^{\text{cdsqc}}$ admits as a left adjoint the functor:
$$ev : \mathscr{S}et^{\text{cdsqc}} \to \mathscr{T}op^{\text{T1}}, T \mapsto T(*)$$
using the previous definition.

Proof. We will build reciprocal bijections:
$$a : \text{Hom}_{\mathscr{S}et^{\text{cds}}}(T, \underline{X}) \leftrightarrows \text{Hom}_{\mathscr{T}op}(T(*), X) : b.$$

Let us build a: consider a natural transformation $\eta : T \to \underline{X}$. Evaluating at the profinite set $*$, we get a map $a(\eta) = \eta_* : T(*) \to X$. Let us show that $a(\eta)$ is continuous. By definition of the topology on $T(*)$, it is sufficient to prove that for any profinite space S and any $\alpha : \underline{S} \to T$, the composite
$$S \xrightarrow{\alpha_*} T(*) \xrightarrow{\eta_*} X$$

is continuous. By Yoneda lemma, this composite corresponds to the image of $\alpha \in T(S)$ by the map $\eta_S : T(S) \to \underline{X}(S) = \mathrm{Hom}_{\mathscr{T}op}(S, X)$, and this concludes.

Now we construct b: consider a continuous map $f : T(*) \to X$. We define a natural transformation $b(f) : T \to \underline{X}$ as follows. For a profinite space S, and $\alpha \in T(S)$ identified with a natural transformation $\alpha : \underline{S} \to T$, we obtain a composite map:

$$S \xrightarrow{\alpha_*} T(*) \xrightarrow{f} X$$

which is continuous as α_* is continuous by definition of the topology on $T(*)$. So we can put $b(f)_S(\alpha) = f \circ \eta_*$. This construction is obviously natural in S, giving the desired natural transformation $b(f)$.

It is now clear that a and b are reciprocal bijections. □

We will say that a compactly generated space X is *accessible* if there exists a cardinal κ such that its topology is the final one for the set of maps $K \to X$ where K runs over all compacta of cardinal less than κ.

Corollary 1.28. *The functor $u : \mathscr{T}op^{\mathrm{T1}} \to \mathscr{S}et^{\mathrm{cdsqc}}$ is fully faithful when restricted to the category of accessible compactly generated spaces. Moreover, for any Fréchet space X, there exists a canonical homeomorphism:*

$$ev \circ u(X) = \underline{X}(*) \simeq k(X) \qquad \text{(1.28.a)}$$

(see Paragraph 1.6 for the functor k, except that one has to change slightly the definition in order to get a functor to accessible compactly generated spaces).

The homeomorphism (1.28.a) is clear from the definitions of the topologies on $X(*)$ and on $k(X)$. So the corollary is immediate as when X is accessible compactly generated, $k(X) \simeq X$.

Remarque 1.29. One can moreover identify the essential image of the restriction of u to *accessible compactly generated and weak Hausdorff spaces*: it is the full subcategory made by the pointwise quasi-compact condensed sets which are quasi-separated as sheaves on $\mathscr{S}et^{\mathrm{pf}}$. We refer the interested reader to [Sch19, Th. 2.16].

Remarque 1.30. In 1967, Steenrod underlines the need for a *convenient category of topological spaces* (see [Ste67]). The main problem is that the category of topological spaces does not admits internal Hom — the so-called *mapping space* of algebraic topology. In that paper, he introduces the category of compactly generated spaces, and advocate the good properties of this category (which we have recalled in Paragraph 1.6).

Given the previous corollary, the category of condensed set can also be interpreted as such a convenient category of spaces.

2. Condensed homological algebra

2.1. **Condensed abelian groups.**

2.1. Recall that a Grothendieck abelian category is an abelian category $\mathscr{A}$ which admits a (small) set of generators and satisfies the following additional properties:

(AB3) all coproducts (and hence all colimits) exists in $\mathscr{A}$.
(AB5) filtered colimits (and hence coproducts) are exact.

Grothendieck also introduced the following supplementary axioms:

(AB3*) all products (and hence all limits) exists in $\mathscr{A}$.
(AB4*) arbitrary products are exact.
(AB6) filtered colimits commute with arbitrary products.

Because of Corollary 1.21, one deduces the following result.

Proposition 2.2. *The category of condensed abelian groups* $\mathscr{A}b^{\mathrm{cds}}$ *is a Grothendieck abelian category which satisfies properties (AB3*), (AB4*), (AB6).*

Proof. This follows from the fact the category $\mathscr{A}b^{\mathrm{cds}}$ can be identified with the category $\mathrm{PSh}^{\mathrm{a}}(\mathscr{T}op^{\mathrm{edK}}, \mathbb{Z})$ of abelian presheaves $(\mathscr{T}op^{\mathrm{edK}})^{op} \to \mathscr{A}b$ which are both accessible and additive, thanks to Corollary 1.21. So $\mathscr{A}b^{\mathrm{cds}}$ inherits all properties stated in the proposition from their analog in $\mathscr{A}b$. □

Remarque 2.3. We will recall from the proof, more precisely from Corollary 1.21, that the category $\mathscr{A}b^{\mathrm{cds}}$ of condensed abelian groups can be identified with the category of additive accessible presheaves of abelian groups on $\mathscr{T}op^{\mathrm{edK}}$. We will denote that category by $\mathrm{PSh}^{\mathrm{a}}(\mathscr{T}op^{\mathrm{edK}}, \mathbb{Z})$.

Let us recall the following basic definition:

Definition 2.4. An object A of a Grothendieck abelian category $\mathscr{A}$ is *compact* (resp. *projective*) if the functor $\mathrm{Hom}_{\mathscr{A}}(-, A)$ commutes with filtered colimits (resp. is exact).

2.5. The inclusion $\mathscr{A}b^{\mathrm{cds}} \to \mathscr{S}et^{\mathrm{cds}}$ admits as usual a left adjoint, the free condensed abelian group functor $T \mapsto \mathbb{Z}[T]$. This can be computed as the sheaf associated with the presheaf $S \mapsto \mathbb{Z}[T(S)]$.

Given now a compacta S, we simply put: $\mathbb{Z}[S] = \mathbb{Z}[\underline{S}]$. By adjunction and the Yoneda lemma (seeing a condensed set as a sheaf on compacta by Prop. 1.19), one gets the relation:

$$\mathrm{Hom}_{\mathscr{A}b^{\mathrm{cds}}}(\mathbb{Z}[S], M) \simeq M(S) =: \Gamma(S, M). \tag{2.5.a}$$

Using Remark 2.3, we now identify the abelian categories $\mathscr{A}b^{\mathrm{cds}}$ and $\mathrm{PSh}^{\mathrm{a}}(\mathscr{T}op^{\mathrm{edK}}, \mathbb{Z})$. But limits and colimits in $\mathrm{PSh}^{\mathrm{a}}(\mathscr{T}op^{\mathrm{edK}}, \mathbb{Z})$ are computed termwise. In particular, for any ed-compacta S, the functor

$$\mathrm{Hom}_{\mathscr{A}b^{\mathrm{cds}}}(\mathbb{Z}[S], -) \simeq \Gamma(S, -)$$

commutes with limits and colimits. One immediately deduces:

Proposition 2.6. *Consider the above notations. For any ed-compacta S, the condensed abelian group $\mathbb{Z}[S]$ is projective and compact. Moreover, the category $\mathscr{A}b^{\mathrm{cds}}$ is generated by the objects $\mathbb{Z}[S]$ for all ed-compactas S.*

As another corollary, we immediately get:

Proposition 2.7. *The category $\mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$ is a compactly generated triangulated category, with generators the objects $\mathbb{Z}[S]$ for S an ed-compacta.*

Moreover, for any complex C of condensed abelian groups, and any ed-compacta S, on has:

$$\mathbf{R}\operatorname{Hom}_{\mathscr{A}b^{\mathrm{cds}}}(\mathbb{Z}[S], C) \simeq C(S).$$

Remarque 2.8. Of course, the above computation is false when S is a profinite space. The problem comes of course from the fact the topology on the profinite sites is non-trivial.

2.2. **Monoidal structure.**

2.9. Each condensed abelian group M can be uniquely written as:

$$M = \varinjlim_{S/M} \mathbb{Z}[S]$$

where the index of the colimit is the subcategory of the comma category $\mathscr{A}b^{\mathrm{cds}}/M$ made by the maps $\mathbb{Z}[S] \to M$.

Proposition 2.10. *There exists a unique closed monoidal structure on $\mathscr{A}b^{\mathrm{cds}}$ such that:*

1. *For ed-compactas S and S', $\mathbb{Z}[S] \otimes \mathbb{Z}[S'] = \mathbb{Z}[S \times S']$.*
2. *For any condensed abelian groups M and N, and any ed-compacta S,*

$$\Gamma(S, \underline{\operatorname{Hom}}(M, N)) = \operatorname{Hom}_{\mathscr{A}b^{\mathrm{cds}}}(\mathbb{Z}[S] \otimes M, N).$$

Moreover, the tensor product of two condensed abelian groups M and N is characterized by the properties that for any ed-compacta S,

$$\Gamma(S, M \otimes N) = M(S) \otimes_{\mathbb{Z}} N(S). \tag{2.10.a}$$

Remarque 2.11. Note in particular that the generators $\mathbb{Z}[S]$ of $\mathscr{A}b^{\mathrm{cds}}$ are not only projective and compact, but also flat for the above monoidal structure. In other words, $\mathscr{A}b^{\mathrm{cds}}$ shares all the good properties of the category of abelian groups (which is generated by the projective, flat and compact object $\mathbb{Z}$).

Corollary 2.12. *Consider the notations or Paragraph 2.5.*

1. *For any condensed sets T and T', $\mathbb{Z}[T] \otimes \mathbb{Z}[T'] \simeq \mathbb{Z}[T \times T']$.*
2. *For any condensed set T, the condensed abelian group $\mathbb{Z}[T]$ is flat.*

The first point follows as $T = \varinjlim_{S/T} (\underline{S})$ and the functor $\mathbb{Z}$ commutes with colimits. The second one from the description (2.10.a) of the tensor product of condensed abelian groups and the fact $\mathbb{Z}[T(S)]$ is a free abelian group (thus flat).

