STABLE MOTIVIC HOMOTOPY THEORY AT INFINITY

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ABSTRACT. In this paper, we initiate a study of motivic homotopy theory at infinity. We use the six functor formalism to give an intrinsic definition of the stable motivic homotopy type at infinity of an algebraic variety. Our main computational tools include cdh-descent for normal crossing divisors, Euler classes, Gysin maps, and homotopy purity. Under ℓ-adic realization, the motive at infinity recovers a formula for vanishing cycles due to Rapoport-Zink; similar results hold for Steenbrink's limiting Hodge structures and Wildeshaus' boundary motives. Under the topological Betti realization, the stable motivic homotopy type at infinity of an algebraic variety recovers the singular complex at infinity of the corresponding analytic space. We coin the notion of homotopically smooth morphisms with respect to a motivic ∞-category and use it to show a generalization to virtual vector bundles of Morel-Voevodsky's purity theorem, which yields an escalated form of Atiyah duality with compact support. Further we study a quadratic refinement of intersection degrees taking values in motivic cohomotopy groups. For relative surfaces, we show the stable motivic homotopy type at infinity witnesses a quadratic version of Mumford's plumbing construction for smooth complex algebraic surfaces. Our main results are also valid for ℓ-adic sheaves, mixed Hodge modules, and more generally motivic ∞-categories.

CONTENTS

1. INTRODUCTION

Topology at infinity is essentially the study of topological properties that persistently occur in complements of compact sets. For example, a space is intuitively simply connected at infinity if one can collapse loops far away from any small subspace. Euclidean space $\mathbf{R}^n$, $n \geq 3$, is the unique open contractible n-manifold that is simply connected at infinity. In particular, the Whitehead manifold is not simply connected at infinity and therefore not homeomorphic to $\mathbf{R}^3$. This article describes our first attempt at extending the theory of topology at infinity for open manifolds to smooth affine varieties. An overriding goal is to develop a study of intrinsic motivic invariants which can distinguish between $\mathbf{A}^1$-contractible varieties. For background on motivic homotopy theory and $\mathbf{A}^1$-contractible varieties we refer to the survey [AØ19]. Our approach makes extensive use of the six-functor formalism in stable motivic homotopy theory, as developed in [Ayo07a, CD19], and we assume some familiarity with this material.

Let S be a qcqs (quasi-compact quasi-separated) base scheme. Its stable motivic homotopy category $\mathrm{SH}(S)$ is a closed symmetric monoidal ∞-category, see, e.g., [DRØ03, Hoy14, Jar00, Rob15]. To any separated S-scheme of finite type $f \colon X \to S$ we define $\Pi_S^\infty(X)$, the *stable motivic homotopy type at infinity of X*, by the exact homotopy sequence

(1.0.0.a) $$\Pi_S^\infty(X) \to f_! f^!(\mathbb{1}_S) \xrightarrow{\alpha_X} f_* f^!(\mathbb{1}_S)$$

Here $\mathbb{1}_S$ is the motivic sphere spectrum over S, $f_! f^!(\mathbb{1}_S)$ is the stable homotopy type of X, $f_* f^!(\mathbb{1}_S)$ is the properly supported stable homotopy type of X. The canonical morphism α_X is obtained from the six-functor formalism for the stable motivic homotopy category $\mathrm{SH}(S)$, which implies the following fundamental properties.

- If X/S is smooth, then $f_! f^!(\mathbb{1}_S) = \Sigma^\infty X_+$ is the motivic suspension spectrum of X
- If X/S is proper, then α_X is an isomorphism
- The morphism α_X is covariant with respect to proper morphisms and contravariant with respect to étale morphisms

With the intrinsic definition of $\Pi_S^\infty(X)$ in (1.0.0.a) we deduce a number of novel properties in the spirit of proper homotopy theory. Let us fix a compactification $\bar{X}$ of X over S and denote by ∂X its reduced *boundary*. Then the induced immersions $j : X \to \bar{X}$, $i : \partial X \to X$ form a diagram of S-schemes

(1.0.0.b)
$$X \overset{j}{\hookrightarrow} \bar{X} \overset{i}{\hookleftarrow} \partial X$$
$$f \searrow \quad \downarrow \quad \swarrow g$$
$$S$$

We observe the stable homotopy type at infinity of X is determined by the data in (1.0.0.b) via a canonical equivalence

(1.0.0.c) $$\Pi_S^\infty(X) \simeq g_* i^* j_* f^!(\mathbb{1}_S)$$

This shows $\Pi_S^\infty(X)$ is independent of the chosen compactification and our construction has properties analogous to Deligne's vanishing cycle functor for étale sheaves, see [DK73]. We may reformulate (1.0.0.c) by means of the canonically induced exact homotopy sequence

(1.0.0.d) $$\Pi_S^\infty(X) \to \Pi_S(\partial X) \oplus \Pi_S(X) \xrightarrow{i_* + j_*} \Pi_S(\bar{X})$$

By a closed pair of S-schemes (Y, W) we mean a closed immersion $W \hookrightarrow Y$ of S-schemes, and a morphism $\phi \colon (Y, W) \to (X, Z)$ is an S-morphism $\phi \colon Y \to X$ such that $\phi^{-1}(Z) = W$.

Suppose X, Y are proper S-schemes, and ϕ induces an isomorphism of formal completions

$$\phi \colon \hat{Y}_W \xrightarrow{\cong} \hat{X}_Z$$

It follows that there exists a canonical equivalence in $\mathrm{SH}(S)$

(1.0.0.e) $$\phi_* \colon \Pi_S^\infty(Y - W) \xrightarrow{\simeq} \Pi_S^\infty(X - Z)$$

This shows the stable motivic homotopy type at infinity functor $\Pi_S^\infty(-)$ satisfies analyticial invariance. The validity of (1.0.0.e) follows by Wildeshaus' proof of [Wil06, Theorem 5.1] and (1.0.0.d) (we leave further details to the interested reader).

Returning to the notation in (1.0.0.b), let us assume $\bar{X}$, ∂X are smooth S-schemes, and write N for the normal bundle of ∂X in $\bar{X}$. In Theorem 3.3.2 we use the Euler class $e(N)$ in $\mathrm{SH}(S)$ to deduce the exact homotopy sequence

(1.0.0.f) $$\Pi_S^\infty(X) \to \Pi_S(\partial X) \xrightarrow{e(N)} \Sigma^\infty \mathrm{Th}_S(N)$$

It is helpful to think of the passage from (1.0.0.a) to (1.0.0.f) in the language of problem solving. Our "problem" is to understand $\Pi_S^\infty(X)$ and the "solution" in the smooth case is the Euler class for the normal bundle of the closed immersion $\partial X \not\hookrightarrow \bar{X}$.

In the following, we further assume $\bar{X}$ is a smooth proper S-scheme and ∂X is a normal crossing divisor on $\bar{X}$. We may write $\partial X = \cup_{i \in I} \partial_i X$ as the union of its irreducible components $\partial_i X$, so there is a canonical closed immersion $\nu_i : \partial_i X \to \bar{X}$. For any subset $J \subset I$, we equip $\partial_J X := \cap_{j \in J} \partial_j X$ with its reduced subscheme structure, where $\cap$ is suggestive notation for fiber products over the boundary ∂X. If $J \subset K$, there is a canonical proper morphism $\nu_K^J : \partial_K X \to \partial_J X$. By means of descent for the cdh-covering

$$\sqcup_{i \in I} \partial_i X \to \partial X$$

we identify $\Pi_S(\partial X)$ with the colimit[1] of the naturally induced diagram in $\mathrm{SH}(S)$

(1.0.0.g) $$\Pi_S(\partial_I X) \longrightarrow \bigoplus_{\sharp J = \sharp I - 1} \Pi_S(\partial_J X) \underset{\cdots}{\overrightarrow{}} \bigoplus_{\sharp J = \sharp I - 2} \Pi_S(\partial_J X) \underset{\overrightarrow{}}{\overrightarrow{\cdots}} \cdots \underset{\overrightarrow{}}{\overrightarrow{\cdots}} \bigoplus_{i \in I} \Pi_S(\partial_i X)$$

The face map on the summand $\Pi_S(\partial_K X)$ is defined by the pushforward maps

$$\sum_{J \subset K, \sharp J = \sharp K - 1} (\nu_K^J)_*$$

Similarly, we identify $\Sigma^\infty \mathrm{Th}_S(N)$ with the limit of the naturally induced diagram in $\mathrm{SH}(S)$

(1.0.0.h) $$\bigoplus_{i \in I} \Sigma^\infty \mathrm{Th}_S(N_i) \underset{\overrightarrow{\cdots}}{\overrightarrow{}} \bigoplus_{\sharp J = 2} \Sigma^\infty \mathrm{Th}_S(N_J) \underset{\overrightarrow{\cdots}}{\overrightarrow{}} \bigoplus_{\sharp J = 3} \Sigma^\infty \mathrm{Th}_S(N_J) \underset{\overrightarrow{}}{\overrightarrow{\cdots}} \cdots \longrightarrow \Sigma^\infty \mathrm{Th}_S(N_I)$$

Here, N_J is the normal bundle of $\partial_J X$ in $\bar{X}$, and the coface map on the summand $\Sigma^\infty \mathrm{Th}_S(N_K)$ is defined by the Gysin maps

$$\sum_{J \subset K, \sharp J = \sharp K - 1} (\nu_K^J)^!$$

Our general computations culminate in Theorem 5.3.2, where we identify $\Pi_S^\infty(X)$ with the homotopy fiber of the map

$$\mathrm{colim}_{n \in (\Delta^{\mathrm{inj}})^{op}} \left(\bigoplus_{J \subset I, \sharp J = n + 1} \Pi_S(\partial_J X) \right) \xrightarrow{\mu} \lim_{n \in \Delta^{\mathrm{inj}}} \left(\bigoplus_{J \subset I, \sharp J = m + 1} \Sigma^\infty \mathrm{Th}_S(N_J) \right)$$

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[1]Limits and colimits in this paper are taken in the sense of ∞-categories.

induced by

$$(\mu_{i,j})_{i,j \in I} \colon \bigoplus_{i \in I} \Pi_S(\partial_i X) \longrightarrow \bigoplus_{j \in I} \Sigma^\infty \operatorname{Th}_S(N_j)$$

More precisely, $\mu_{i,j}$ is shorthand for the composite map

$$\Pi_S(\partial_i X) \xrightarrow{\nu_{i*}} \Pi_S(\bar{X}) \to \Sigma^\infty \left(\frac{\bar{X}}{\bar{X} \smallsetminus \partial_j X} \right) \xrightarrow{\simeq} \Sigma^\infty \operatorname{Th}_S(N_j)$$

Here, for the rightmost map, we make use of the homotopy purity equivalence

(1.0.0.i)
$$\frac{\bar{X}}{\bar{X} \smallsetminus \partial_j X} \simeq \operatorname{Th}_S(N_j)$$

for normal bundles of closed immersions [MV99, Theorem 3.2.23].

To refine these techniques we develop a theory of duality with compact support. In the process, we generalize the homotopy purity theorem and give new examples of rigid objects. Our approach is based on the notion of a homotopically smooth morphism. If $f : X \to S$ is a smoothable lci morphism with virtual bundle τ_f over X, we say that f is *homotopically smooth* (*h-smooth*) if the naturally induced morphism

$$\mathfrak{p}_f : \operatorname{Th}(\tau_f) \to f^!(\mathbb{1}_S)$$

is an isomorphism (see Definition 4.1.7 for details). Any closed immersion between smooth varieties over a field is h-smooth. When f is h-smooth and $i : Z \to X$ is a closed immersion with Z/S h-smooth, Theorem 4.1.12 shows the relative purity isomorphism

$$\Pi_S(X/X - Z, v) \simeq \Pi_S(Z, i^*v + N_i)$$

Here, v is a virtual vector bundle over X and N_i is the (necessarily regular) normal bundle of $i : Z \to X$. Under the additional assumption that $\Pi_S(X, v)$ is rigid, we show in Corollary 4.2.5 the duality with compact support isomorphism

$$\Pi_S(X, v)^\vee \simeq \Pi_S^c(X, -v - \tau_f)$$

This duality isomorphism can be seen as a motivic analog of classical topological results due to Atiyah [Ati61, §3], Milnor-Spanier [MS60, Lemma 2]. As an application, we identify the stable motivic homotopy type at infinity of hyperplane arrangements in Section 4.3.