2.13. Recall the following definitions of Quillen:

- The left derived functor $\otimes^{\mathbf{L}}$ of $\otimes$, if it exists is the left Kan extension of $\otimes : \mathrm{C}(\mathscr{A}b^{\mathrm{cds}}) \times \mathrm{C}(\mathscr{A}b^{\mathrm{cds}}) \to \mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$ along the canonical projection $\mathrm{C}(\mathscr{A}b^{\mathrm{cds}}) \times \mathrm{C}(\mathscr{A}b^{\mathrm{cds}}) \to \mathscr{D}(\mathscr{A}b^{\mathrm{cds}}) \times \mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$.
- The right derived functor $\mathbf{R}\,\underline{\mathrm{Hom}}_{\mathscr{A}b^{\mathrm{cds}}})$ is the right Kan extension of the functor $\underline{\mathrm{Hom}}_{\mathscr{A}b^{\mathrm{cds}}} : \mathrm{C}(\mathscr{A}b^{\mathrm{cds}})^{op} \times \mathrm{C}(\mathscr{A}b^{\mathrm{cds}}) \to \mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$ along the canonical projection $\mathrm{C}(\mathscr{A}b^{\mathrm{cds}})^{op} \times \mathrm{C}(\mathscr{A}b^{\mathrm{cds}}) \to \mathscr{D}(\mathscr{A}b^{\mathrm{cds}})^{op} \times \mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$.

With suitable boundedness assumptions, the existence of these derived functors will rightly follows from classical homological algebra and the good properties of $\mathscr{A}b^{\mathrm{cds}}$. Recall that more recent techniques allows to bypass the boundedness assumptions:

Proposition 2.14. *The left and right derived functors $\otimes^{\mathbf{L}}$ and $\mathbf{R}\,\underline{\mathrm{Hom}}$ exists.*

The historical reference to get the proposition is [Spa88]: one uses the existence of K-projective, K-injective and K-flat resolutions for unbounded complexes.

Remarque 2.15. A more systematic approach was (slowly) developed thanks to Quillen's theory of model categories.

So let us mention the fact that one can get a good model structure on the category $\mathrm{C}(\mathscr{A}b^{\mathrm{cds}})$, which uses the good properties of condensed abelian groups. One can apply the constructions of [CD09] to one of the underlying site of condensed sets (till the end of this remark, all numbered references will refer to the results of [CD09]). We will get the so called *descent model structure* on $\mathrm{C}(\mathscr{A}b^{\mathrm{cds}})$ with respect to the following choice of generators $\mathcal{G}$ and hypercovers $\mathcal{H}$ (see Definition 2.2, Example 2.3 and Theorem 2.5):

(1) $\mathcal{G}$ is the set of $\mathbb{Z}[S]$ where S is an ed-compacta, and $\mathcal{H}$ reduced to 0.[8]
(2) $\mathcal{G}$ is the set of $\mathbb{Z}[S]$ where S is a profinite space, and $\mathcal{H}$ is the made by the complex $\mathrm{Cone}\big(\mathbb{Z}(S_\bullet) \to \mathbb{Z}[S]\big)$ where $S_\bullet \to S$ is an hypercover of S in the profinite site.

This gives two model structures on $\mathrm{C}(\mathscr{A}b^{\mathrm{cds}})$, with weak equivalences the quasi-isomorphisms, which we will call the ed-model structure and the profinite model structure.[9]

The ed-model structure is nothing else than the *projective model structure* on $\mathrm{C}(\mathrm{PSh}^{\mathrm{a}}(\mathscr{T}op^{\mathrm{edK}}, \mathbb{Z}))$. In other words, fibrations are the epimorphisms (see Cor. 5.5) and cofibrant objects are K-projective complexes. The profinite model structure can be described as follows:

(1) bounded below cofibrant objects are exactly the complexes which are degree-wise coproducts of $\mathbb{Z}[S]$ for S profinite.
(2) Fibrant objects are complexes C which satisfies "profinite descent"; *i.e.* one of the following equivalent conditions (see Th. 2.5):

[8]This corresponds to the fact the hypercovers on the effective site of ed-compacta are all split.
[9]In addition, these model structures are proper and cellular (see Theorem 2.5).

(i) For any profinite space S, and any integer $n \in \mathbb{Z}$, the canonical map:

$$H^n(C(S)) \to H^n_{\mathrm{cds}}(S, C)$$

is an isomorphism.

(ii) For any hypercover $S_\bullet \xrightarrow{p} S$ of a profinite space S, the induced map:

$$p^* : C(S) \to \mathrm{Tot}^{\oplus} C(S_\bullet)$$

is a quasi-isomorphism.

Moreover, fibrations are epimorphisms of complexes whose kernel is fibrant (Cor. 5.5).

These model structures are compatible with the monoidal structure, in the sense that they are monoidal model structures which satisfies in addition the monoid axiom. This follows from the fact the chosen generators $\mathbb{Z}[S]$ (either for ed-compactas S or for profinite spaces S) are flat; thus we can apply Prop. 3.2 and Cor. 3.5.

Note in addition that this approach would work well if one considers condensed modules over a condensed ring R. In fact, this case could also be handled with the help of the monoid axiom.

2.3. Associated derived ∞-category.

2.16. There are several ways of constructing the derived ∞-category of condensed abelian groups.

The simplest way is to consider the symmetric monoidal category of abelian presheaves on $\mathscr{S}et^{\mathrm{pf}}$, seen as a presentable, stable, monoidal ∞-category by the monoidal nerve construction. Then we localize this with respect to (condensed) hypercovers, which are stable under tensor products. By the way, it can be easier to restrict to the site of ed-compacta and then localize with respect to Čech hypercovers.

Second, one can consider the hypercompletion of the ∞-topos of sheaves on $\mathscr{S}et^{\mathrm{pf}}$ for the condensed topology with topology of surjective maps (Definition 1.17), and then one restricts this ∞-category to accessible and abelian group objects. It is not so easy to deduce that this is indeed a monoidal ∞-category.

Finally, an old-school method is to consider the symmetric monoidal model structure on $\mathrm{C}(\mathscr{A}b^{\mathrm{cds}})$ explained in Remark 2.15.

3. EMBD-resolution of abelian groups

The aim of this preliminary section is to prove the following result which improves earlier results of Eilenberg-MacLane (1950), Breen (1977) and was apparently already proved by Deligne.

Theorem 3.1. *Let $\mathscr{T}$ be an arbitrary topos. Then there exists a (covariantly) functorial resolution of abelian groups A in $\mathscr{T}$ of the form:*

(3.1.a) $$\ldots \to F_n(A) \to \ldots \to F_2(A) \to F_1(A) \to A$$

where:

(1) $F_1(A) \to F_0(A) \to A$ *is the canonical left exact sequence:*

$$\mathbb{Z}[A \times A] \xrightarrow{d_1} \mathbb{Z}[A] \xrightarrow{\epsilon} A \tag{3.1.b}$$

such that:

- d_1 *is obtained by sheafification of the map on sections over objects of a given site which sends a generator* (a, b) *to the formal sum* $[a + b] - [a] - [b]$*;*
- ϵ *is the sum map.*

(2) *for any* $n > 0$*,* $F_n(A)$ *is a finite sum of abelian groups of the form* $\mathbb{Z}[A^r]$ *for some integer* $r > 0$*.*

3.2. We will apply this theorem to the case of a condensed abelian group A.

Moreover, the general theorem results from the case where $\mathscr{T}$ is the topos $\mathscr{S}et$, that is to say in the case of (usual!) abelian groups.

Indeed, assume we have a functorial resolution $F_\bullet(A) \to A$ in abelian groups A, satisfying the properties (1) and (2) stated in the above theorem. Then given any sheaf of abelian groups ϕ on a site $\mathscr{S}$ underlying the topos $\mathscr{T}$, one gets a resolution for any object S of $\mathscr{S}$:

$$F_\bullet(\phi(S)) \to \phi(S)$$

which is functorial in S. Sheafifying the corresponding natural transformation, we obtain the desired resolution $F_\bullet(\phi) \to \phi$.

To conclude this preliminary paragraph, the idea to get the functorial resolution for abelian groups is first to treat the case of free abelian groups of finite type by gluing together (Lemmas 3.19 and 3.21) resolutions obtained from homotopy theory (see Lemma 3.13). Then the functoriality of this first kind of resolutions allows to extend it formally to all abelian groups (see Paragraph 3.22).

3.1. **Eilenberg-MacLane spaces and resolutions.**

3.3. Let us start with some recall of algebraic topology. First the so-called Dold-Kan equivalence gives an equivalence of categories:

$$N : \Delta^{op}\mathscr{A}b \leftrightarrows \mathrm{C}_{\geq 0}(\mathscr{A}b) : K$$

between simplical abelian groups and complexes of abelian groups concentrated in non-negative homological degrees. We use the formulation of Dold and Puppe (see [DP61].[10] Given any simplicial abelian group $A_\bullet$, $N(A_\bullet)$ is called the *normalized complex* associated with $A_\bullet$, and is canonical quasi-isomorphic to the usual "chain" complex associated with $A_\bullet$.[11]

[10]The equivalence of the associated homotopy categories was obtained independently by Dold and Kan in 1958.

[11]Equal to A_n in degree n and with differentials the alternate sum of the (relevant) face maps.

Definition 3.4. for any integer $n \geq 0$ and any abelian group A, one defines the n-th *Eilenberg-MacLane* simplicial space associated with A by the formula:

$$K(A, n) := K(A[-n]).$$

Its homotopy type is uniquely characterized by the properties:

- $\pi_n(K(A, n)) = A$;
- $\forall i \geq n$, $\pi_i(K(A, n)) = 0$.

Remarque 3.5. According to [Bre78, Remark 2.4], the construction of the functor K shows the following properties of the simplicial abelian group $K(A, n)$:

(1) $K(A, n)_i = 0$ if $i < n$;
(2) $K(A, n)_i = A^{\Delta(i,n)}$ if $i \geq n$, where $\Delta(i, n)$ is the set of morphisms in Δ from $[i]$ to $[n]$.