In Section 6, we consider the case of a compactified smooth surface whose boundary is a union of rational curves. More precisely, let X/S be a smooth family of surfaces over S with a smooth proper compactification $\bar{X}/S$, whose boundary ∂X is a normal crossing divisor on $\bar{X}$, see Definition 5.1.1. We assume in addition that each branch $\partial_i X$, with its reduced schematic structure, is a rational curve over S. This setup gives is a quadratic generalization of Mumford's plumbing game [Mum61] using Chow-Witt groups. While Mumford uses orientations on the normal bundles of the branches, which are copies of the projective line, much of the subtleties in our setting comes from working with twisted Milnor-Witt K-theory sheaves in order to compute the quadratic degree maps of the intersections of the branches taking values in the Grothendieck-Witt ring. We decompose $\Pi_S(\partial X)$ into its combinatorial part $\mathcal{D}_X$, which is an Artin motivic spectrum over S, and its geometrical part $\oplus_{i \in I} \mathbb{1}_S(1)[2]$. In Theorem 6.2.7, we show $\Pi_S^\infty(X)$ is the homotopy fiber of a map

$$\mathcal{D}_X \oplus \bigoplus_{i \in I} \mathbb{1}_S(1)[2] \xrightarrow{\begin{pmatrix} a & b \\ b' & \mu \end{pmatrix}} \mathcal{D}_X' \oplus \bigoplus_{j \in I} \operatorname{Th}(N_j^0)$$

The map μ is computed by a "quadratic Mumford matrix" defined by the composite

(1.0.0.j) $$\mu_{ij} : \mathbb{1}_S(1)[2] \hookrightarrow \Pi_S(\partial_i X) \xrightarrow{\nu_{i*}} \Pi_S(\bar{X}) \xrightarrow{\nu_j^!} \Pi_S(\partial_j X, \langle N_j \rangle) \twoheadrightarrow \mathrm{Th}(N_j^0)$$

When $S = \mathrm{Spec}(\mathcal{O})$ is a semi-local essentially smooth scheme of a field, we can interpret μ_{ij} as the class of a quadratic form $(\partial_i X, \partial_j X)_{quad} \in \mathrm{GW}(\mathcal{O})$ called the *quadratic degree* of the intersections of the divisors $\partial_i X$ and $\partial_j X$. The close connection with quadratic forms arises since elements of the ith Chow-Witt group is represented by formal sums of subvarieties Z of codimenison i equipped with an element of $\mathrm{GW}(k(Z))$. Moreover, the rank of the quadratic degree equals the corresponding Mumford degree.

Further we specialize our results to motives. The above formula implies that the motive at infinity, i.e., Wildeshaus' boundary motive, of X is an Artin-Tate motive. When S is a finite field, a global field, or a number ring, we have the motivic t-structure on rational Artin-Tate motives at our disposal (see [Lev93] for the case of fields, and [Sch11] for number rings). Our main result can be restated (see Corollary 6.3.3) in terms of abelian Artin-Tate mixed motives, by asserting the existence of an exact sequence

$$0 \to \underline{\mathrm{H}}_3^\infty(X) \to \bigoplus_{i \in I} \mathbb{1}_S(2) \xrightarrow{\sum_{i<j} p_{ij}^{i!} - p_{ij}^{j!}} \bigoplus_{i<j} M_S(\partial_{ij}X)(2)$$

$$\to \underline{\mathrm{H}}_2^\infty(X) \to \bigoplus_{i \in I} \mathbb{1}_S(1) \xrightarrow{\mu} \bigoplus_{j \in I} \mathbb{1}_S(1)$$

$$\to \underline{\mathrm{H}}_1^\infty(X) \to \bigoplus_{i<j} M_S(\partial_{ij}X) \xrightarrow{\sum_{i<j} p_{ij*}^i - p_{ij*}^j} \bigoplus_{i \in I} \mathbb{1}_S \to \underline{\mathrm{H}}_0^\infty(X) \to 0$$

Here, μ is the Mumford matrix[2] and $\underline{\mathrm{H}}_i^\infty(X)$ is the ith homology motive at infinity of X. One can extract the weight filtration of explicit examples of abelian mixed motives from the above exact sequence. This gives an abelian motivic extension of Mumford's work to the arithmetic case. Due to its motivic nature, it has the advantage of giving universal formulas in the various realizations (ℓ-adic, rigid, syntomic,...) of motives. For example, this informs us about Galois representations attached to $\underline{\mathrm{H}}_i(X)$.

We illustrate the general theory with examples of $\mathbf{A}^1$-equivalent smooth affine surfaces with non-isomorphic stable motivic homotopy types at infinity. For any integer $n > 0$, the Danielewski surface D_n is the closed subscheme of $\mathbf{A}^3$ cut out by the equation $x^n z = y(y-1)$, see [Dan88]. We note that D_1 is the Jouanolou device over $\mathbf{P}^1$; in fact, D_n is $\mathbf{A}^1$-equivalent to $\mathbf{P}^1$ for all n, see [AØ19, §3.4]. Over any field k, one can distinguish between $\Pi_k^\infty(D_m)$ and $\Pi_k^\infty(D_n)$ for $m \neq n$ by viewing Danielewski surfaces as affine modifications of $\mathbf{A}^2$. We refer to Section 6.5 for precise statements and further examples, [DPØ19] for background on $\mathbf{A}^1$-contractibility of affine modifications, and [Fie94] for Fieseler's work on first homology at infinity of Danielewski surfaces over the complex numbers. The technique of affine modifications gives an affirmative answer to Problem 3.4.5 in [AØ19]. Our viewpoint, however, is that Π_k^∞ is the first step towards a refined invariant in unstable motivic homotopy theory, see Example 4.2.6. The problem of defining a notion of unstable motivic homotopy types at infinity captures the tension between unstable and stable motivic homotopy theory. For example, the six functor formalism is not available in the unstable setting. To remedy this one can take into account all possible smooth compactifications. Nonetheless some of the techniques developed

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[2]Due to the hypothesis on S, one has $\mathrm{Hom}(\mathbb{1}_S(1), \mathbb{1}_S(1)) \cong \mathbb{Q}$.

in this paper will carry over to unstable motivic homotopy categories, e.g., the calculations in Section 5.1 hold in the cdh-topology, and one can expect more developments along these lines.

Remark 1.0.1. This paper's results hold more generally for any motivic ∞-category such as triangulated and abelian mixed motives, Artin-Tate motives, étale motives, torsion and ℓ-adic categories, mixed Hodge modules, etc., in place of SH. In the case there exists a realization functor that commutes with the six operations, e.g., the Betti or ℓ-adic realizations, then this follows from the universality of SH.

Conventions. Our results are couched in the axiomatic setting of [CD19], [Kha16] which complements [Ayo07a]. We fix a *motivic ∞-category* ([CD19, Definition 2.4.45]) $\mathscr{T}$ over the category of qcqs schemes, i.e., a *monoidal stable homotopy functor* according to [Ayo07a]. Our primary example is the motivic stable homotopy category SH. In the language of presentable stable monoidal ∞-categories [Kha16], SH is the initial motivic ∞-category. Thus there is a unique morphism of motivic ∞-categories SH $\to \mathscr{T}$. To maintain intuition, we shall refer to the objects of $\mathscr{T}(S)$ as $\mathscr{T}$-spectra over S.

2. Preliminaries and notation

We work over quasi-coherent and quasi-compact base schemes S. The natural framework for this paper is Morel-Voevodsky's stable homotopy category SH(S) over S. Owing to the works [Ayo07a, Ayo07b], [CD19], for varying S, these categories satisfy *Grothendieck' six functor formalism* which we will use extensively. The elimination of the noetherian hypothesis was achieved in [Hoy14, Appendix C]. We will freely use the terminology and notations of [CD19] (see Section A.5 for a summary). In particular, SH(S) is a triangulated category and a closed symmetric monoidal category; moreover, the two structures are compatible, see e.g., [BRTV18], [DRØ03], [Hoy14], [Jar00], [Voe98].

In Section 6 we employ the fact that SH(S) admits a stable ∞-categorical enhancement. This means that the six functors can be viewed as ∞-functors, and commutative diagrams involving Grothendieck's six functor formalism can be enhanced into homotopy coherent diagrams. This useful point of view started, to our knowledge, with the paper [LZ12] on étale ℓ-adic sheaves. Following Lurie's idea, systematic treatment is given in [GR17, Chapter 3, Appendix]. Within the framework of motivic categories, this was exploited in Khan's thesis [Kha16]. For a recent reference, we refer to [BRTV18, Appendix A].

As noted in Remark 1.0.1, our main results are also valid in arbitrary triangulated motivic ∞-categories such as the following examples.
- DM$_{\mathbb{Q}}$ the ∞-category of rational mixed motives: see [CD19]
- DM the ∞-category of integral motives: we take as a model the category of modules over Spitzweck motivic cohomology ring spectrum relative to $\mathbb{Z}$: see [Spi18][3]

---

[3]This viewpoint was first advocated in [RØ06, RØ08]. If one restricts to schemes over a prime field k and inverts the characteristic exponent of k, one can take the so-called cdh-motives as defined in [CD16] (using cdh-sheaves with transfers).

- $DM_{\text{ét}}$ the ∞-category of étale mixed motives: see [Ayo14, CD15]
- $D_c^b(-_{\text{ét}}, \mathbb{Z}_\ell)$ the ∞-category of integral ℓ-adic étale sheaves: see [BBD82], [CD15, 7.2.18]
- DH the ∞-category of motivic Hodge modules, which should correspond to complexes of Saito's mixed Hodge modules of geometric origin (obtained by realization of mixed motives): see [Dre18]

Recall that an object M of a symmetric monoidal category $\mathcal{M}$ is *rigid* (or *strongly dualizable*) with dual N if the functor $(M \otimes -)$ is left and right adjoint to $(N \otimes -)$.[4] It is well-known that for X/S smooth and proper, with tangent bundle T_X, the spectrum $\Sigma^\infty X_+$ over S is rigid with dual $\Sigma^\infty \text{Th}(-T_X)$, see e.g., [CD19, 2.4.31].[5]

3. STABLE HOMOTOPY AT INFINITY

3.1. **Main definitions and examples.** This section defines the stable motivic homotopy type at infinity for separated schemes defined over a quasi-compact quasi-separated (=qcqs) base scheme S. We begin with a "classical" definition of stable motivic homotopy types, based on the analogy with Voevodsky's theory of motivic complexes ([VSF00, Chapter 5]).

Definition 3.1.1. Let $f \colon X \to S$ be a separated morphism. The stable motivic homotopy type $\Pi_S(X)$ and the properly supported stable motivic homotopy type $\Pi_S^c(X)$ of X in $\text{SH}(S)$ are defined by

$$\Pi_S(X) := f_! f^!(\mathbb{1}_S)$$
$$\Pi_S^c(X) := f_* f^!(\mathbb{1}_S)$$

The exchange natural transformation $\alpha_f : f_! \to f_*$ induces a canonical map

$$\alpha_X : \Pi_S(X) \to \Pi_S^c(X)$$

Remark 3.1.2. Suppose k is a field. The corresponding cohomological notation for the motivic complex (resp. properly supported cohomological complex) of X/k is $f_*(\mathbb{1}_X)$ (resp. $f_!(\mathbb{1}_X)$). Definition 3.1.1 complies with more classical invariants and allows for a duality theory.

3.1.3. The standard properties of the six functor formalism for $\text{SH}(S)$ implies:

- If X/S is smooth then $\Pi_S(X) = \Sigma^\infty X_+$
- If X/S is proper then α_X is an isomorphism
- $\Pi_S(-)$ (resp. $\Pi_S^c(-)$) is covariant functorial for all morphisms (resp. proper morphisms) and contravariant functorial with respect to étale morphisms
- α_X is natural with respect to proper covariance and étale contravariance

Definition 3.1.4. Let $f \colon X \to S$ be a separated morphism. The *stable homotopy type at infinity* of X/S is the homotopy fiber of α_X so that there is an exact homotopy sequence

$$\Pi_S^\infty(X) \to \Pi_S(X) \xrightarrow{\alpha_X} \Pi_S^c(X)$$

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[4]This notion was introduced in [DP80], but it appeared earlier in the theory of pure motives [Dem71, Prop. 4] as a key property in the theory of Tannakian categories [SR72]. We choose to write rigid since it is shorter than strongly dualizable. Note that *dualizable* is ambiguous as it could also refer to the property $DD(M) \simeq M$ for a dualizing functor D in Grothendieck-style duality. Recall, for example, that for spectra over a scheme of characteristic 0, the latter property amounts to constructibility (see [Ayo07a]).

[5]This is in fact an easy consequence of the six functor formalism, as formulated in [CD19, Theorem 2.4.50], given that $\Sigma^\infty X_+ \simeq f_! f^!(\mathbb{1}_S)$, where $f : X \to S$ is the projection map: indeed, one deduces from the projection formula and the purity property that $f_! f^!(\mathbb{1}_S)$ is rigid with dual $f_*(\mathbb{1}_X) \simeq f_\sharp(\text{Th}_X(-T_X))$.

Example 3.1.5. *Motivic realization.* Suppose S is a perfect field k with characteristic exponent p. In this case, see [HKØ17], [RØ08], there is a realization functor to motives

(3.1.5.a) $$\mathrm{SH}(k) \to \mathrm{DM}(k)[1/p]$$

If $X \to \mathrm{Spec}(k)$ is smooth, the motivic realization functor sends $\Pi_k(X)$ to $M(X)$, Voevodsky's homological motive of X, and it sends $\Pi_k^c(X)$ to $M^c(X)$, Voevodsky's homological motive of X with compact support (see [CD15, Proposition 8.10]). It follows that the motivic realization functor sends $\Pi_k^\infty(X)$ to *the boundary motive $\partial M(X)$* (see Wildeshaus [Wil06]). In particular, the Betti or ℓ-adic cohomology of $\Pi_k^\infty(X)$ coincides with the so-called *interior cohomology* of X.

We generalize the above discussion to arbitrary base schemes in Section 6.

Example 3.1.6. *Betti realization.* If the field k admits a complex embedding, there is a Betti realization functor to the stable homotopy category

(3.1.6.a) $$\mathrm{SH}(k) \to \mathrm{SH}^{top}$$

Owing to Ayoub's enhancement of (3.1.6.a) to an arbitrary base scheme using the technique of analytical sheaves [Ayo10], one derives that $\Pi_S(X)$ realizes to the singular chain complex $S_*(X)$, and $\Pi_S^c(X)$ realizes to the Borel-Moore singular chain complex $S_*^{BM}(X)$. Example 3.1.5 implies the Betti realization of $\Pi_S^\infty(X)$ coincides with the singular complex at infinity, $S_*^\infty(X)$, whose homology groups are the singular homology groups at infinity $H_*^\infty(X)$. Indeed, for a topological space W, there is a distinguished triangle of chain complexes of abelian groups

(3.1.6.b) $$S_*^\infty(W) \to S_*(W) \xrightarrow{\alpha_W} S_*^{lf}(W) \to S_*^\infty(W)[1]$$

Here, $S_*(W)$ (resp. $S_*^{lf}(W)$) be the complex of singular chains (resp. locally finite singular chains) on W, see [HR96, Chapter 3]. If W is locally contractible and σ-compact, then $S_*^{lf}(W)$ is quasi-isomorphic to the Borel-Moore complex of W.

3.2. **Computation via compactifications.**

3.2.1. Let $f: X \to S$ be a separated morphism. We fix an arbitrary compactification $\bar{X}$ of X over S and denote by $\partial X = (\bar{X} - X)_{red}$ its *boundary*. Consider the immersions $j : X \to \bar{X}$, $i : \partial X \to \bar{X}$. Using the localization property of SH, one derives the exact homotopy sequence

(3.2.1.a) $$\Pi_S^c(\partial X) \xrightarrow{i_*} \Pi_S^c(\bar{X}) \xrightarrow{j^*} \Pi_S^c(X)$$

Moreover, using properness of $\bar{X}$ and ∂X, there is a naturally induced commutative diagram with vertical isomorphisms

(3.2.1.b)
$$\begin{array}{ccc} \Pi_S(\partial X) & \xrightarrow{i_*} & \Pi_S(\bar{X}) \\ {\scriptstyle \alpha_{\partial X}} \downarrow {\scriptstyle \sim} & & {\scriptstyle \sim} \downarrow {\scriptstyle \alpha_{\bar{X}}} \\ \Pi_S^c(\partial X) & \xrightarrow{i_*} & \Pi_S^c(\bar{X}) \end{array}$$

We define the relative stable motivic homotopy type $\Pi_S(\bar{X}, X)$ of the pair $(\bar{X}, X)$ by the exact homotopy sequence

(3.2.1.c) $$\Pi_S(X) \xrightarrow{j_*} \Pi_S(\bar{X}) \xrightarrow{\pi} \Pi_S(\bar{X}, X)$$

In the next result, directly inspired by [Wil13, Theorem 1.6], we relate (3.2.1.a) and (3.2.1.c) to the stable motivic homotopy type at infinity.

Proposition 3.2.2. *There exist canonical maps β and δ in $\mathrm{SH}(S)$ rendering the following diagram of homotopy exact columns and rows commutative*

(3.2.2.a)

$$
\begin{array}{ccccc}
0 & \longrightarrow & \Pi_S(\partial X) & \Longrightarrow & \Pi_S(\partial X) \\
\downarrow & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \beta} \\
\Pi_S(X) & \xrightarrow{\ j_*\ } & \Pi_S(\bar{X}) & \xrightarrow{\ \pi\ } & \Pi_S(\bar{X}, X) \\
\| & & \downarrow{\scriptstyle j^*} & & \downarrow{\scriptstyle \delta} \\
\Pi_S(X) & \xrightarrow{\ \alpha_X\ } & \Pi_S^c(X) & \longrightarrow & \Pi_S^\infty(X)[1]
\end{array}
$$

The maps β and δ are natural in $(\bar{X}, X, \partial X)$, covariantly functorial with respect to proper maps, and contravariantly functorial with respect to étale maps (see 3.2.4).

Proof. To define β and δ canonically we rely on the six functors formalism. By the localization property, there exist exact homotopy sequences of functors

(3.2.2.b)
$$
i_* i^! \xrightarrow{\ ad_i'\ } 1 \xrightarrow{\ ad_j\ } j_* j^*
$$

(3.2.2.c)
$$
j_! j^* \xrightarrow{\ ad_j'\ } 1 \xrightarrow{\ ad_i\ } i_* i^*
$$

A combination of (3.2.2.b) and (3.2.2.c) yields the commutative diagram

(3.2.2.d)

$$
\begin{array}{ccccc}
0 & \longrightarrow & i_* i^! & =\!=\!=\!= & i_* i^! \\
\downarrow & & \downarrow{\scriptstyle ad_i'} & & \downarrow{\scriptstyle i_* i^*.ad_i'} \\
j_! j^* & \xrightarrow{\ ad_j'\ } & 1 & \xrightarrow{\ ad_i\ } & i_* i^* \\
\| & & \downarrow{\scriptstyle ad_j} & & \downarrow{\scriptstyle i_* i^*.ad_j} \\
j_! j^* & \xrightarrow{\ ad_j'\cdot j_* j^*\ } & j_* j^* & \xrightarrow{\ ad_i \cdot j_* j^*\ } & i_* i^* j_* j^*
\end{array}
$$

Here we us the fact that i_* and j_* are fully faithful, so that $j_! j^* j_* j^* \simeq j_! j^*$ and $i_* i^* i_* i^! \simeq i_* i^!$. The bottom row is obtained from (3.2.2.c) by post-composition with $j_* j^*$, and the rightmost column is obtained from (3.2.2.b) by pre-composition with $i_* i^*$. Thus, all columns and rows in (3.2.2.d) are homotopy exact. Let $\bar{f} : \bar{X} \to S$ be the structure map. Inserting $\bar{f}^!(\mathbb{1}_S)$ into (3.2.2.d) and applying $\bar{f}_! = \bar{f}_*$ yields the desired diagram (3.2.2.a). $\qquad\square$

Remark 3.2.3. In the setting of Proposition 3.2.2 there are structure maps

$$
\begin{array}{ccc}
X & \xhookrightarrow{\ j\ } \bar{X} \xhookleftarrow{\ i\ } & \partial X \\
 & {\scriptstyle f}\searrow \ \downarrow{\scriptstyle \bar{f}}\ \swarrow{\scriptstyle g} & \\
 & S &
\end{array}
$$

The proof of Proposition 3.2.2 gives the following diagram of exact homotopy sequence

$$
\begin{array}{ccccc}
\Pi_S(\partial X) & \xrightarrow{\ \beta\ } & \Pi_S(\bar{X}, X) & \xrightarrow{\ \delta\ } & \Pi_S^\infty(X)[1] \\
{\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sim} \\
g_! g^!(\mathbb{1}_S) & \longrightarrow & g_! i^* \bar{f}^!(\mathbb{1}_S) & \longrightarrow & g_* i^* j_* f^!(\mathbb{1}_S) \\
\| & & \| & & \| \\
\bar{f}_! i_* i^! \bar{f}^!(\mathbb{1}_S) & \xrightarrow{\ ad_i'\ } & \bar{f}_! i_* i^* \bar{f}^!(\mathbb{1}_S) & \xrightarrow{\ ad_j\ } & \bar{f}_! i_* i^* j_* j^* \bar{f}^!(\mathbb{1}_S)
\end{array}
$$

That is, β (resp. δ) is induced by the counit map ad'_i (resp. unit map ad_j) of the adjunction (i_*, i^*) (resp. (j^*, j_*)). Here we use simplified notation for the morphisms in the lower rows. In particular, we obtain the following interesting formula for the stable homotopy type at infinity of X

(3.2.3.a) $$\Pi_S^\infty(X) \simeq g_* i^* j_* f^!(\mathbb{1}_S)[-1]$$

This shows $\Pi_S^\infty(-)$ is analogous to the vanishing cycle functor for étale sheaves [DK73].