Example 3.6. Recall the topological realization functor $S_\bullet \mapsto |S_\bullet|$ induces an equivalence of model categories between simplicial sets and CW-complexes. As spaces, one has: $K(\mathbb{Z}, 1) = S^1$, $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$.

For abelian groups A and B, one also gets:

$$K(A \times B, n) = K(A, n) \times K(B, n). \tag{3.6.a}$$

Theorem 3.7. (1) *For any abelian group A, and any couple of integers $i \leq n$, one has:*

$$H_i(\mathbb{Z}[K(A, n)]) = H_i^{\text{sing}}(K(A, n), \mathbb{Z}) = \begin{cases} 0 & \textit{if } i < n, \\ A & \textit{if } i = n. \end{cases}$$

(2) *The space $K(A, n)$ represents the functor $H^n_{\text{sing}}(-, A)$, there exists an isomorphism functorial in the CW-complex X:*

$$[X, K(A, n)] \simeq H^n(X, A).$$

where [-,-] means homotopy classes of maps.

Point (1) follows from Hurewicz theorem. Point (2) is classical.

Remarque 3.8. It follows from Yoneda lemma that

$$H^i_{sing}(K(A, n), B) = [K(A, n), K(B, i)]$$

is isomorphic to the natural transformations of (abelian presheaves):

$$H^n_{\text{sing}}(-, A) \to H^i_{\text{sing}}(-, B);$$

in other words, the *additive operations* of type (n, i) on singular cohomology. In the case $A = B = \mathbb{F}_p$, This explains the interest on determining the cohomology of Eilenberg-MacLane spaces.

We end-(up with the following finiteness theorem.

Theorem 3.9 (Eilenberg-MacLane, Serre)**.** *Let A be an abelian group of finite type. Then for all couple of integers $(i,n) \in \mathbb{N} \times \mathbb{N}^*$, $H_i\big(K(A,n),\mathbb{Z}\big)$ is of finite type.*

Proof. This was proved by Serre in his Ph. D. thesis (see [Ser51]), as a simple corollary of the now called the Serre spectral sequence and Serre's classes theory (applied to the class of finite type abelian groups).

Let us first start to note that the theorem is well-known for $n = 1$. Either you use the fact that $K(A,1) = BA$ and therefore $H_i^{\text{sing}}\big(K(A,1),\mathbb{Z}\big) = H_i(A,\mathbb{Z})$, and use standard facts about group cohomology. Or you can use the fact $K(A,n)$ is a finite CW-complex, as A is of finite presentation; then singular homology can be computed as cell homology, which is of finite type as an abelian group.

Then we use induction on $n > 1$. The inductive step uses the path fibration:

$$\Omega K(\pi,n) \to PK(\pi,n) \xrightarrow{p} K(\pi,n) \tag{3.9.a}$$

where $PK(\pi,n) = \text{Map}([0,1],K(\pi,n))$ is the associated path space sending 0 to the base point of $K(\pi,n)$, p is the evaluation map at 1. Then p is a Serre fibration with fiber the loop space $\Omega K(\pi,n) = \text{Map}(S^1,K(\pi,n))$. But, by uniqueness of Eilenberg-MacLane space (stated in Definition 3.4), we get $\Omega K(\pi,n) = K(\pi,n-1)$. Moreover, the path space $PK(\pi,n)$ is homotopically contractible. Then we can use the Serre spectral sequence associated with the fibration p:

$$E^2_{p,q} = H_p\big(K(\pi,n),H_q(K(\pi,n-1),\mathbb{Z})\big) \Rightarrow H_{p+q}(PK(\pi,n),\mathbb{Z}) = H_{p+q}(*,\mathbb{Z}).$$

Thus we get a spectral sequence converging to a graded abelian group which vanish except in degree 0.

It is easy now to do the inductive step on $n > 1$. So we want to prove $H_i\big(K(A,n),\mathbb{Z}\big)$ is of finite type for all $i \geq 0$. This is obviously true for $i = 0$ so we make an induction on $i > 0$. So we are reduced to prove $H_i\big(K(\pi,n),\mathbb{Z})\big)$ is of finite type, assuming it is true for $n' < n$ and $i' < i$.

We argue by contradiction and assume it is not true. Then $E^2_{i,0} = H_i\big(K(\pi,n),\mathbb{Z})\big)$ is not of finite type. But $E^3_{i,0}$ is the kernel of the map:

$$d^2_{i,0} : E^2_{i,0} \to E^2_{i-2,1} = H_{i-2}\big(K(\pi,n),H_1(K(\pi,n-1),\mathbb{Z})\big).$$

As by induction, $H_1(K(\pi,n-1),\mathbb{Z})$ is of finite type, and $H_{i-2}\big(K(\pi,n),\mathbb{Z}\big)$ is of finite type, we deduce that $E^2_{i-2,1}$ is of finite type. Thus the kernel of $d^2_{i,0}$, the term $E^3_{i,0}$ is not of finite type. Arguing in the same way by considering

$$d^3_{i,0} : E^3_{i,0} \to E^3_{i-3,2} = H_{i-3}\big(K(\pi,n),H_2(K(\pi,n-1),\mathbb{Z})\big)$$

we found that for all $r > 1$, $E^3_{i,0}$ is not of finite type. But this is a contradiction as the spectral sequence collapses and $E^\infty_{i,0}$ must be 0 as a graded piece of the null group. □

Remarque 3.10. Serre's class theory can be stated as follows:

Let $\mathcal{C}$ be a class of A-modules stable under cokernel, kernel and extensions. Let $p : X \to B$ be a Serre fibration of pointed spaces with fiber F. Then if two of the free homology groups $H_*(F, A)$, $H_*(B, A)$, $H_*(X, A)$ belong to $\mathcal{C}^{\mathbb{Z}}$, the other one belongs also to $\mathcal{C}^{\mathbb{Z}}$.

According to the previous theorem, for all (i, n), the abelian groups:

$$M_i^n := H_i\big(K(\mathbb{Z}, n), \mathbb{Z}\big) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{if } i < n, \\ \text{finitely generated} & \text{otherwise.} \end{cases} \tag{3.10.a}$$

Proposition 3.11. *For any free abelian group P, any integer $n > 0$, and any integer $i < 2n$, there exists an isomorphism functorial in P:*

$$H_i\big(K(P, n), \mathbb{Z}\big) \simeq M_i^n \otimes_{\mathbb{Z}} P.$$

Proof. Note that $K(\mathbb{Z}^r, n) = K(\mathbb{Z}, n)^r$. The proof is a consequence of the Künneth formula and formula (3.6.a). Let us write $M_i^n(r) = H_i\big(K(\mathbb{Z}^r, n), \mathbb{Z}\big)$, and prove by induction on $r \geq 1$ that: for any $i < 2n$, $M_i^n(r) \simeq M_i^n \otimes \mathbb{Z}^r$.

The case $r = 1$ is trivial. For the induction, we note from (3.6.a):

$$M_i^n(r) = H_i\big(K(\mathbb{Z}^r, n), \mathbb{Z}\big) = H_i\big(K(\mathbb{Z}^{r-1}, n) \times K(Z, n), \mathbb{Z}\big).$$

In particular, the Künneth short exact sequence, taking into account the vanishing of $M_i^n(s)$ for $0 < i < n$, gives:

$$\begin{aligned} 0 \to & M_i^n(r-1) \otimes_{\mathbb{Z}} M_0^n \oplus M_0^n(r-1) \otimes_{\mathbb{Z}} M_i^n \xrightarrow{a} M_i^n(r) \\ & \longrightarrow \mathrm{Tor}_1(M_i^n(r-1), M_0^n) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(M_0^n(r-1), M_i^n) \to 0. \end{aligned}$$

As $M_0^n(s) = \mathbb{Z}$, we deduce an isomorphism

$$M_i^n(r) \xrightarrow{a^{-1}} M_i^n(r-1) \oplus M_i^n \simeq M_i^n \otimes_{\mathbb{Z}} \mathbb{Z}^r$$

where the last isomorphism follows from the induction step. This isomorphism can be shown to be independent of the chosen base of $P \simeq \mathbb{Z}^r$, and then functorial. □

Remarque 3.12. One can organize the Eilenberg-MacLane spaces as an Ω-spectrum given the canonical weak equivalence:

$$K(A, n) \xrightarrow{\sim} \Omega K(A, n+1) \leftrightarrows \Sigma K(A, n) \xrightarrow{\sigma}_n K(A, n+1)$$

denoted by HA, which according to Theorem 3.7 represents singular cohomology with coefficients in A in the stable homotopy category.

According to Eilenberg and MacLane (see also [Ser51, VI. 2, Prop. 2]), one deduces from the fiber sequence (3.9.a) a canonical suspension map for $r < n$:

$$H_{n+r}\big(K(P, n), \mathbb{Z}\big) \xrightarrow{\Sigma} H_{n+r+1}\big(K(P, n+1), \mathbb{Z}\big)$$

which is an isomorphism. In other words, $M^n_{n+r} = M^{n+1}_{n+1+r}$. This implies that the groups M^n_i appearing in the above corollary do not depend on $n > 0$. So we will put as in [Sch19, 4.16]:

$$M_0 = \mathbb{Z}, \forall r > 0, M_r = M^1_{1+r}. \tag{3.12.a}$$

(?) According to Cartan's computation of the homology of Eilenberg-MacLane spaces, one seems to obtain, still for $r < n$:

$$H_{n+r}\big(K(P,n),\mathbb{Z}\big) \simeq \pi_r(H\mathbb{Z} \wedge H\mathbb{Z}).$$

Moreover, $\pi_*(H\mathbb{Z}\wedge H\mathbb{Z})$ is a co-Hopf algebra (rather an ind-co-Hopf algebra) which is the Milnor dual of the integral Steenrod algebra $[S^* \wedge H\mathbb{Z}, H\mathbb{Z}]$.[12]

As an application one gets the following lemma, which is the key step for Scholze's inductive construction.