3.2.4. *Functoriality properties.* Suppose

$$
\begin{array}{ccc}
Y & \overset{l}{\hookrightarrow} \bar{Y} \overset{k}{\hookleftarrow} & \partial Y \\
{\scriptstyle f}\downarrow & {\scriptstyle \bar{f}}\downarrow & \downarrow{\scriptstyle g} \\
X & \overset{j}{\hookrightarrow} \bar{X} \overset{i}{\hookleftarrow} & \partial X
\end{array}
$$

is a commutative diagram of S-schemes, where $\bar{X}$, $\bar{Y}$ are proper, i, k closed immersions, and j, l open immersions. Then, if f is proper, there is a commutative diagram

$$
\begin{array}{ccc}
\Pi_S(\partial Y) \overset{\beta_Y}{\longrightarrow} & \Pi_S(\bar{Y}, Y) \overset{\delta_Y}{\longrightarrow} & \Pi_S^\infty(Y)[1] \\
{\scriptstyle g_*}\downarrow & {\scriptstyle (\bar{f}, f)_*}\downarrow & \downarrow{\scriptstyle f_*} \\
\Pi_S(\partial X) \overset{\beta_X}{\longrightarrow} & \Pi_S(\bar{X}, X) \overset{\delta_X}{\longrightarrow} & \Pi_S^\infty(X)[1]
\end{array}
$$

Moreover, if $\bar{f}, f, g$ are étale, we obtain the commutative diagram

$$
\begin{array}{ccc}
\Pi_S(\partial X) \overset{\beta_X}{\longrightarrow} & \Pi_S(\bar{X}, X) \overset{\delta_X}{\longrightarrow} & \Pi_S^\infty(X)[1] \\
{\scriptstyle g^*}\downarrow & {\scriptstyle (\bar{f}, f)^*}\downarrow & \downarrow{\scriptstyle f^*} \\
\Pi_S(\partial Y) \overset{\beta_Y}{\longrightarrow} & \Pi_S(\bar{Y}, Y) \overset{\delta_Y}{\longrightarrow} & \Pi_S^\infty(Y)[1]
\end{array}
$$

Remark 3.2.5. Another way of stating Proposition 3.2.2 is that $\Pi_S^\infty(X)$ is the homotopy fiber of the canonical map

(3.2.5.a) $$\Pi_S(\partial X) \oplus \Pi_S(X) \overset{i_* + j_*}{\longrightarrow} \Pi_S(\bar{X})$$

Under motivic realization, (3.2.5.a) becomes the fundamental formula for the boundary motive in [Wil06, Proposition 2.4]. Following the proof method of [Wil06, Theorem 5.1], we deduce the theorem of analyticial invariance (1.0.0.e) for $\Pi_S^\infty(-)$.

3.3. **The case of smooth compactifications.**

3.3.1. *Abstract Euler classes.* Recall that for a vector bundle V over a scheme Z, the (abstract) Euler class[6] $e(V)$ defined in [DJK18] is the map in $\mathrm{SH}(Z)$

$$e(V) : \mathbb{1}_Z \simeq \mathrm{Th}(0_X) \overset{s_*}{\longrightarrow} \mathrm{Th}_Z(V)$$

induced by the zero section s of V. If $g : Z \to S$ is smooth, the left adjoint $g_\sharp$ to g^* yields by abuse of notation a map

$$e(V) : \Pi_S(Z) \to \mathrm{Th}(V)$$

in $\mathrm{SH}(S)$. We develop a technique for computing stable motivic homotopy types at infinity involving abstract Euler classes for normal bundles.

---

[6]It can be interpreted as a twisted cohomology class $e(V) \in H^0(Z, \langle V \rangle)$, in the 0-th stable cohomology group, which generalizes the well-known Euler classes in oriented and $\mathbf{SL}^c$-oriented cohomology theories.

Theorem 3.3.2. *Let X be an S-scheme with compactification $\bar{X}$ and boundary $\partial X = \bar{X} - X$. Suppose $\bar{X}$, ∂X are smooth S-schemes, and let N denote the normal bundle of ∂X in $\bar{X}$. Then there exists an exact homotopy sequence in $\mathrm{SH}(S)$*

$$(3.3.2.a) \qquad \Pi_S^\infty(X) \to \Pi_S(\partial X) \xrightarrow{e(N)} \mathrm{Th}(N)$$

Proof. The exact homotopy sequence (3.3.2.a) follows from Proposition 3.2.2 and an analysis of $\beta_X \colon \Pi_S(\partial X) \to \Pi_S(\bar{X}, X) \simeq \Sigma^\infty(\bar{X}/\bar{X} - \partial X)$. Owing to [DJK18], the latter map coincides with the composite

$$(3.3.2.b) \qquad \Sigma^\infty \partial X_+ \xrightarrow{i_*} \Sigma^\infty \bar{X}_+ \to \Sigma^\infty(\bar{X}/\bar{X} - \partial X)$$

Homotopy purity for closed embeddings [MV99, Theorem 3.2.23] identifies (3.3.2.b) with

$$(3.3.2.c) \qquad \Sigma^\infty \partial X_+ \xrightarrow{i_*} \Sigma^\infty \bar{X}_+ \xrightarrow{i^*} \mathrm{Th}(N)$$

The excess intersection formula in [DJK18, 3.2.6] equates the composite $i^* i_*$ in (3.3.2.c) with the abstract Euler class $e(N)$. $\qquad\qquad\square$

Corollary 3.3.3. *Under the assumptions in Theorem 3.3.2 there exists a canonical isomorphism*

$$\Pi_S^\infty(X) \simeq \Pi_S(N^\times)$$

Here, $N^\times$ is the complement of the zero section of the normal bundle N. If N contains a trivial direct factor, or more generally the Euler class $e(N) = 0$, then

$$\Pi_S^\infty(X) \simeq \Pi_S(\partial X) \oplus \mathrm{Th}(N)[-1]$$

Therefore, if N is trivial of rank d, we have

$$\Pi_S^\infty(X) \simeq \Pi_S(\partial X) \oplus \mathbb{1}(d)[2d-1]$$

Example 3.3.4. For $d > 0$ the stable homotopy type at infinity of the affine space $\mathbf{A}^d$ is given by

$$\Pi_S^\infty(\mathbf{A}^d) \simeq \mathbb{1} \oplus \mathbb{1}(d)[2d-1]$$

This follows since $\mathbf{A}^d$'s Euler class vanishes.

Remark 3.3.5. We note that $\Pi_k^\infty(X)$ is a strictly finer invariant than its motivic realization, the boundary motive $\partial M(X)$ in Example 3.1.5. Indeed, Theorem 3.3.2 shows there is an exact homotopy sequence in $\mathrm{DM}(k)[1/p]$

$$\partial M(X) \to M(\partial X) \xrightarrow{\tilde{c}_r(N)} M(\partial X)(r)[2r]$$

Here, r is the rank of N and the map $\tilde{c}_r(N)$ is induced by multiplication with the top Chern class $c_r(N) \in \mathrm{CH}^r(\partial X) \simeq \mathrm{Hom}(M(\partial X), \mathbb{1}(r)[2r])$. In particular, the sequence splits if $c_r(N) = 0$. However, the vanishing of the homotopy Euler class $e(N)$, which implies the vanishing of the Euler class in Chow-Witt groups, is a strictly stronger condition than the vanishing of the top Chern class. For the smooth affine quadric 5-fold $Q \colon x_1 y_1 + x_2 y_2 + x_3 y_3 = 1$ in $\mathbf{A}^6$, the kernel of the surjection $(x_1, x_2, x_3) \colon k[Q]^3 \to k[Q]$ defines a nontrivial and stably trivial vector bundle ξ of rank 2 on Q. Since $\det(\xi)$ is trivial, ξ is orientable. While ξ's Chern classes are trivial, ξ's Euler class in $\widetilde{\mathrm{CH}}^2(Q) = \mathrm{K}_{-1}^{\mathrm{MW}}(k)$ equals η, see the case $n = 2$ in [AF14, Lemma 3.5].

4. DUALITY AND FUNDAMENTAL CLASS OF THE DIAGONAL

In this section, we develop computational techniques based on the notion of homotopically smooth morphisms with respect to a motivic ∞-category. Our method involves duality with compact support instead of compactifications, as in Section 3. In the process, we generalize the Morel-Voevodsky homotopy purity theorem [MV99] and give new examples of rigid objects; see [NSØ09, Remark 8.2] for examples of compact non-rigid objects.

4.1. **Homotopical smoothness and generalized purity.**

4.1.1. In the setting of stable motivic homotopy, we shall use the technique of twisted Thom spaces to deduce the duality results we use in our computations. Recall the Thom space functor, from vector bundles over a scheme X to the pointed homotopy category over X, is given by

$$\mathrm{Th} : E \mapsto \Sigma^\infty \mathrm{Th}(E), \mathrm{Th}(E) = E/E^\times$$

It sends direct sums to tensor products. Owing to [CD19, Remark 2.4.15] this functor admits an extension to a monoidal functor, called the Thom spectrum functor

$$\mathrm{Th} : \underline{\mathrm{K}}(X) \to (\mathrm{Pic}(\mathrm{SH}(X)), \otimes)$$

from the Picard category of virtual bundles over X to the monoidal category of $\otimes$-invertible objects in $\mathrm{SH}(X)$. When $v = \langle E \rangle$ is the virtual bundle associated to a vector bundle, one has $\mathrm{Th}(v) = \Sigma^\infty \mathrm{Th}(E)$. Given a smooth morphism $p : X \to S$ it is sometimes convenient to simply "forget the base" of Thom spectra; we set $\mathrm{Th}_S(v) := p_\sharp \mathrm{Th}(v)$.

Using this observation, we extend Definition 3.1.1 as follows.

Definition 4.1.2. Let $f : X \to S$ be a separable morphism and v a virtual vector bundle over X. One associates to X/S and v the following motivic spectra over S:

- Homotopy: $\Pi_S(X, v) = f_!(\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S))$
- Cohomotopy: $\mathrm{H}_S(X, v) = f_*(\mathrm{Th}(v) \otimes f^*(\mathbb{1}_S)) \simeq f_*(\mathrm{Th}(v))$
- Borel-Moore (or (co)compactly supported) homotopy: $\Pi_S^c(X, v) = f_*(\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S))$
- Compactly supported cohomotopy: $\mathrm{H}_S^c(X, v) = f_!(\mathrm{Th}(v) \otimes f^*(\mathbb{1}_S)) \simeq f_!(\mathrm{Th}(v))$

Definition 4.1.2 specializes to Definition 3.1.1 in the case of the trivial vector bundle $v = 0$, i.e., we have $\Pi_S(X, 0) = \Pi_S(X)$ and $\Pi_S^c(X, 0) = \Pi_S^c(X)$. As in the previous case, the natural transformation $\alpha_f : f_! \to f_*$ yields canonical maps:

- $\alpha_{X/S} : \Pi_S(X, v) \to \Pi_S^c(X, v)$
- $\alpha'_{X/S} : \mathrm{H}_S^c(X, v) \to \mathrm{H}_S(X, v)$ ("forgetting compact support")

Both $\alpha_{X/S}$ and $\alpha'_{X/S}$ are isomorphisms whenever X/S is proper.

Remark 4.1.3. Definition 4.1.2 can be adapted to an arbitrary triangulated motivic category $\mathscr{T}$ in the sense of [CD19]. This generality is not strictly needed if there exists a realization functor $\mathrm{SH}(S) \to \mathscr{T}$ commuting with the six operations, as in the case of motives $\mathrm{DM}_t(S)$. Twisting by vector bundles is not pertinent for motives since there exists well-behaved Thom isomorphisms $\mathrm{Th}(v) \simeq \mathbb{1}_S(r)[2r]$, where r is the rank of v.[7] The four theories in Definition 4.1.2 realize to the homological motive $M_S(X)$, the cohomological motive $h_S(X)$ ("Chow motive" when X/S is smooth proper), the "homological motive with compact support" $M_S^c(X)$ in Voevodsky's terminology, and the cohomological motive with compact support $h_S^c(X)$, respectively.

4.1.4. *Natural functoriality*: Let $f : Y \to X$ be a separable morphism of S-schemes. Our choice of terminology can be explained by the following naturally induced maps:

---

[7]On the contrary, in motivic homotopy theory the consideration of twists is indispensable.

- $f_* : \Pi_S(Y, f^{-1}v) \to \Pi_S(X, v)$
- $f^* : \mathrm{H}_S(X, v) \to \mathrm{H}_S(Y, f^{-1}v)$
- $f_* : \Pi_S^c(Y, f^{-1}v) \to \Pi_S^c(X, v)$
- $f^* : \mathrm{H}_S(X, v) \to \mathrm{H}_S(Y, f^{-1}v)$

In addition, when f is proper then the comparison maps $\alpha_{X/S}$ and $\alpha'_{X/S}$ are compatible with f_*, f^*, f_*, f^*.

Remark 4.1.5. Note, in particular, that homotopy twisted by some virtual bundle w on Y, with or without compact support, is not covariant functorial unless w is the pullback of some virtual bundle on X.

4.1.6. *Exceptional functoriality (Gysin maps)*: Due to the existence of the fundamental classes introduced in [DJK18] the four theories in Definition 4.1.2 satisfy exceptional functoriality.[8] Let $f : Y \to X$ be a smoothable lci morphism, i.e., f factors as a regular closed immersion followed by a smooth morphism, with cotangent complex L_f. We refer to the associated virtual bundle τ_f as the virtual tangent bundle of f. One deduces, from the system of fundamental classes in [DJK18, Theorem 3.3.2], the canonical natural transformation

$$(4.1.6.a) \qquad\qquad \mathfrak{p}_f : \mathrm{Th}(\tau_f) \otimes f^* \to f^!$$

By adjunction, one deduces trace and cotrace maps (see §4.3.4 in *loc. cit.*)

$$\mathrm{tr}_f : f_!(\mathrm{Th}(\tau_f) \otimes f^*) \to Id$$

$$\mathrm{cotr}_f : Id \to f_*(\mathrm{Th}(-\tau_f) \otimes f^!)$$

These maps induce the following *Gysin maps*:

- $f^! : \Pi_S(X, v) \to \Pi_S(Y, f^{-1}v + \tau_f)$, when f is proper
- $f_! : \mathrm{H}_S(Y, f^{-1}v + \tau_f) \to \mathrm{H}_S(X, v)$, when f is proper
- $f^! : \Pi_S^c(X, v) \to \Pi_S^c(Y, f^{-1}v + \tau_f)$
- $f_! : \mathrm{H}_S^c(Y, f^{-1}v + \tau_f) \to \mathrm{H}_S^c(X, v)$

Again, assuming f is proper, the comparison maps $\alpha_{X/S}$ and $\alpha'_{X/S}$ are compatible with the above Gysin morphisms in the obvious sense.

The following definition is a variant of [DJK18, Definition 4.3.7].

Definition 4.1.7. Let $f : X \to S$ be a smoothable lci morphism with virtual bundle τ_f over X associated to the cotangent complex L_f. We say that f is *homotopically smooth* (*h-smooth*) with respect to the motivic ∞-category $\mathscr{T}$ if the natural transformation (4.1.6.a) evaluated at the sphere spectrum $\mathbb{1}_S$

$$\mathfrak{p}_f : \mathrm{Th}(\tau_f) \to f^!(\mathbb{1}_S)$$

is an isomorphism.

4.1.8. One gets the following basic properties of h-smoothness: considering composable lci smoothable morphisms f, g, $h = f \circ g$ (which is also lci smoothable), if f and g (resp. f and h) are h-smooth, then so is h (resp. g). Moreover, if $g^!$ is conservative, g and h being h-smooth implies f is h-smooth. On the other hand, h-smoothness is not stable under base change.

Example 4.1.9. Here are examples, in order of difficulty, of cases where $f : X \to S$ is h-smooth:

- f is smooth
- X, S are smooth over some base B and f is a morphism of B-schemes

---

[8]See [DJK18, 4.3.4] for the general case of a triangulated motivic category.

- X, S are regular over a field k and $\mathscr{T}$ is continuous, see [DFKJ20, Appendix A] (all our examples are continuous in this sense)

Note, in particular, that a closed immersion between smooth varieties over a field is h-smooth. On the other hand, not all regular closed immersions are smooth. For example, the immersion $i : D \to X$ of a relative normal crossing divisor D over S into a smooth S-scheme in the sense of Definition 5.1.1 is not h-smooth unless D is smooth over S. Indeed, in Section 6, we show

$$i^!(\mathbb{1}_X) \simeq \mathrm{colim}_{n \in \Delta^{\mathrm{inj}}} \bigoplus_{J \subset I, \sharp J = n} i_{J!}(\mathrm{Th}(N_J))$$

which does not coincide with $\mathrm{Th}(N_D X)$ if there is a non-trivial intersection on the branches of D (consider the pullback to any such intersection).

Remark 4.1.10. It is expected that any morphism between regular schemes is h-smooth with respect to $\mathrm{SH}(S)$. The said assertion holds and is called "absolute purity" in the case when $\mathscr{T}$ is $\mathrm{SH}(S) \otimes \mathbb{Q}$ ([DFKJ20, Theorem 3.6]), rational mixed motives $\mathrm{DM}_{\mathbb{Q}}(S)$ ([CD19, 14.4.1]), and étale integral motives $\mathrm{DM}_{\text{ét}}(S)$ ([CD16, 5.6.2]).

Example 4.1.11. The h-smoothness hypothesis allows one to compare the four different theories from Definition 4.1.2 and generalize the smooth case. More precisely, if $f : X \to S$ is h-smooth, the purity isomorphism induces isomorphisms

$$\Pi_S(X, v) = f_!\big(\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S)\big) \xrightarrow{\mathfrak{p}_f} f_!\big(\mathrm{Th}(v) \otimes \mathrm{Th}(\tau_f)\big) = \mathrm{H}_S^c(X, v + \tau_f)$$

$$\Pi_S^c(X, v) = f_*\big(\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S)\big) \xrightarrow{\mathfrak{p}_f} f_*\big(\mathrm{Th}(v) \otimes \mathrm{Th}(\tau_f)\big) = \mathrm{H}_S(X, v + \tau_f)$$

Moreover, these isomorphisms transform the usual functoriality (resp. Gysin maps) in the source to the Gysin maps (resp. usual functoriality) on the target. This follows by considering the "associativity formula" for fundamental classes in [DJK18, Theorem 3.3.2].

Given this new notion of h-smoothness, we extend the Morel-Voevodsky homotopy purity theorem as follows.[9]

Theorem 4.1.12. *Let $f : X \to S$ be an h-smooth morphism with virtual tangent bundle τ_f, and v a virtual vector bundle over X. Let $i : Z \to X$ be a closed immersion and set $g = f \circ i$. We let $\Pi_S(X/X - Z, v)$ denote the homotopy cofiber of the canonical map $j_* : \Pi_S(X - Z, v) \to \Pi_S(X, v)$.*
Then the purity isomorphism $\mathfrak{p}_f$ induces an isomorphism

$$\Pi_S(X/X - Z, v) \simeq \mathrm{H}_S^c(Z, i^*v + i^*\tau_f)$$

Moreover, if Z/S is h-smooth and N_i denotes the normal bundle associated with the (necessarily regular) closed immersion i, there exists a (relative) purity isomorphism

$$\Pi_S(X/X - Z, v) \simeq \Pi_S(Z, i^*v + N_i)$$

Proof. For the closed immersion i we have the associated localization homotopy exact sequence

$$j_! j^! \to Id \to i_* i^*$$

By inserting $\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S)$ and applying $f_!$ we get the homotopy exact sequence

$$\Pi_S(X - Z, j^*v) \to \Pi_S(X, v) \to q_!\big(\mathrm{Th}(i^*v) \otimes i^* f^!(\mathbb{1}_S)\big)$$

Here, we use the identifications

$$f_! j_! j^!\big(\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S)\big) \simeq h_!\big(\mathrm{Th}(j^*v) \otimes j^! f^!(\mathbb{1}_S)\big) = \Pi_S(X - Z, j^*v)$$

$$f_! i_* i^*\big(\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S)\big) \simeq f_! i_*\big(\mathrm{Th}(i^*v) \otimes i^* f^!(\mathbb{1}_S)\big) = q_!\big(\mathrm{Th}(i^*v) \otimes i^* f^!(\mathbb{1}_S)\big)$$

---

[9]When Z has crossing singularities, a more aesthetically pleasing formulation will be given in Theorem 5.2.5.

In particular, there is an isomorphism

$$\Pi_S(X/X - Z, v) \simeq q_! \big(\mathrm{Th}(i^*v) \otimes i^* f^!(\mathbb{1}_S) \big)$$

The purity isomorphism yields the desired isomorphism

$$
\begin{aligned}
\Pi_S(X/X - Z, v) &\simeq q_! \big(\mathrm{Th}(i^*v) \otimes i^* f^!(\mathbb{1}_S) \big) \\
&\xrightarrow{\mathfrak{p}_f} q_! \big(\mathrm{Th}(i^*v) \otimes i^* \mathrm{Th}(\tau_f) \otimes f^*(\mathbb{1}_S) \big) \\
&= q_! \big(\mathrm{Th}(i^*v + i^*\tau_f) \big) = \mathrm{H}_S^c(Z, i^*v + i^*\tau_f)
\end{aligned}
$$

As explained in Example 4.1.11, the second isomorphism in Theorem 4.1.12 is now a direct consequence of the h-smoothness property of Z/S. □

4.2. **Homotopy type at infinity and the fundamental class of the diagonal.**

Lemma 4.2.1. *Let $f : X \to S$ be a separated morphism and v a virtual vector bundle over X.*

(1) There exists a canonical isomorphism

$$\underline{\mathrm{Hom}}(\mathrm{H}_S^c(X, v), \mathbb{1}_S) \xrightarrow{\simeq} \Pi_S^c(X, -v)$$

The isomorphism is functorial in X for both the natural functoriality with respect to proper maps (4.1.4) and the Gysin morphisms with respect to smoothable lci morphisms (4.1.6).

(2) Assume, in addition, that f is h-smooth (4.1.7). Then there exists an isomorphism

$$\underline{\mathrm{Hom}}(\Pi_S(X, v), \mathbb{1}_S) \xrightarrow{\simeq} \mathrm{H}_S(X, -v)$$

which is again functorial with respect to the natural functoriality and the Gysin maps.

Proof. To prove the isomorphism in (1) we use

$$
\underline{\mathrm{Hom}}(\mathrm{H}_S^c(X, v), \mathbb{1}_S) = \underline{\mathrm{Hom}}(f_!(\mathrm{Th}(v)), \mathbb{1}_S) \xrightarrow{(a)} f_* \underline{\mathrm{Hom}}\big(\mathrm{Th}(v), f^!(\mathbb{1}_S) \big)
$$

$$
\xrightarrow{(b)} f_* \big(\mathrm{Th}(-v) \otimes f^!(\mathbb{1}_S) \big) = \Pi_S^c(X, -v)
$$

Here, (a) (resp. (b)) follows from the internal interpretation of the fact that $f^!$ is right adjoint to $f_!$ (resp. $\mathrm{Th}(v)$ is $\otimes$-invertible). The functoriality statement is obvious by construction.