Lemma 3.13. *Let P be a free abelian group and n be an integer. Then there exists a complex of abelian groups $C(P)$, functorial in P, of the form:*

$$\ldots \to \mathbb{Z}[P^{\Delta(n+r,n)}] \to \ldots \to \mathbb{Z}[P] \to P \to 0$$

where P is placed in homological degree -1 and such that:

$$H_r(C(P)) = \begin{cases} 0 & r = -1, \\ P \otimes_{\mathbb{Z}} M_r & 0 \leq r < n. \end{cases}$$

We have just glued together the naive (homological) truncation $\tau^n_{\geq n}\mathbb{Z}[K(P,n)]$ along the isomorphism $H_n(\mathbb{Z}[K(P,n)]) \simeq P$, and applied Remark 3.5 together with Proposition 3.11.

3.2. **Pseudo-coherence.**

3.14. In all this subsection, we fix a Grothendieck abelian category $\mathscr{A}$ with a fixed set of compact projective generators $\mathscr{A}_0$.

Definition 3.15. Let A be an object of $\mathscr{A}$ and $n \geq -1$ an integer. An $\mathscr{A}_0$-*resolution* of A of length n is a complex C of objects of $\mathscr{A}$ together with a quasi-isomorphism $C \to A$ and such that:

$$C_i = \begin{cases} \text{finite sum of objects of } \mathscr{A}_0 & 0 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Concretely, this corresponds to a resolution of A of the form:

$$\oplus_j P_{n,j} \to \ldots \to \oplus_j P_{0,j} \to A \to 0.$$

We have the following lemma.

[12] I could not find a proper reference for these results.

Lemma 3.16. *Let A be an object of $\mathscr{A}$ and $n > 0$ an integer.*
Then the following conditions are equivalent:

(i) *For all $i < n$, the functor $\mathrm{Ext}^i_{\mathscr{A}}(A, -)$ commutes with filtered colimits.*
(ii) *A admits an $\mathscr{A}_0$-resolution of lenght n.*
(ii') *For any integer $-1 \leq r \leq n$, any $\mathscr{A}_0$-resolution of length r of A can be extended to an $\mathscr{A}_0$-resolution of length n.*

Proof. It is clear that (ii') implies (ii) implies (i). Let us prove (i) implies (ii') by induction on n. Case $n = 1$. In any case, as $\mathscr{A}_0$ generates $\mathscr{A}$, A is a quotient of a sum of objects of $\mathscr{A}_0$:

$$\phi : \oplus_{i \in I} P_i \twoheadrightarrow A.$$

By assumption A is compact. This implies that the above map factorizes through a finite sum so that we get the case $r = 0$. to get the case $r = 1$, we look at an epimorphism ϕ as above where I is finite. Then the kernel of ϕ is compact so that we that, applying the same reasoning, it is a quotient of a finite sum of objects in $\mathscr{A}_0$ and we deduce the desired $\mathscr{A}_0$-resolution of A of length 1.

The inductive step for $n > 1$ follows essentially from the same argument: one remarks that given an $\mathscr{A}_0$-resolution C of length r of $\mathscr{A}$, the kernel of the r-differential of C is $n-1$-pseudocoherent, so that it admits a $\mathscr{A}_0$-resolution of length 2, which glued with C gives a $\mathscr{A}_0$-resolution of length $r + 1$. ☐

Definition 3.17. Let A be an object of $\mathscr{A}$ and $n \geq 0$ an integer. One says A is n-pseudocoherent if the equivalent conditions of the preceding lemma are satisfied.

Moreover, one says A is 0-pseudocoherent if it is finitely generated. Finally, we say that A is pseudocoherent if it is n-pseudocoherent for all $n \geq 0$.

Example 3.18. In particular, 1-pseudocoherent means that A is compact in $\mathscr{A}$ while pseudocoherent implies that A is compact in $\mathscr{D}(\mathscr{A})$. Any compact projective object of A is pseudocoherent.

We will use the following lemma.

Lemma 3.19. *Let A be an object of $\mathscr{A}$ such that there exists a complex C of the form*

$$\dots P_n \to \dots \to P_0 \to A \to 0$$

where A is placed in homological degree -1 and such that:

(1) *for all $i \geq 0$, $C_i = P_i$ is compact and projective;*
(2) *$H_{-1}(C) = 0$* i.e. *$P_0 \to A$ is an epimorphism;*
(3) *for all $0 \leq i < n$, $H_i(C)$ is $(n - 1 - i)$-pseudocoherent.*

Then A is n-pseudocoherent.

Proof. One replaces C by its naive homological (homological) truncation $\tau^n_{\leq n} C$, and argue by diagram chase as in the proof of the preceding proposition. ☐

3.3. **Proof of the main theorem.**

3.20. Let Latt be the category of lattices *i.e.* finite free abelian groups. Then the category $\mathscr{F}(\text{Latt}, \mathscr{A}b)$ of covariant functors from Latt to $\mathscr{A}b$ is a Grothendieck abelian category generated by the functors $\phi_n : P \mapsto \mathbb{Z}[P^n]$ for $n \geq 0$. Moreover, ϕ_n are compact and projective.

Indeed, Latt is equivalent to the category Latt_0 whose objects are integers $n \geq 0$ and morphisms are matrices $M_{n,m}(\mathbb{Z})$. Then $\mathscr{F}(\text{Latt}, \mathscr{A}b)$ is nothing else that the category of abelian presheaves on Latt_0^{op} which is generated by the compact projective objects of presheaves represented $\mathbb{Z}[\mathbb{Z}^n]$, that is the functor $\text{Hom}_{\mathscr{A}} b(\mathbb{Z}[\mathbb{Z}^n], -)$ which is obviously ϕ_n.

Applying Definition 3.17 to the category $\mathscr{F}(\text{Latt}, \mathscr{A}b)$ with the set of generators $\{\phi_n, n \geq 0\}$, we can formulate the key lemma:

Lemma 3.21. *The functor* ϕ : Latt $\rightarrow \mathscr{A}b, P \mapsto P$ *is pseudocoherent in the abelian category* $\mathscr{F}(\text{Latt}, \mathscr{A}b)$.

Proof. We show by induction on n that ϕ is n-pseudocoherent.

According to the classical resolution:

(3.21.a) $$\mathbb{Z}[P^2] \xrightarrow{d_1} \mathbb{Z}[P] \xrightarrow{\epsilon} P \rightarrow 0$$

as defined in (3.1.b), we know that ϕ is 1-pseudo-coherent.

To prove n-pseudocoherence by induction, we can now apply Lemma 3.19 to obtain the existence of a complex $C(P)$, functorial in P, which satisfies the assumptions of Lemma 3.13. So the lemma, applied to the functor ϕ, implies that ϕ is n-pseudocoherent, as required. □

3.22 (Proof of Theorem 3.1)**.** Consider a latice P. The preceding lemma (and Lemma 3.16) shows the existence of functorial resolutions $F_\bullet(P) \rightarrow P$ where $F_i(P)$ is a finite sum of $\mathbb{Z}[P^r]$. Moreover, extending the resolution (3.21.a), we can assume $F_\bullet A$ is of the form required in Theorem 3.1.

We fix such a functor in P, $F_\bullet(P) \xrightarrow{\epsilon} P$. Restricted to Latt_0, we see that the differentials f_i of $F_\bullet$ are given by a finite sum of matrices $M_{n_i,m_i}(\mathbb{Z})$. In particular, the complex $F_\bullet$ with its augmentation ϵ, is given by universal formulas which therefore extends to any abelian group A, and gives a functor in A: $F_\bullet(A) \rightarrow A$, whose first terms are again the short exact sequence:

$$\mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A] \rightarrow A \rightarrow 0.$$

It remains to show that $F_\bullet(A) \xrightarrow{\epsilon_A} (A)$ is a quasi-isomorphism. The case where A is free finite follows from the finite free case by passage to filtered colimits. The case of an arbitrary abelian group A can be reduced to the latter, by using the functorially $F_\bullet(A)$ and a simplicial resolution of A by free abelian groups.

The methods used allow to deduce the following lemma, needed later.

Lemma 3.23. *Consider a functorial resolution $F_\bullet(A) \to A$ as in Theorem 3.1.*

Then for any $n > 0$, the maps $n = n.Id_{F_\bullet(A)}$ and $[n] = F_\bullet(n.Id_A)$ are homotopic via a functorial homotopy.

Proof. In fact, given the assumptions Theorem 3.1, we have a natural transformation $F_\bullet(-) \to \phi$ which is in fact a projective resolution of ϕ in the abelian category $\mathscr{F}(\mathrm{Latt}, \mathscr{A}b)$. Moreover, the maps $n.Id_{\phi(P)}$ and $\phi(n.Id_P)$ are both equal to $n.Id_P : P \to P$ so that the maps n and $[n]$ lift the same morphisms on ϕ. Standard homological algebra, in the abelian category $\mathscr{F}(\mathrm{Latt}, \mathscr{A}b)$, gives a homotopy equivalence between α and β as required. □

4. The derived embedding

4.1. Locally compact abelian groups.

Definition 4.1. A Locally compact abelian group will be a topological abelian group which is locally compact and Hausdorff (Definitions 1.1 and 1.3). We will use the acronym for these groups: LCA. The corresponding additive category will be denoted by LCA.

Example 4.2. LCA groups are basic objects of harmonic analysis. Familiar examples are: $\mathbb{Z}$ (discrete), $\mathbb{Q}$, $\mathbb{R}$, the group circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $\mathbb{Q}_p$, $\hat{\mathbb{Z}}$, the group of adeles $\mathbb{A}$.

The category LCA is stable under quotient by closed subgroups and admits all colimits. It follows that it admits kernels and cokernels. Nevertheless, it fails to be abelian as the canonical map

$$\mathrm{coKer}(f) \to \mathrm{Im}(f)$$

is in general not an isomorphism (a morphism for which this happens is called *strict*). Such a category is called *quasi-abelian* following a theory established by Schneiders.

Let us recall the following classifications of LCA groups (see [Bou67, II, §2, no 2, Prop. 3]).

Theorem 4.3. *Let A be an LCA group. Then there exists an integer $n \geq 0$, an LCA group A' which contains a compact open subgroup, and an isomorphism:*

$$A \simeq \mathbb{R}^n \times A'.$$

Moreover:

- *n is uniquely determined by these properties.*
- *A' is an extension of a discrete abelian group by a compact abelian group.*

4.4. Recall that the category of Hausdorff locally compact spaces is cartesian closed. Moreover, given LCA groups A and B, the mapping space $\mathrm{Map}(A, B)$ defined in Paragraph 1.5 has a natural topological group structure, so it belongs to LCA. We can formulate a satisfactory duality theory without a priori having a monoidal structure, originally established by Pontryagin (see [Bou67, II, §1, no 5, Th. 2 and II, §1, no 9, Prop. 11]).