To deduce (2), we establish the isomorphisms

$$
\underline{\mathrm{Hom}}(\Pi_S(X, v), \mathbb{1}_S) = \underline{\mathrm{Hom}}\big(f_! \big(\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S) \big), \mathbb{1}_S \big) \xrightarrow{(a)} f_* \underline{\mathrm{Hom}}\big(\mathrm{Th}(v) \otimes f^!(\mathbb{1}_S), f^!(\mathbb{1}_S) \big)
$$

$$
\xrightarrow{(b)} f_* \Big(\mathrm{Th}(-v) \otimes \underline{\mathrm{Hom}}\big(f^!(\mathbb{1}_S), f^!(\mathbb{1}_S) \big) \Big),
$$

$$
\xrightarrow{(c)} f_* \big(\mathrm{Th}(-v) \otimes \mathbb{1}_S \big) = \mathrm{H}_S(X, -v)
$$

Here, (a) and (b) are justified as in (1), and (c) follows from the fact that the canonical map

$$(4.2.1.a) \qquad\qquad \mathbb{1}_X \to \underline{\mathrm{Hom}}\big(f^!(\mathbb{1}_S), f^!(\mathbb{1}_S) \big)$$

obtained by adjunction is an isomorphism because f is h-smooth. The functoriality statement follows from the definitions. □

Remark 4.2.2. We say that f is a *pre-dualizing* morphism if the map (4.2.1.a) is an isomorphism. This condition implies the isomorphism in (2). The notion of a pre-dualizing morphism is crucially linked with Grothendieck-Verdier duality, as shown by [CD19, 4.4.11]. In fact, if $f^!(\mathbb{1}_S)$ is a dualizing object ([CD19, Definition 4.4.4]), it follows that f is pre-dualizing. Assuming that S is a smooth $\mathbb{Q}$-scheme, it follows from [Ayo07a] that any separated morphism $f : X \to S$ is pre-dualizing.

Theorem 4.2.3. *Let $f : X \to S$ be an h-smooth morphism with virtual tangent bundle τ_f. Then the purity isomorphism $\mathfrak{p}_f$ induces a canonical isomorphism*

$$\underline{\mathrm{Hom}}\left(\Pi_S(X, v), \mathbb{1}_S\right) \simeq \Pi_S^c(X, -v - \tau_f)$$

In addition, assume that f is smooth. Then the map α'_f obtained by adjunction from the composite

$$\Pi_S(X) \xrightarrow{\alpha_f} \Pi_S^c(X) \simeq \underline{\mathrm{Hom}}\left(\Pi_S(X, -\tau_f), \mathbb{1}_S\right)$$

fits into the commutative diagram

$$
\begin{array}{ccc}
\Pi_S(X) \otimes \Pi_S(X, -\tau_f) & \xrightarrow{\alpha'_f} & \mathbb{1}_S \\
\simeq \downarrow & & \uparrow f_* \\
\Pi_S(X \times_S X, -p_1^* \tau_f) & \xrightarrow{\delta^!} & \Pi_S(X)
\end{array}
$$

where the left vertical map is the Künneth isomorphism.[10]

If f is smooth, the above commutative diagram shows the map α'_f can be computed under the canonical isomorphism

$$
\begin{aligned}
[\Pi_S(X), \Pi_S^c(X)] \simeq & [\Pi_S(X) \otimes \Pi_S(X, -\tau_f), \mathbb{1}_S] \\
\simeq & [X \times_S X, \mathrm{Th}_S(p_1^*)] =: \Pi^{2d,d}(X \times_S X, p_1^*(\tau_f))
\end{aligned}
$$

as the fundamental class of the diagonal

$$\eta_{X \times_S X}(\Delta_{X/S}) = \delta_*(1)$$

Remark 4.2.4. We note that when f is an h-smooth closed immersion $i : Z \to X$, the diagonal $\delta : Z \to Z \times_S Z$ is an isomorphism. In particular, the normal bundle of δ is unrelated to the virtual tangent space $-N_Z S$ of Z/S. For this reason, the preceding formula cannot be true for arbitrary h-smooth morphisms. We expect ideas in derived algebraic geometry can fix this defect.

Proof. The first isomorphism is a combination of the second isomorphism in Lemma 4.2.1 and the second isomorphism of Example 4.1.11

$$\underline{\mathrm{Hom}}\left(\Pi_S(X, v), \mathbb{1}_S\right) \simeq \mathrm{H}_S(X, -v) \overset{\mathfrak{p}_f^{-1}}{\simeq} \Pi_S^c(X, -v - \tau_f)$$

The commutativity of the square follows from the following facts:

- If $f_1 : X \times_S X \to X$ is the projection on the first factor, the associativity formula in [DJK18, Theorem 3.3.2] shows there is an equality of fundamental classes $\eta_\delta . \eta_{f_1} = 1$
- The assumption that f is smooth implies the cartesian square

$$
\begin{array}{ccc}
X \times_S X & \xrightarrow{f_1} & X \\
f_2 \downarrow & \Delta & \downarrow f \\
X & \xrightarrow{f} & S
\end{array}
$$

  is Tor-independent. Thus the transversal base change formula in [DJK18, Theorem 3.3.2] implies the equality $\Delta^*(\eta_f) = \eta_{f_1}$

$\square$

---

[10]Which in our case is a tautology by definition of the tensor product of spectra and the fact that X/S is smooth.

Theorem 4.2.3 implies the following more potent form of duality (see the end of Section 2 for a review). We give concrete examples in Theorem 5.2.1.

Corollary 4.2.5 (Duality with compact support)**.** *Let X be an h-smooth scheme over S and v a virtual bundle over X such that $\Pi_S(X, v)$ is rigid. Then there exists an isomorphism*

$$\Pi_S(X, v)^\vee \simeq \Pi_S^c(X, -v - \tau_f)$$

In particular, the Borel-Moore homotopy object $\Pi_S^c(X, -v - \tau_f)$ is rigid.

Example 4.2.6*.* Suppose X is a d-dimensional smooth affine $\mathbf{A}^1$-contractible k-scheme. Owing to Atiyah duality in Theorem 4.2.3 there is an exact homotopy sequence

(4.2.6.a) $$\Pi_k^\infty(X) \to \Pi_k(X) \to \Pi_k(X, -T_X)^\vee$$

The assumption on X implies the tangent bundle T_X is trivial. Since $\Pi_k(X) \simeq \mathbb{1}$ we deduce the equivalences

$$\Pi_k(X, -T_X)^\vee \simeq (\Pi_k(X)(-d)[-2d])^\vee \simeq \Pi_k(X)(d)[2d] \simeq \mathbb{1}(d)[2d]$$

Morel's connectivity theorem shows the map $\mathbb{1} \to \mathbb{1}(d)[2d]$ is zero for $d > 0$. Hence there is an equivalence

$$\Pi_k^\infty(X) \simeq \mathbb{1} \oplus \mathbb{1}(d)[2d - 1]$$

In conclusion, the stable motivic homotopy type at infinity cannot distinguish between smooth affine $\mathbf{A}^1$-contractible k-schemes of the same dimension. This is analogous to the classification of Euclidean spaces among open contractible manifolds, see [AØ19], which cannot be achieved by invariants of the stable topological homotopy category.

4.3. **Stable motivic homotopy type at infinity of hyperplane arrangements.**

4.3.1. In the following, we use Theorem 4.2.3 to compute the stable motivic homotopy type at infinity of hyperplane arrangements. We say a smooth S-scheme X with structural morphism f is *stably $\mathbf{A}^1$-contractible* if $f_* : \Sigma^\infty X_+ \to \mathbb{1}_S$ is an equivalence. Vector bundles give basic examples over S. To illustrate the preceding results, we determine the stable homotopy of a normal crossing S-scheme.

Proposition 4.3.2. *Let S be a stably $\mathbf{A}^1$-contractible smooth scheme over a field k, and X be a smooth affine and stably $\mathbf{A}^1$-contractible S-scheme of dimension d. Suppose the closed subscheme $D = \cup_{i\in I} D_i$ of X is a smooth reduced crossing S-scheme in the sense of Definition 5.1.1. In addition, assume that for any $J \subset I$, every connected component of D_J is stably $\mathbf{A}^1$-contractible over S. For a subset $J \subset I$ we set $n_J = \sharp J$, and for any generic point x of D_J we let $c_J(x)$ denote the codimension of x in X.*
Then there exists a canonical isomorphism

$$\Pi_S(X - D) \simeq \bigoplus_{J \subset I, x \in D_J^{(0)}} \mathbb{1}_S\big(c_J(x)\big)\big[2c_J(x) - n_J\big]$$

If D is a normal crossing divisor and $m(n)$ is the sum of the number of connected components of all codimension n subschemes D_J of X, then the isomorphism takes the form

$$\Pi_S(X - D) \simeq \bigoplus_{n=0}^{d} m(n).\mathbb{1}_S(n)[n]$$

Proof. According to Corollary 5.1.13, one obtains that $\Pi_S(X - D)$ is the homotopy limit of the (finite) tower

(4.3.2.a) $$\Pi_S(X) \to \oplus_{i\in I}\Pi_S(D_i, N_i) \to \cdots \to \oplus_{J\subset I, \sharp J = n}\Pi_S(D_J, N_J) \to \cdots$$

Suppose x is a generic point on $D_J \neq \varnothing$, for $J \subset I$, and write $D_J(x)$ for the associated connected component. By assumption, $D_J(x)$ is smooth and stably $\mathbf{A}^1$-contractible over k. Thus there is an isomorphism

$$\mathbb{Z} = K_0(k) = \mathrm{Hom}_{\mathrm{SH}(k)}(\mathbb{1}_k, \mathbf{KGL}) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{SH}(k)}(\Sigma^\infty D_{J+}, \mathbf{KGL}) = K_0(D_J)$$

In particular, the rank $c_J(x)$ vector bundle $N_J|_{D_J(x)}$ is stably trivial. Since $D_J(x)/S$ is stably $\mathbf{A}^1$-contractible, it follows that

$$\Pi_S(D_J, N_J) \simeq \oplus_x \Pi_S(D_J(x), N_J|_{D_J(x)}) \simeq \oplus_x \mathbb{1}_S(c_J(x))[2c_J(x)]$$

To deduce the first assertion it suffices to show that the morphisms in (4.3.2.a) are zero. Recall that these maps are sums of Gysin morphisms $\nu_K^{J!}$ for $J, K \subset I$, $K = J \cup \{k\}$, $\nu_K^J : D_K \to D_J$. We are reduced to consider maps of the form

(4.3.2.b) $$\mathbb{1}_S(c_J(x))[2c_J(x)] \to \mathbb{1}_S(c_K(y))[2c_J(y)]$$

Here, x (resp. y) is a generic point of D_J (resp. D_K). Since D_J is nowhere dense in D_K, all such maps belong to some stable cohomotopy group $\pi^{2r,r}(S)$ for $r > 0$. The assumption that S is stably $\mathbf{A}^1$-contractible over k implies $\pi^{2r,r}(S) \simeq \pi^{2r,r}(k)$. Morel's $\mathbf{A}^1$-connectivity theorem shows the latter group is trivial. It follows that the map (4.3.2.b) is zero.

For the second assertion, it suffices to note that if D is a normal crossing divisor, then $c_J = n_J$ for any $J \subset I$. $\qquad\square$

Example 4.3.3. Let S/k be a stably $\mathbf{A}^1$-contractible smooth scheme. Proposition 4.3.2 applies to the basic example when $X \simeq \mathbf{A}_S^d$ and D is a hyperplane arrangement in X.

Owing to Corollary 4.2.5 and Proposition 4.3.2, we deduce the following computation of stable motivic homotopy types at infinity.

Proposition 4.3.4. *Under the assumptions in Proposition 4.3.2 the object $\Pi_S(X - D)$ is rigid (see Section 2) and there are isomorphisms*

$$\Pi_S^c(X - D) \simeq \Pi_S(X - D)^\vee(d)[2d] \simeq \bigoplus_{K | D_K \neq \varnothing} \mathbb{1}_S(d - c_K)[2(d - c_K) + n_K]$$

Moreover, there exists a canonical isomorphism

$$\Pi_S^\infty(X - D) \simeq \bigoplus_{J | D_J \neq \varnothing} \mathbb{1}_S(c_J)[2c_J - n_J] \oplus \bigoplus_{K | D_K \neq \varnothing} \mathbb{1}_S(d - c_K)[2(d - c_K) + n_K - 1]$$

Finally, if D is a normal crossing divisor in X relative to S, then we have

$$\Pi_S^\infty(X - D) \simeq \bigoplus_{n=0}^d m(n).\mathbb{1}_S(n)[n] \oplus \bigoplus_{n=0}^d m(n).\mathbb{1}_S(d - n)[2d - n - 1]$$

5. NORMAL CROSSING BOUNDARIES

5.1. **Homotopy type of crossing singularities.** We start with various notion of singularities extending the classical notion of normal crossing divisor.

Definition 5.1.1. Let X is an S-scheme with irreducible components $X_\bullet = (X_i)_{i \in I}$. For $J \subset I$, let $X_J = \cap_{j \in J} X_j$, where $\cap$ means fiber product over X and $X_J' = (X_J)_{red}$. We say that X has *smooth* (resp. *regular*, *h-smooth*) *crossing* over S if, for any non-empty $J \subset I$, X_J' is a smooth (resp. regular, h-smooth, Definition 4.1.7) S-scheme.

Suppose, in addition, that X is a sub-scheme of a smooth (resp. regular, h-smooth) S-scheme Ω. Then X is called a normal (resp. regular, h-normal) crossing subscheme of Ω if X is a smooth (resp. regular, h-smooth) crossing S-scheme, and for all $J \subset I$ there is the equality

$$\mathrm{codim}_\Omega(X_J) = \sum_{j \in J} \mathrm{codim}_\Omega(X_j)$$

Remark 5.1.2.

(1) When X is a divisor of Ω, i.e., every irreducible component of X has codimension 1 in Ω, we recover the classical notion of a normal crossing divisor relative to the base S.
(2) Working with the reduced scheme X'_J instead of X_J allows to consider multiplicities on the intersections of the irreducible components of X. That is, we don't restrict to simple normal crossing divisors.

5.1.3. Let X be a noetherian scheme and consider a finite closed cover[11] of X, i.e., a surjective map

$$p : \sqcup_{i \in I} X_i \to X$$

obtained from closed immersions $X_i \to X$. For the sake of intuitive notation, let $\cap = \times_X$ be shorthand for the fiber product of closed X-schemes. The Čech simplicial X-scheme $\check{S}_*(X_\bullet/X)$ associated with p takes the form

(5.1.3.a) $$\check{S}_n(X_\bullet/X) = \bigsqcup_{(j_0,\ldots,j_n) \in I^{n+1}} X_{j_0} \cap \ldots \cap X_{j_n}$$

The degeneracy maps $\delta_n^i : \check{S}_n(X_*) \to \check{S}_{n-1}(X_*)$ are given by the appropriate closed immersions. One can simplify $\check{S}_*(X_\bullet/X)$ by setting $X_J = \cap_{j \in J} X_j$ for each subset $J \subset I$. To proceed we consider semi-simplicial schemes.[12] Let us choose a total order on the finite set I, and define the ordered Čech semi-simplicial X-scheme $\check{S}_*^{\mathrm{ord}}(X_\bullet)$ associated to $X_\bullet/X$ by setting

(5.1.3.b) $$\check{S}_n^{\mathrm{ord}}(X_\bullet/X) := \bigsqcup_{J \subset I, \sharp J = n+1} X_J$$

The degeneracy maps in (5.1.3.b) admit the following description. To any subset $J \subset I$ of cardinality $n+1$, we associate the uniquely defined n-tuple $(j_0, \ldots, j_n) \in I^{n+1}$ such that $J = \{j_0, \ldots, j_n\}$ and $j_0 < \ldots < j_n$. This gives a canonical embedding $\check{S}_*^{\mathrm{ord}}(X_\bullet/X) \subset \check{S}_*(X_\bullet/X)$ of $\mathbb{N}$-graded X-schemes. In degree n, the factor X_J maps by the identity to $X_{j_0} \times_X \ldots \times_X X_{j_n}$. The degeneracy maps δ_n^i in the simplicial structure on $\check{S}_*(X_\bullet/X)$ preserve $\check{S}_*^{\mathrm{ord}}(X_\bullet/X)$. This yields the desired semi-simplicial structure on $\check{S}_*^{\mathrm{ord}}(X_\bullet/X)$. Moreover, the map p induces an augmentation $\check{S}_*^{\mathrm{ord}}(X_\bullet/X) \to X$.

Remark 5.1.4. The ordered Čech semi-simplicial scheme is much smaller than $\check{S}_*(X_\bullet/X)$. It is bounded by the cardinality c of the index set I in the sense that $\check{S}_n^{\mathrm{ord}}(X_\bullet/X) = \varnothing$ for all $n > c$ (a sum indexed by the empty set is the empty scheme).

5.1.5. Owing to [BRTV18, Appendix A] the stable homotopy category $\mathrm{SH}(S)$ has an underlying stable ∞-category $\mathscr{SH}(S)$, and the six functor formalism is compatible with the ∞-structure. We shall use a particular case of the said functoriality. Let Sch_S (resp. $\mathrm{Sch}_S^{\mathrm{tf}}$) be the category of

---

[11]In applications, it is the set of irreducible components, but the flexibility with closed covers simplifies the proofs.

[12]Let Δ^{inj} be the category of finite ordered sets with morphisms only the injective maps. A semi-simplicial object in a category $\mathscr{C}$ is a contravariant functor from Δ^{inj} to $\mathscr{C}$. There is a forgetful functor from simplicial objects to semi-simplicial objects that forgets the degeneracy maps.

separable schemes (resp. schemes separated of finite type) over S. To any motivic spectrum $\mathbb{E}$ over S, we associate the contravariant ∞-functor

$$\mathrm{Sch}_S^{op} \xrightarrow{\Pi'_S(-,\mathbb{E})} \mathscr{SH}(S), (f : X \to S) \mapsto f_* f^*(\mathbb{E})$$

This is the ∞-categorical version of the notion of relative cohomology with coefficients in $\mathbb{E}$. Dually, we obtain the covariant ∞-functor

$$\mathrm{Sch}_S^{tf} \xrightarrow{\Pi_S(-,\mathbb{E})} \mathscr{SH}(S), (f : X \to S) \mapsto f_! f^!(\mathbb{E})$$

5.1.6. Let us return to the setup in 5.1.3, with $X = \sqcup X_{i \in I}$. Suppose $p \colon X \to S$ is a morphism of finite type. For any $J \subset I$, we let $p_J \colon X_J \to S$ denote the obvious composition. To the ordered Čech complex $\check{S}_n^{\mathrm{ord}}(X_\bullet/X)$ and $\mathbb{E}$ we associate the functors

$$(\Delta^{\mathrm{inj}})^{op} \to \mathrm{Sch}_S^{tf} \xrightarrow{\Pi_S(-,\mathbb{E})} \mathscr{SH}(S)$$

$$\Delta^{\mathrm{inj}} \to \mathrm{Sch}_S^{op} \xrightarrow{\Pi'_S(-,\mathbb{E})} \mathscr{SH}(S)$$

By using the augmentation map to X, we obtain canonical maps involving the limit and colimit of the preceding functors

$$\pi_{X_\bullet/X,\mathbb{E}} : \mathrm{colim}_{n \in (\Delta^{\mathrm{inj}})^{op}} \left(\oplus_{J \subset I, \sharp J = n+1} \Pi_S(X_J, \mathbb{E})\right) \to \Pi_S(X, \mathbb{E})$$

$$\pi'_{X_\bullet/X,\mathbb{E}} : \Pi'_S(X, \mathbb{E}) \to \lim_{n \in \Delta^{\mathrm{inj}}} \left(\oplus_{J \subset I, \sharp J = n+1} \Pi'_S(X_J, \mathbb{E})\right)$$

We show the colimit (resp. limit) can be viewed as the "standard" resolution of homology (resp. cohomology) of X/S with coefficients in $\mathbb{E}$.

Theorem 5.1.7. *Under the above assumptions, $\pi_{X_\bullet/X,\mathbb{E}}$ and $\pi'_{X_\bullet/X,\mathbb{E}}$ are isomorphisms in* $\mathrm{SH}(S)$.

Proof. First we reduce to the case when X, and similarly each X_i, is reduced. We consider the nil-immersion $\nu : X_{red} \to X$. The localization property of SH implies the adjunction (ν^*, ν_*) is an equivalence. Moreover, ν^* is right adjoint to $\nu_* = \nu_!$, so that $(\nu_!, \nu^!)$ is an equivalence of categories. This implies $\nu_* : \Pi_S(X_{red}, \mathbb{E}) \to \Pi_S(X, \mathbb{E})$ is an isomorphism.

Let us consider $\pi_{X/S,\mathbb{E}}$. For any $J \subset I$, there is an isomorphism $p_{J!} p_J^! \simeq p_! i_{J!} i_J^! p^!$, where $i_J : X_J \to X$ is the canonical closed immersion. By replacing $\mathbb{E}$ with $p^!(\mathbb{E})$, we have reduced to the case $X = S$. There is, see for example [DFKJ20, B.20], a conservative family of functors

$$i_x^! \colon \mathscr{SH}(X) \to \mathscr{SH}\left(\mathrm{Spec}(\kappa(x))\right), x \in X$$

Therefore, it suffices to show $i_x^!\left(\pi_{X_\bullet/X,\mathbb{E}}\right)$ is an isomorphism for all $x \in X$. Given $J \subset I$, we consider the following cartesian square

$$
\begin{array}{ccc}
X_J' & \xrightarrow{i_x'} & X_J \\
{\scriptstyle i_J'}\downarrow & & \downarrow{\scriptstyle i_J} \\
\{x\} & \xrightarrow{i_x} & X
\end{array}
$$

Proper base change for i_J implies there is an isomorphism

$$i_x^! i_{J!} i_J^! \simeq i_{J!}' i_x'^! I_J^!$$

Moreover, we have

$$i_{J!}' i_x'^! I_J^! \simeq i_{J!}' i_J'^! i_x^!$$

For the map in question, we deduce

$$i_x^! \left(\pi_{X_\bullet/X,\mathbb{E}} \right) = \pi_{X_\bullet \times_X \{x\}/\{x\}, i_x^! \mathbb{E}}$$

We may therefore assume $X = \{x\}$ is a point. In this case, since the X_i's are closed reduced subschemes of the reduced scheme $\{x\}$, the closed cover $p : \sqcup_{i \in I} X_i \to X = \{x\}$ is given by a sum of identity maps. To proceed, one can, for example, construct an explicit homotopy contraction of the semi-simplicial augmented pointed X-scheme

$$\check{\mathrm{S}}_*^{\mathrm{ord}}(X_\bullet/\{x\})_+ \to \{x\}_+$$

The proof for the map $\pi'_{X_\bullet/X,\mathbb{E}}$ is entirely analogous. In this case, we employ, see for example [CD19, Proposition 4.3.17], the conservative family of functors

$$i_x^* : \mathscr{SH}(X) \to \mathscr{SH}\left(\mathrm{Spec}(\kappa(x)) \right), x \in X$$

$\square$

Remark 5.1.8. The proof method of Theorem 5.1.7 can be used to show the isomorphisms

(5.1.8.a) $$\mathrm{colim}_{n \in (\Delta^{\mathrm{inj}})^{op}} \left(\oplus_{i_0 \leq \ldots \leq i_n} \Pi_S(X_{i_0,\ldots,i_n}, \mathbb{E}) \right) \simeq \Pi_S(X, \mathbb{E})$$

(5.1.8.b) $$\mathrm{colim}_{n \in (\Delta)^{op}} \left(\oplus_{(i_0,\ldots,i_n) \in I^n} \Pi_S(X_{i_0,\ldots,i_n}, \mathbb{E}) \right) \simeq \Pi_S(X, \mathbb{E})$$

The isomorphism (5.1.8.b) is a manifestation of cdh-descent for the *triangulated motivic category* SH, see [CD19, Proposition 3.3.10]. In (5.1.8.b) we may replace Δ by Δ^{inj} since the inclusion functor $\Delta^{\mathrm{inj}} \subset \Delta$ is final. We leave the analogous dual statements for $\Pi'_S(X, \mathbb{E})$ to the reader.