Theorem 4.5. *The functor* $\mathbb{D} = \mathrm{Map}(-, \mathbb{T})$ *is a contravariant auto-equivalence of the additive category* LCA.

Moreover, an LCA group A is compact (resp. discrete) if and only if $\mathbb{D}(A)$ is discrete (resp. compact). In particular, $\mathbb{D}$ induces an anti-equivalence of categories between compact abelian groups and discrete abelian groups.

Remarque 4.6. As a corollary of this duality theory, we can try to define a tensor product in LCA by the formula:

$$A \otimes B = \mathrm{Map}\big(A, \mathbb{D}(B)\big).$$

In [HS07, 3.14], it is shown this defines a symmetric monoidal, with internal Hom given by $\mathrm{Map}(-,-,)$, provided one restricts to LCA groups with a finiteness condition (see Definition 2.6 of *loc. cit.*).

4.7. Recall the adjunction (ev, u) between condensed set and Fréchet spaces from Proposition 1.27. As a right adjoint, the functor u commutes with products so that it respects the categorical structure of abelian group. In particular, it induces a faithful functor:[13]

$$u : \mathrm{LCA} \to \mathscr{A}b^{\mathrm{cds}}, A \mapsto \underline{A}.$$

Note that u is left exact according to Proposition 1.27. We have in addition the following easy lemma.

Lemma 4.8. *The functor* $u : \mathrm{LCA} \to \mathscr{A}b^{\mathrm{cds}}$ *is strictly exact.*

Proof. We need to prove that u sends a strict surjection $\phi : B \to A$ of LCA groups to an epimorphism $\underline{\phi} : \underline{B} \to \underline{A}$ of condensed abelian groups. But ϕ being strict surjective means that it is open. Let us prove that $\underline{\phi}$ is an epimorphism. Taking section in an ed-compacta S, we have to prove that:

$$\mathrm{Hom}_{\mathscr{T}op}(S, B) \xrightarrow{\phi_*} \mathrm{Hom}_{\mathscr{T}op}(S, A)$$

is surjective. Let $f : S \to A$ be a continuous map. Then $K = g(S)$ is closed in A as S is compact. In particular, there is an open neighborhood U of 0 in A such that $K \subset U$. Moreover, as B is locally compact and ϕ is open, we can assume there exists an open neighborhood U of 0 in A such that $V = \phi(U)$. We now

[13]One can show that this functor admits a left adjoint, but it is not induced in general by the functor ev of Corollary 1.28. In particular, we cannot understand its effect on a condensed abelian group of the form $\underline{A}$.

contemplate the following commutative diagram:

$$\begin{array}{ccccccc} T & \twoheadrightarrow & L & \hookrightarrow & V & \hookrightarrow & B \\ \downarrow{\scriptstyle \phi'} \uparrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} \\ S & \twoheadrightarrow & K & \hookrightarrow & U & \hookrightarrow & A \end{array} \qquad f : S \to A$$

where the first two squares are defined to be cartesian: $L = \phi^{-1}(K)$, closed in V is compact; $T = S \times_K L$ is compact as it is closed in $S \times L$ which is compact. Therefore, ϕ' is surjective between compacta, and as S is ed-compacta, it admits the indicated splitting. The commutativity of the diagram thus shows, as required, that $f = \phi \circ g$ where $g : S \to B$ is the obvious composite map. □

We also have the following compatibility.

Proposition 4.9. *For any compactly generated LCA groups A and B, there exists a natural isomorphism of condensed abelian groups:*

$$\nu : \underline{\mathrm{Hom}}(\underline{A}, \underline{B}) \xrightarrow{\sim} \underline{\mathrm{Map}(A, B)}.$$

Proof. We first build the natural transformation ν_S evaluated at a profinite space S, as follows:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\underline{A}, \underline{B})(S) = \mathrm{Hom}_{\mathscr{A}b^{\mathrm{cds}}}(\underline{A} \otimes \mathbb{Z}[\underline{S}], \underline{B}) & \xrightarrow{\Gamma(*,-)} & \mathrm{Hom}_{\mathscr{A}b}(A \otimes \mathbb{Z}[S], B) \\ & & \| \\ & & \mathrm{Hom}_{\mathscr{A}b}(\mathbb{Z}[S], B^A) \\ & & \| \, (*) \\ & {\scriptstyle \nu_S} \searrow & \mathrm{Hom}_{\mathscr{S}et}(S, B^A) \\ & & \cup \\ & & \mathrm{Hom}_{\mathscr{T}op}(S, \mathrm{Map}(A, B)). \end{array}$$

In other words, we have to check that the image $S \to B^A$ of any natural transformation $\underline{A} \otimes \mathbb{Z}[\underline{S}] \to \underline{B}$ under the functor $\Gamma(*, -)$ is continuous, for the compact-open topology on B^A. Note that if we show the existence of ν_S, it will necessary be a morphism of abelian group are all the maps involved in the preceding diagram are morphisms of abelian groups, except $(*)$ and the inclusion. But the composition of the inclusion and the identification $(*)$ respects the group structure.

To do the preceding check, we use the epimorphism of condensed abelian group $\mathbb{Z}[\underline{A}] \to \underline{A}$. After tensoring with $\mathbb{Z}[\underline{S}]$, it induces an epimorphism

$$\epsilon : \mathbb{Z}[\underline{A \times S}] \simeq \mathbb{Z}[\underline{A}] \otimes \mathbb{Z}[\underline{S}] \to \underline{A} \otimes \mathbb{Z}[\underline{S}],$$

and then the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{A}b^{\mathrm{cds}}}(\underline{A}\otimes\mathbb{Z}[\underline{S}],\underline{B}) & \xrightarrow{\Gamma(*,-)} & \mathrm{Hom}_{\mathscr{S}et}(S,B^{A}) \\
{\scriptstyle \epsilon_*}\big\downarrow & & \cup \\
\mathrm{Hom}_{\mathscr{A}b^{\mathrm{cds}}}(\mathbb{Z}[\underline{A\times S}],\underline{B}) & & \mathrm{Hom}_{\mathscr{T}op}(S,\mathrm{Map}(A,B)) \\
\| & & \| \\
\mathrm{Hom}_{\mathscr{S}et^{\mathrm{cds}}}(\underline{A\times S},\underline{B}) & \xrightarrow[\sim]{\Gamma(*,-)} & \mathrm{Hom}_{\mathscr{T}op}(A\times S,B).
\end{array}
$$

where we have used the fact A and B are compactly generated to get $\underline{A}(*)=A$ and $\underline{B}(*)=B$: Corollary 1.28.

The same corollary implies that the lower $\Gamma(*,-)$ is an isomorphism as indicated. In particular, ν_S is injective. It remains to show it is surjective. This amounts to show that the above map ϵ_* is surjective.

For this, we use the resolution obtained by applying resolutions (3.1.b) points by points (for example on ed-compacta S):

$$\mathbb{Z}[\underline{A}\times\underline{A}]\to\mathbb{Z}[\underline{A}]\to\underline{A}\to 0.$$

Tensoring with $\mathbb{Z}[S]$ gives an exact sequence of condensed abelian groups:

$$\mathbb{Z}[\underline{A\times A\times S}])\xrightarrow{d_1}\mathbb{Z}[\underline{A\times S}]\xrightarrow{\epsilon}\underline{A}\otimes\mathbb{Z}[S]\to 0.$$

Given a natural transformation of condensed abelian groups, $\alpha:\mathbb{Z}[\underline{A\times S}]\to\underline{B}$, it is clear that the dotted arrow exists fulfilling the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathbb{Z}[\underline{A\times A\times S}] & \xrightarrow{d_1} & \mathbb{Z}[\underline{A\times S}] & \xrightarrow{\epsilon} & \underline{A}\otimes\mathbb{Z}[S] & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\big\downarrow & \swarrow & & & \\
 & & \underline{B} & & & &
\end{array}
$$

as the composite $\alpha\circ d_1$ uniquely corresponds to a (continuous) map $A\times A\times S\to B$ which is zero given the fact for any S, $\alpha_s:A\times\{s\}\to B$ is a morphism of abelian groups. ☐

4.2. **Discrete abelian groups.**

4.10. Let S be a space and A a discrete abelian group. The sheaf cohomology of S with coefficients in A can be defined through topos theory by associating to S its site S_{top} of open subsets, equipped with the topology generated by covering families. Then the sheaf A_S on S_{top} associated with the constant presheaf with values A on S_{top} is an abelian object of the associated topos $\tilde{S}_{\mathrm{top}}$.

The sheaf cohomology of S with coefficients in A can be defined as:

$$H^i(S_{\mathrm{top}},A)=H^i\mathbf{R}\Gamma(S,A_S)\simeq\mathrm{Ext}^i(\mathbb{Z}[S],A_S),$$

respectively the i-th cohomology of the derived functor associated with the global section functor, the i-th extension group in the Grothendieck abelian category $\mathrm{Sh}(S,\mathbb{Z})$ of abelian sheaves on S_{top}.

As in any topos, this cohomology group can be computed using hypercoverings. As an approximation of this computation, one can define Čech cohomology with coefficients in A by restraining attention to those hypercoverings associated with covers by the Čech construction:

$$\check{H}^i(S_{\mathrm{top}}, A) = \varinjlim_{U_\bullet/S} H^i\big(\Gamma(\check{S}(U_\infty/S), A)\big)$$

where the limit runs over covers ordered by refinement, $U_\infty = \sum_i U_i$ and $\check{S}(U_\infty/S)$ being the associated Čech simplicial set,[14] and $\Gamma(\check{S}(U_\infty/S), A)$ means the complex associated with cosimplicial abelian group obtained by applying $\Gamma(-, A)$ termwise, being understood that this functor sends a sum of open subschemes U of S is sent to a product of $\Gamma(U, A_S)$.

Recall the following classical theorem due to Godement ([God58, Th. 5.10.1]).

Proposition 4.11. *Let X be a paracompact*[15] *Hausdorff space. Then the natural comparison map:*

$$\check{H}^i(S_{\mathrm{top}}, A) \to H^i(S_{\mathrm{top}}, A)$$

is an isomorphism.

For clarity of the following discussion we will use the notations:

$$H^*_{\mathrm{shv}}(S, A) := H^*(S_{\mathrm{top}}, A),\ \check{H}^*_{\mathrm{shv}}(S, A) = \check{H}^*(S_{\mathrm{top}}, A) \tag{4.11.a}$$

4.12. Given again a space S and a discrete abelian group, we have already used the singular cohomology of X with coefficients in A, which following the definition of Eilenberg is:

$$H^i_{\mathrm{sing}}(S, A) = H^i \operatorname{Hom}\big(S^{\mathrm{sing}}_\bullet(S), A\big)$$

where $S^{\mathrm{sing}}_\bullet(X) = \operatorname{Hom}_{\mathscr{T}\!op}(\Delta^\bullet, S))$ is the singular chain complex, $\Delta^\bullet$ being the standard cosimplicial topological space, and $\operatorname{Hom}(-, A)$ sends a simplicial set to the complex associated with the cosimplicial A-module obtained by applying the usual $\operatorname{Hom}(-, A)$ termwise.

When X is a CW-complex, these groups are easily computable in terms of the associated cell-complex. We refer the reader to the classical [ES45]. In simplicial language, one writes X as the (topological) realization $|S_\bullet|$ of a simplicial set $S_\bullet$, and one uses the following isomorphism:

$$H^i_{\mathrm{sing}}(S, A) \simeq H^i A[S_\bullet].$$

where $A[S_\bullet]$ stands for the complex associated with the abelian simplicial group obtained by applyign the free A-module functor termwise so $S_\bullet$. This result in his final elaboration is due to Quillen (see in particular [Qui67, II. §3, Th. 1])

They agree with sheaf cohomology for CW-complexes, and slightly more. Here is a very general result (see the main theorem of [Sel16]).

---

[14] Recall $\check{S}_i(U_\infty/S) = (U_\infty)^{i+1}_S$, $(i+1)$ cartesian power over S;

[15] Recall that X is paracompact if every covering of X admits a refinment such that any point $x \in X$ is covered only by a finite number of open of the refinement.

Proposition 4.13. *Let S be a semi-locally contractible*[16] *space. Then for any $i \geq 0$, there exists a canonical isomorphism, functorial in S:*

$$H^i_{\text{sing}}(S, A) \simeq H^i_{\text{shv}}(S, A).$$

4.14. Let again S be a space and A a (discrete) abelian group. Then, using the functor of Proposition 1.27, one can go to the condensed site and topos. Note that, $\underline{A}$ is a condensed abelian group, so we can define the *condensed cohomology of S with coefficient in A* as:

$$H^i_{\text{cds}}(S, A) := H^i(\underline{S}, \underline{A}) = H^i\mathbf{R}\Gamma(\underline{S}, \underline{A}) = \operatorname{Ext}^i\big(\mathbb{Z}(\underline{S}), \underline{A}\big),$$

as in the case recalled in 4.10.

The following result is attributed to Dyckhoff by Scholze, but in fact one can apply Deligne's cohomological descent theory for topological spaces, which says that proper surjective covers are universal cohomological descent covers (see [Del74, 5.3.5.III]).

Proposition 4.15. *Let S be a compacta and A a discrete abelian group. Then for all $i \geq 0$, there exists an isomorphism in S:*

$$H^i_{\text{shv}}(S, A) \xrightarrow{\sim} H^i_{\text{cds}}(S, A).$$

Moreover, when S is profinite, one gets:

$$H^i_{\text{cds}}(S, A) \simeq \begin{cases} \operatorname{Hom}_{\mathscr{T}op}(S, A) & i = 0, \\ 0 & i > 0. \end{cases} \tag{4.15.a}$$

Proof. Let us first remark that, when S is a compacta, one has:

$$\Gamma(\underline{S}, \underline{A}) = \operatorname{Hom}_{\mathscr{T}op}(S, A) = \Gamma(S_{top}, A_S).$$

(By the way, this group is made by the locally constant functions from S to A.) This already settle the case $i = 0$.

Let us start with case S is profinite.

We check the computations stated in (4.15.a) for each cohomology theory. Writing $S = \varprojlim_\lambda S_\lambda$, with S_λ finite discrete, one gets for any $i > 0$:

$$H^i_{\text{shv}}(S, A) \simeq \varinjlim_\lambda H^i_{\text{shv}}(S_\lambda, A) = 0.$$

Moreover, $H^0(S, A) = \Gamma(S, A_S)$ is the abelian group of locally constant function on S with values in A, that is $\operatorname{Hom}_{\mathscr{T}op}(S, A)$ as expected.

Concerning the condensed cohomology we need only to check vanishing of Čech cohomology in degree $i > 0$. Let $p : S' \to S$ be a surjective morphism of profinite spaces. As this p can be written as a projective limit of surjective morphisms $p_\lambda : S'_\lambda \to S_\lambda$, for finite discrete spaces S'_λ and S_λ, the Čech complex associated

[16] *i.e.* any open subset $U \subset S$ admits an open cover $(W_i)_i$ such that for all i, the inclusion $W_i \to U$ is homotopic to a constant map; this is true in particular for CW-complexes!

with S'/S is the filtered limit of the Čech complex associated with S'_λ/S_λ. This concludes as p_λ splits.

To get the comparison isomorphism for a general compacta S, one remarks that, thanks to the case of a profinite S, the complex $\mathbf{R}\Gamma(S,\mathbb{Z})$ can be computed by the complex $\Gamma(S_\bullet, M)$ associated with any hypercover $p : S_\bullet \to S$ whose terms are profinite spaces. But applying Deligne's descent theorem to any p (or [Dyc76, 3.11] for p a "projective resolution") and the computation of sheaf cohomology of profinite spaces, we get an isomorphism:

$$H^i_{\text{shv}}(S, M) \simeq H^i\Gamma(S_\bullet, M).$$

□

Remarque 4.16. It seems that one can get a more conceptual proof. In fat, let us consider the big site $\mathscr{T}op$ of all topological spaces with proper surjective covers. Then we get a fully faithful continuous map of sites $\rho : \mathscr{T}op^{\mathrm{K}} \to \mathscr{T}op$. This implies that sheaves on $\mathscr{T}op^{\mathrm{K}}$ form a full subcategory in that of sheaves on $\mathscr{T}op$. If we are interested only in cohomology, we get an adjunction: an adjunction:

$$\rho_\sharp : \mathrm{Sh}(\mathscr{T}op^{\mathrm{K}}, \mathbb{Z}) \leftrightarrows \mathrm{Sh}(\mathscr{T}op, \mathbb{Z}) : \rho^*$$

such that $\rho_\sharp$ is exact and faithful. Applying Corollary 1.21, one gets the adjunction:

$$\rho_\sharp : \mathrm{Sh}(\mathscr{S}et^{\mathrm{pf}}, \mathbb{Z}) \leftrightarrows \mathrm{Sh}(\mathscr{T}op, \mathbb{Z}) : \rho^*$$

with the same properties. Then condensed abelian groups is the full subcategory of the left hand-side made by the so-called "accessible sheaves". In any case, if follows from the exactness property and the adjunction that for any compacta S,

$$H^i(\underline{S}, A) \simeq H^i(\mathscr{T}op/S, A).$$

The right hand-side can be identified with the sheaf cohomology if S with coefficients in A (by Deligne's theorem).

On the contrary, is X is a topological space which is not a compacta, then one can restrict the sheaf represented by X to the condensed site, but then the cohomology of this restriction computed in the condensed site will be very different from the same cohomology computed in the big site $\mathscr{T}op$ (that is, the sheaf cohomology).

Remarque 4.17. Let us draw a conclusion.

If S is a compacta, there exists canonical isomorphisms:

$$H^i_{\text{cds}}(S, A) \simeq H^i_{\text{shv}}(S, M) \simeq \check{H}^i_{\text{shv}}(S, M).$$

If S is a finite CW-complex, one gets canonical isomorphisms:

$$H^i_{\text{cds}}(S, A) \simeq H^i_{\text{sing}}(S, M) \simeq \check{H}^i_{cell}(S, M).$$

Note however that is S is profinite, one gets:

$$H^0_{\text{cds}}(S, A) = \mathrm{Hom}_{\mathscr{T}op}(S, M)$$

while

$$H^0_{\mathrm{sing}}(S,A) = \mathrm{Hom}_{\mathscr{S}et}(S,M).$$

So sheaf cohomology is in general closer to condensed cohomology of a discrete abelian group.

4.18. Consider now the obvious short exact sequence of LCA groups:

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{T} \to 0.$$

Applying the functor $u : \mathrm{LCA} \to \mathscr{A}b^{\mathrm{cds}}$, one gets an exact sequence in $\mathscr{A}b^{\mathrm{cds}}$ (as it is right exact and respects strict monomorphisms):

$$0 \to \underline{\mathbb{Z}} \to \underline{\mathbb{R}} \to \underline{\mathbb{T}} \to 0.$$

In particular, for any discrete abelian group A, one gets an exact triangle in $\mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$:

$$\mathbf{R}\,\underline{\mathrm{Hom}}(\underline{\mathbb{R}},\underline{A})[-1] \to \underline{A}[-1] \xrightarrow{\partial} \mathbf{R}\,\underline{\mathrm{Hom}}(\underline{\mathbb{T}},\underline{A}) \to \mathbf{R}\,\underline{\mathrm{Hom}}(\underline{\mathbb{R}},\underline{A}) \tag{4.18.a}$$

where we have used the canonical isomorphism $\underline{A}[-1] = \mathbf{R}\,\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}[1],\underline{A})$.

Consider an arbitrary set I. For $i \in I$, we denote by $p_i : \underline{\mathbb{T}}^I \to \mathbb{T}$ the projection on the i-th factor. By taking the sump over the $i \in I$ of the following composite maps:

$$\underline{A}[-1] \xrightarrow{\partial} \mathbf{R}\,\underline{\mathrm{Hom}}(\underline{\mathbb{T}},\underline{A}) \xrightarrow{p_i^*} \mathbf{R}\,\underline{\mathrm{Hom}}\left(\underline{\mathbb{T}}^I,\underline{A}\right)$$

we get a morphism in $\mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$:

$$\phi_I : \bigoplus_I \underline{A}[-1] \to \mathbf{R}\,\underline{\mathrm{Hom}}\left(\underline{\mathbb{T}}^I,\underline{A}\right)$$

The following result is a particular case of the main theorem we want to prove.

Theorem 4.19. *With the above notation, the maps ϕ_I is an isomorphism.*

Proof. Assume first that I is finite. By additivity of $\mathbf{R}\,\underline{\mathrm{Hom}}(-,\underline{A})$, we are reduced to the case $I = \{*\}$. Then the exact triangle (4.18.a) reduces to prove that $\mathbf{R}\,\underline{\mathrm{Hom}}(\underline{\mathbb{R}},\underline{A}) = 0$. According to Theorem 3.1, we get a quasi-isomorphism if $\mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$:

$$F_\bullet(\underline{\mathbb{R}}) \xrightarrow{\sim} \underline{\mathbb{R}}$$

so that it suffices to show: $\mathbf{R}\,\underline{\mathrm{Hom}}(F_\bullet(\underline{\mathbb{R}}),\underline{A}) = 0$. In fact, by functoriality of the resolution $F_\bullet \to Id$, this amounts to prove that the following map, induced by $\mathbb{R} \to 0$,

$$\mathbf{R}\,\underline{\mathrm{Hom}}(F_\bullet(0),\underline{A}) \to \mathbf{R}\,\underline{\mathrm{Hom}}(F_\bullet(\underline{\mathbb{R}}),\underline{A})$$

is a quasi-isomorphism. As the complex on the left-hand side is concentrated in non-negative degrees, it is sufficient by *dévissage* to prove that for any $n \geq 0$ the canonical map

$$\mathbf{R}\,\underline{\mathrm{Hom}}(F_n(0),\underline{A}) \to \mathbf{R}\,\underline{\mathrm{Hom}}(F_n(\underline{\mathbb{R}}),\underline{A})$$

is an isomorphism. Now $F_n(\underline{\mathbb{R}})$ is a finite sum of condensed abelian groups of the form $\mathbb{Z}(\underline{\mathbb{R}}^r)$, so it is sufficient to prove that for any $r \geq 0$, the canonical map

$$\mathbf{R}\,\underline{\mathrm{Hom}}(\underline{\mathbb{Z}}, \underline{A}) \to \mathbf{R}\,\underline{\mathrm{Hom}}(\mathbb{Z}[\underline{\mathbb{R}^r}], \underline{A}).$$

Taking sections over an arbitrary ed-compacta S, we are thus reduced to show the following map, induced by the canonical projection, is an isomorphism:

$$\mathbf{R}\Gamma(\underline{S}, \underline{A}) = \mathbf{R}\,\underline{\mathrm{Hom}}(\mathbb{Z}[\underline{S}], \underline{A}) \longrightarrow \mathbf{R}\,\mathrm{Hom}\left(\mathbb{Z}[\underline{S \times \mathbb{R}^r}], \underline{A}\right) = \mathbf{R}\Gamma(\underline{S \times \mathbb{R}^r}, \underline{A}).$$

As a space, one has, in the category of compactly generated spaces:

$$\mathbb{R} = \varinjlim_{n>0} \left([-n, n]\right).$$

In general, the functor u does not commute with colimits. However, as the preceding one is indexed by a countable set and the transition maps are closed inclusions, we derive from [BS15, Lem. 4.3.7] that u commutes with the above particular colimit, yielding an isomorphism:

$$\underline{\mathbb{R}} \simeq \varinjlim_{n>0} \left(\underline{[-n, n]}\right).$$

In particular, in the category of condensed abelian groups:

$$\mathbb{Z}[\underline{S \times \mathbb{R}^r}] = \varinjlim_{n>0} \mathbb{Z}[\underline{S \times [-n, n]^r}].$$

As filtered colimits are exact, this expression is also true in $\mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$ provided we take the homotopy colimit. As the functor $\mathbf{R}\,\mathrm{Hom}\left(-, \underline{A}\right)$ sends with homotopy colimits to homotopy limits, we get:

$$\mathbf{R}\,\mathrm{Hom}\left(\mathbb{Z}[\underline{S \times \mathbb{R}^r}], \underline{A}\right) = \mathrm{holim}_{n>0}\,\mathbf{R}\,\mathrm{Hom}\left(\mathbb{Z}[\underline{S \times [-n, n]^r}], \underline{A}\right)$$

so that it is sufficient to prove that the following map, induced by the projection, is a quasi-isomorphism:

$$\mathbf{R}\Gamma(S, \underline{A}) = \mathbf{R}\Gamma(\underline{S}, \underline{A}) \longrightarrow \mathbf{R}\Gamma(\underline{S \times [-n, n]^r}, \underline{A}) = \mathbf{R}\Gamma(S \times [-n, n]^r, \underline{A})$$

where we have used the fact $S \times [-n, n]$ is a compacta, and Proposition 1.19 to be able to forget the underlines. Taking cohomology in degree $i \in \mathbb{Z}$, and applying Proposition 4.15, it is now sufficient to apply homotopy invariance of sheaf cohomology, to conclude the case where I is finite.

The general case directly follows (and is equivalent to) the following lemma:

Lemma 4.20. *Let $\mathcal{P}^f(I)$ be the set of finite subsets J of I ordered by inclusion. Then the canonical map:*

$$\mathrm{hocolim}_{J \in \mathcal{P}^f(I)}\,\mathbf{R}\,\underline{\mathrm{Hom}}\left(\underline{\mathbb{T}}^J, \underline{A}\right) \to \mathbf{R}\,\underline{\mathrm{Hom}}\left(\underline{\mathbb{T}}^I, \underline{A}\right)$$

is an isomorphism in $\mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$.

First note ty Tychonoff theorem implies $\mathbb{T}^I$ is a compacta. Applying again Theorem 3.1, we get a commutative diagram:

$$\begin{array}{ccc} F_\bullet(\underline{\mathbb{T}^I}) & \longrightarrow & \underline{\mathbb{T}^I} \\ \downarrow & & \downarrow \\ F_\bullet(\underline{\mathbb{T}^J}) & \longrightarrow & \underline{\mathbb{T}^J} \end{array}$$

where the vertical maps are induced by projection and the horizontal maps are quasi-isomorphisms. As in the preceding reasoning, using *dévissage* and the particular form of $F_\bullet(-)$, we are reduced to prove that for any ed-compacta S, the canonical map:

$$\varinjlim_{J \in \mathcal{P}^f(I)} H^i_{\mathrm{shv}}\big(S \times \mathbb{T}^J, A\big) \longrightarrow H^i_{\mathrm{shv}}\big(S \times \mathbb{T}^I, A\big)$$

is an isomorphism. This follows by comparison with Čech cohomology (Prop. 4.11). □

4.3. The real case.

4.21. Let S be a compacta. Using Corollary 1.28, we get the following computation:

$$\Gamma(S, \underline{\mathbb{R}}) = \mathrm{Hom}_{\mathscr{S}et^{\mathrm{cds}}}(\underline{S}, \underline{\mathbb{R}}) = \mathrm{Hom}_{\mathscr{T}op}(S, \mathbb{R}) =: C(S, \mathbb{R})$$

where $C(S, \mathbb{R})$ is the set of continuous functions from S to $\mathbb{R}$. This is a Banach space.

Then the cohomology group $H^i_{\mathrm{cds}}(S, \underline{\mathbb{R}}) = H^i \mathbf{R}\Gamma(S, \underline{\mathbb{R}})$ can be computed by taking the colimit over the hypercovers $S_\bullet \to S$ (for the condensed topology on $\mathscr{S}et^{\mathrm{pf}}$) of S of the i-th cohomology of the complexes:

$$C(S_0, \mathbb{R}) \to C(S_1, \mathbb{R}) \to C(S_2, \mathbb{R}) \to \dots$$

where the differential d_n is, as usual, the alternate sum of the $(\delta_i^n)^*$. Obviously, each differential is continuous and $C(S_\bullet, \mathbb{R})$ is then a graded differential Banach space (in fact a Banach differential graded algebra).

Theorem 4.22. *Let S be a compacta. Then for any $i \geq 0$, one has:*

$$H^i_{\mathrm{cds}}(S, \underline{\mathbb{R}}) = \begin{cases} C(S, \mathbb{R}) & i = 0, \\ 0 & i \neq 0. \end{cases}$$

Moreover, for any (bounded) hypercover $S_\bullet \to S$ by profinite spaces, the augmented Banach differential complex:

$$0 \to C(S, \mathbb{R}) \to C(S_0, \mathbb{R}) \to C(S_1, \mathbb{R}) \to \dots \tag{4.22.a}$$

satisfies the following stronger exactness property:

$$\forall i \geq 0, \forall f \in C(S_i, \mathbb{R}), \forall \epsilon > 0, \exists g \in C(S_{i-1}, \mathbb{R}) \mid dg = f, \|g\| \leq (1 + \epsilon).\, \|f\|\,,$$

where we have put $S_{-1} = S$.

Proof. We need only to prove the second statement.

Let us start by the case where S and all S_i are finite. So we are considering hypercovers $S_\bullet/S$ on the site of finite discrete sets with covers the surjective maps. As all these covers are split, all hyper-covers are homotopy contractible (see for example [Con03, §5]). This implies that the complex (4.22.c) is homotopy contractible and the contracting homotopy $h_i : C(S_i, \mathbb{R}) \to C(S_i, \mathbb{R})$ is induced by pullback along a map $\nu_i : S_{i-1} \to S_i$. Given any $f \in C(S_i, \mathbb{R})$, such that $d(f) = 0$, one can put $g = h_i(f) = f \circ \nu_i$ such that: $f = d(g)$. But by definition of the sup-norm, one gets:

$$\|g\| = \|f \circ \nu_i\| \leq \|f\| . \tag{4.22.b}$$

Let us now assume that S and the S_i are profinite, and consider a cycle $f \in C(S_i)$, $d(f) = 0$. By boundedness of the hypercover, the map $p : S_\bullet \to S$ can be written as a projective limit: $p_\lambda : S_\bullet, \lambda \to S_\lambda$ such that each spaces S_λ and the $S_{i,\lambda}$ are finite discrete. Then the complex is isometric with the completion of the filtered colimit of the complexes of normed $\mathbb{R}$-vector spaces:

$$0 \to C(S_\lambda, \mathbb{R}) \to C(S_{0,\lambda}, \mathbb{R}) \to C(S_{1,\lambda}, \mathbb{R}) \to \dots \tag{4.22.c}$$

Moreover, each map:

$$\varinjlim_{\lambda} C(S_{i,\lambda}, \mathbb{R}) \to C(S_i, \mathbb{R})$$

is injective, dense, isometric, and the right-hand side is the completion of the left hand-side. Using this description, we can approximate f by a cycle h in some $C(S_{i,\lambda}, \mathbb{R})$, $\|f - h\| \leq \epsilon. \|f\|$. According to (4.22.b), it will be the boundary of an element g_0, $g_0 = d(h)$ such that:

$$\|g_0\| \leq \|h\| \leq (1 + \epsilon). \|f\| .$$

We can now iterate this process by putting $f_0 = f$, $f_1 = f - dg_0$, so that $\|f_1\| \leq \epsilon \|f\|$. We therefore build sequences $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}$ such that:

$$f_{n+1} = f_n - d(g_n), \|f_{n+1}\| \leq \epsilon. \|f_n\| , \|g_n\| \leq (1 + \epsilon). \|f_n\|$$

This condition ensures convergence of the power series:

$$g = \sum_{i=0}^{\infty} g_i$$

and the properties:

$$f = d(g),\ \|g\| \leq \frac{1+\epsilon}{1-\epsilon}. \|f\| , \tag{4.22.d}$$

and this concludes.

In the general case, we have to assume S is a compacta and the S_i are profinite, considering $f \in C(S_i, \mathbb{R})$, $d(f) = 0$. Arguing at points $s \in S$, we can apply the profinite case to the cycle $f_s \in C(S_i \times_S s, \mathbb{R})$, so that we find $g_s \in C(S_{i-1} \times_S s, \mathbb{R})$ such that:

$$f_s = d(f_s), \|g_s\| \leq (1+\epsilon).\, \|f_s\|\,.$$

By Tietze extension theorem, we can extend g_s to $\tilde{g}_s \in C(S_{i-1}, \mathbb{R})$ and find an open neighborhood U_s of s in S such that:

$$\left\| \big(f - d(\tilde{g}_s)\big)|_{U_s} \right\| \leq \epsilon.\, \|f\|\,, \|\tilde{g}_s\| \leq \|g_s\| \leq (1+\epsilon).\, \|f\|\,.$$

By compactness, S_{i-1} is covered by a finite number of the U_s, and using partitions of unity allows to glue the corresponding $\tilde{g}_s$ to get a function $g_0 \in C(S_{i-1} \times_S s, \mathbb{R})$ such that:

$$\|f - d(g_0)\| \leq \epsilon.\, \|f\|\,, \|g_0\| \leq (1+\epsilon).\, \|f\|\,.$$

Iterating the process with $f_1 = f - d(g_0)$, we obtain as in the profinite case an element $g = \sum_{i=0}^{+\infty} g_i$ satisfying properties (4.22.d). ☐

We are now ready to prove the last theorem:

Theorem 4.23. *For any set I, one has in $\mathscr{D}(\mathscr{A}b^{\mathrm{cds}})$:*

$$\mathbf{R}\,\underline{\mathrm{Hom}}\left(\underline{\mathbb{T}}^I, \underline{\mathbb{R}}\right) = 0.$$

Proof. Let us put $A = \underline{\mathbb{T}}^I$. As A is compact, multiplication by 2^n stays bounded in A while it is unbounded in $\mathbb{R}$. In particular any continuous map $A \to \mathbb{R}$ must be 0.

We key argument to extend this idea to higher cohomology is Lemma 3.23. So let us consider a resolution $F_\bullet(\underline{A}) \to \underline{A}$. According to that lemma, the two multiplication maps 2 and [2] on $F_\bullet(\underline{A})$ are homotopic through a functorial homotopy $h_\bullet$.

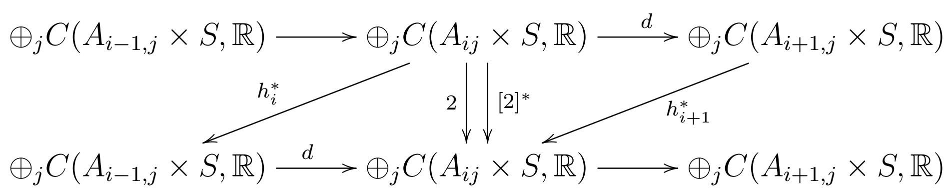
Now we need to prove that:

$$\mathbf{R}\,\underline{\mathrm{Hom}}\left(F_\bullet(\underline{A}), \underline{\mathbb{R}}\right) = 0.$$

To fix notations, write $F_i(\underline{A}) = \oplus_j \mathbb{Z}[\underline{A_{ij}}]$, where $A_{ij} = A^{r_{ij}}$ for some integer $r_{ij} > 0$. Then, taking sections over a profinite space S, we have to prove the exactness of the complex:

$$\ldots \to \oplus_j C(A_{ij} \times S, \mathbb{R}) \xrightarrow{d} \oplus_j C(A_{i+1,j} \times S, \mathbb{R}) \to \ldots$$

Note that the homotopy equivalence $h_\bullet$ induces a homotopy in the complex as follows:



Consider a cycle $f \in \oplus_j C(A_{ij} \times S, \mathbb{R})$: $d(f) = 0$. According to the above pictured homotopy, we get:

$$2.f = [2]^*(f) + d(h_i(f)) \Leftrightarrow f = \frac{1}{2}.[2]^*(f) + d(\frac{1}{2}.h_i^*(f)).$$

Iterating gives:

$$f = \frac{1}{2^N}.[2^N]^*(f) + d\left(\sum_{n=1}^{N} \frac{1}{2^n}.h_i^*\big([2^{n-1}]^* f\big)\right).$$

As $[2]^*$ is induced by multiplication by 2 on each topological abelian group A_{ij}, one gets:

$$\left\|[2^N]^*(f) = f(2^N.)\right\| \leq \|f\|.$$

Moreover, as $h_i : \oplus_j \mathbb{Z}[\underline{A}_{i-1,j}] \to \oplus_j \mathbb{Z}[\underline{A}_{ij}]$ is a morphism of condensed abelian groups, the induced map h_i^* is a continuous map of Banach algebra, so it must have bounded norm. As a consequence, the power series:

$$\sum_{n=1}^{+\infty} \frac{1}{2^n}.h_i^*\big([2^{n-1}]^* f\big)$$

converges absolutely, say to g. As d is continuous, taking limits gives: $f = d(g)$, as required. ☐

4.4. **Main theorem.**

4.24. Recall that $\mathscr{D}^b(\mathrm{LCA})$ is the Verdier quotient of the category $K^b(\mathrm{LCA})$ with respect to strict acyclic complexes. The additive functor u induces a triangulated functor:

$$K^b(\mathrm{LCA}) \to K^b(\mathscr{A}b^{\mathrm{cds}}) \to D^b(\mathscr{A}b^{\mathrm{cds}}).$$

As u is strictly exact (Lemma 4.8), the above functor induces an additive functor:

$$D^b(\mathrm{LCA}) \to D^b(\mathscr{A}b^{\mathrm{cds}})$$

still denoted by u. As a corollary of the previous computations and the ones of [HS07], we get:

Theorem 4.25. *The functor $u : D^b(\mathrm{LCA}) \to D^b(\mathscr{A}b^{\mathrm{cds}})$ is fully faithful.*

Proof. We are reduced to prove that the map induced by u:

$$\mathbf{R}\,\mathrm{Hom}_{\mathrm{LCA}}(A, B) \to \mathbf{R}\,\mathrm{Hom}_{\mathscr{A}b^{\mathrm{cds}}}(\underline{A}, \underline{B})$$

is an isomorphism.

By the classification of LCA groups (Th. 4.3), we are reduced to the case where A (resp. B) is $\mathbb{R}^n$, discrete or compact.

The case where A (resp. B) is discrete reduces to the case of $\mathbb{Z}$ by choosing a presentation, which is clear. The case where A (resp. B) is $\mathbb{R}^n$ reduces to that of $\mathbb{R}$ by additivity.

Consider the remaining cases for A. The case $A = \mathbb{R}$ reduces to that of $\mathbb{R}/\mathbb{Z}$ given what we said about the discrete case. So we are reduced to the case where A is a compacta. Now $\mathbb{D}(A)$ admits a free resolution in LCA of the form:

$$\mathbb{Z}^J \to \mathbb{Z}^i \to \mathbb{D}(A) \to 0.$$

Taking duals and applying Theorem 4.5, we get a resolution:

$$0 \to A \to \mathbb{T}^I \to \mathbb{T}^J$$

so that we are reduced to the case $A = \mathbb{T}^I$ for an arbitrary set I.

Now, if B is compact, we reduced using the above trick to the case $B = \mathbb{T}^J$ for an arbitrary set J. Using (AB4*), one reduces top the case $B = \mathbb{T}$. But this case follows the cases where B is $\mathbb{R}$ or $\mathbb{Z}$.

Finally:

- $A = \mathbb{T}^I$, $B = \mathbb{Z}$: we apply Theorem 4.19 and [HS07, Ex. 4.11].
- $A = \mathbb{T}^I$, $B = \mathbb{R}$: we apply Theorem 4.23 and [HS07, Th. 4.15(vii)].

□

An important corollary for us is:

Corollary 4.26. *For any LCA groups A, B and any $i > 1$, one has:*

$$\underline{\mathrm{Ext}}^i_{\mathscr{A}b^{\mathrm{cds}}}(\underline{A}, \underline{B}) = 0.$$

This is [HS07, 4.15(iv)].

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