Remark 5.1.9. Theorem 5.1.7 holds more generally for any triangulated motivic category $\mathscr{T}$ in the sense of [CD19, 2.4.45] which admits an ∞-categorical enhancement as in [Kha16]. As shown in *loc. cit.*, such an enhancement exists if $\mathscr{T}$ is defined via a model Sm-premotivic structure, as in [CD19, §1.3.d]. Main examples include the motivic stable homotopy category with coefficients (see [Ayo07b, §4.5]), the $\mathbf{A}^1$-derived category (see [CD19, 5.3.31]) rational (or Beilinson) motives $\mathrm{DM}_\mathbb{Q}$ (see [CD19, Part IV]), étale motives $\mathrm{DM}_{\mathrm{ét}}$ (see [CD16]), cdh-motives (see [CD15]), and étale torsion and ℓ-adic sheaves (see [LZ12] respectively [CD16, §7.2]).

Example 5.1.10. Suppose X has smooth reduced crossings over S, and let $X_\bullet$ be the closed cover by the irreducible components of X. Set $c = \sharp I$ following Definition 5.1.1. In this case, the isomorphism $\pi_{X_\bullet/X, \mathbb{1}_S}$ from Theorem 5.1.7 amounts to the exact homotopy sequence

(5.1.10.a) $$\Sigma^\infty X'_{I+} \to \ldots \bigoplus_{J \subset I, \sharp J = 2} \Sigma^\infty X'_{J+} \rightrightarrows \bigoplus_{i \in I} \Sigma^\infty X'_{i+} \to \Pi_S(X)$$

of length at most $c + 1$ in $\mathrm{SH}(S)$ (note that X'_I can be empty). For $J \subset K$ there is a naturally induced inclusion $\nu_K^J : X'_K \to X'_J$. The face map

$$\delta_n^k : \bigoplus_{K \subset I, \sharp K = n+1} \Sigma^\infty X'_{K+} \to \bigoplus_{J \subset I, \sharp J = n} \Sigma^\infty X'_{J+}$$

in (5.1.10.a) is defined by the formula

$$\delta_n^k = \sum_{J = \{i_0 < \ldots < \widehat{i_k} < \ldots < i_n\} \subset K = \{i_0 < \ldots < i_n\}} (\nu_K^J)_*$$

We obtain a similar formula when considering a twist by a virtual bundle v over X

(5.1.10.b) $$\Sigma^\infty \mathrm{Th}_S(v_I) \to \ldots \bigoplus_{J \subset I, \sharp J = 2} \Sigma^\infty \mathrm{Th}_S(v_J) \rightrightarrows \bigoplus_{i \in I} \Sigma^\infty \mathrm{Th}_S(v_i) \to \Pi_S(X, v)$$

where v_J is the pullback of v to X'_J for $J \subset I$.

Remark 5.1.11. Continuing with Example 5.1.10, the S-scheme X defines a sheaf of sets $\underline{X}$ on Sm_S. We claim the preceding computation yields an isomorphism $\Pi_S(X) \simeq \Sigma^\infty \underline{X}_+$. A proof uses the $\mathbf{P}^1$-stable $\mathbf{A}^1$-homotopy category $\underline{\mathrm{SH}}_{\mathrm{cdh}}(S)$ over S for the big cdh site[13] in the style of [CD19, §6.1].[14] Theorem 5.1.7 holds in $\underline{\mathrm{SH}}_{\mathrm{cdh}}(S)$ due to cdh-descent, and then the comparison reduces to the smooth case, which holds by the general properties of an enlargement.

Example 5.1.12. With the same assumptions as in Example 5.1.10 and following Remark 5.1.9, the étale or rational motive $M_S(X) = f_! f^!(\mathbb{1}_S)$ of X/S is isomorphic to the complex of sheaves

$$M_S(X_I') \xrightarrow{d_{c-2}} \ldots \xrightarrow{d_1} \bigoplus_{J \subset I, \sharp J = 2} M_S(X_J') \xrightarrow{d_0} \bigoplus_{i \in I} M_S(X_i')$$

The differentials are given by the alternating sums $d_i = \sum_{k=0}^n (-1)^k \delta_n^k$. Equivalently, $M_S(X)$ is the motivic complex associated with the étale sheaf (Nisnevich sheaf for rational motives) represented by X on Sm_S, according to Remark 5.1.9.

This formula is a motivic relative version of the classical computation of the homology of a normal crossing scheme. It actually gives back the known formulas by realization of motives (Betti, étale, etc...). A dual formula, see also Theorem 5.2.1, holds for computing the relative Chow motive $h_S(\underline{X}) = p_* p^*(\mathbb{1}_S)$, $p : X \to S$, by considering the isomorphism $\pi'(X_\bullet/X, \mathbb{1}_S)$ of Theorem 5.1.7: $h_S(\underline{X})$ is quasi-isomorphic to the image of the preceding complex under the (derived) internal Hom functor $\mathbf{R}\underline{\mathrm{Hom}}(-, \mathbb{1}_S)$.

As a consequence of Theorem 5.1.7, we obtain the following generalization of a computation due to Rappoport and Zink (see Remark 5.1.14 for details).

Corollary 5.1.13. *Let $i : Z \to X$ be a closed immersion, $U = X - Z$ and $j : U \to X$ the complementary open immersion. For the decomposition into irreducible components $Z = \cup_{i \in I} Z_i$ we set $c = \sharp I$, $Z_J = \cap_{j \in J} Z_j$, where $\cap$ is the fiber product over X, and $Z_J' = Z_{J,red}$. Assume one of the following conditions holds:*

(1) *$\mathscr{T} = \mathrm{SH}_{\mathbb{Q}}, \mathrm{DM}_{\mathbb{Q}}, \mathrm{DM}_{\acute{e}t}, D_c^b(-_{\acute{e}t}, \mathbb{Z}_\ell), \mathrm{DH}(X)$, X is regular and D is a regular crossing subscheme of X*

(2) *$\mathscr{T}$ is any of the triangulated motivic categories in Section 2, X is h-smooth over a scheme S (see Definition 4.1.7) and Z is an h-smooth crossing subscheme of X/S*

Then the object $i^* j_*(\mathbb{1}_U)$ of $\mathscr{T}(Z)$ is isomorphic to the colimit of the following augmented semi-simplicial object of length at most $c + 1$ in the underlying ∞-category

$$\nu_{I*} \mathrm{Th}(-N_I) \to \ldots \bigoplus_{J \subset I, \sharp J = 2} \nu_{J*}(\mathrm{Th}(-N_J)) \rightrightarrows \bigoplus_{i \in I} \nu_{i*}(\mathrm{Th} -(N_i)) \xrightarrow{\epsilon} \mathbb{1}_Z$$

Here, N_J is the normal bundle of the regular closed immersion $Z_J' \to X$, and the face maps are defined by the Gysin maps

$$\delta_n^k = \sum_{J = \{i_0 < \ldots < \widehat{i_k} < \ldots < i_n\} \subset K = \{i_0 < \ldots < i_n\}} (\nu_K^J)_!$$

Here, $(\nu_K^J)_! : \nu_{K*}(\mathrm{Th}(N_K)) \to \nu_{J*}(\mathrm{Th}(N_J))$ is the Gysin map associated with the closed immersion ν_J^K relative to X.[15] Similarly, the Gysin maps $\nu_{i!} : \nu_{i*}(\mathrm{Th}(N_i)) \to \mathbb{1}_Z$ for all $\nu_i : Z_i \to Z$, $i \in I$, define

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[13]The site of finite type S-scheme endowed with the cdh-topology.

[14]In the terminology of *loc. cit.*, one gets an *enlargement* of motivic triangulated categories: see Definition 1.4.13.

[15]In fact, one can interpret $\nu_{J*}(\mathrm{Th}(N_J))$ as the twisted cohomology $\Pi_X'(Z_J, \langle N_J \rangle)$, so that $(\nu_J^K)_!$ is the Gysin map associated with the proper morphism of X-schemes ν_J^K.

the augmentation map

$$\epsilon = \sum_{i\in I} \nu_{i!}$$

Dually, in the underlying ∞-category, the object $i^! j_!(\mathbb{1}_U)$ in $\mathscr{T}(U)$ is isomorphic to the limit of the following augmented semi-cosimplicial object (of length at most $c+1$)

$$\mathbb{1}_Z \to \bigoplus_{i\in I} \nu_{i!}(\mathrm{Th}(N_i)) \rightrightarrows \bigoplus_{J\subset I, \sharp J=2} \nu_{J!}(\mathrm{Th}(N_J))\ldots \to \nu_{I!}\,\mathrm{Th}(N_I)$$

Proof. We note that $Z'_J \to X$ is a regular closed immersion so that the normal bundle N_J exists. The distinguished localization triangle associated with the closed immersion i corresponds to an exact homotopy sequence in the underlying ∞-category

$$i_! i^!(\mathbb{1}_S) \to \mathbb{1}_S \to j_* j^*(\mathbb{1}_S) = j_*(\mathbb{1}_U)$$

Applying i^* and Theorem 5.1.7 to the left-hand term, we deduce the exact homotopy sequence

$$\mathrm{colim}_{n\in(\Delta^{\mathrm{inj}})^{op}} \left(\nu_{J*} i_J^!(\mathbb{1}_S)\right) \to \mathbb{1}_Z \to i^* j_*(\mathbb{1}_U)$$

To conclude, we use the purity isomorphism $i_J^!(\mathbb{1}_S) \simeq \mathrm{Th}(-N_J)$, which is absolute purity in case of assumption (1) and the fact i_J is h-smooth in case of assumption (2) (apply 4.1.8).

The result for $i^! j_!(\mathbb{1}_U)$ follows by applying the same argument to the dual exact homotopy sequence

$$j_! j^!(\mathbb{1}_S) \to \mathbb{1}_S \to i_* i^*(\mathbb{1}_S)$$

$\square$

Remark 5.1.14. In the situation of Corollary 5.1.13 we consider the cases, (1) $\mathscr{T} = \mathrm{DM}_{\mathbb{Q}}, \mathrm{DM}_{\text{ét}}$, (2) $\mathscr{T} = \mathrm{DM}$, and assume $Z = D$ is a normal crossing divisor on X. Then, in the underlying motivic ∞-category, the motive $i^* j_*(\mathbb{1}_U)$ is the colimit of the complex

(5.1.14.a) $$\nu_{I*}(\mathbb{1})\langle c\rangle \xrightarrow{d_{c-2}} \ldots \bigoplus_{J\subset I, \sharp J=2} \nu_{J*}(\mathbb{1})\langle 2\rangle \xrightarrow{d_0} \bigoplus_{i\in I} \nu_{i*}(\mathbb{1})\langle 1\rangle \xrightarrow{\epsilon} \mathbb{1}_D$$

Here, $d_n = \sum_k (-1)^k \delta_n^k$ is an alternate sum of Gysin maps associated with the relevant closed immersions. The computation for (5.1.14.a) specializes under ℓ-adic realization to the Rapoport-Zink formula for vanishing cycles [RZ82, Lemma 2.5]. A similar remark applies to Steenbrink's limit Hodge structure [Ste76], except that our computation for motives does not account for the action of the monodromy operator.

Corollary 5.1.15. *Consider the assumptions of Corollary 5.1.13, and assume in addition (in case (1)) that X be an S-scheme with projection map $\bar{p} : X \to S$.*

Then $\Pi_S(U)$ is isomorphic to the limit of the augmented semi-simplicial diagram

(5.1.15.a) $$\Pi_S(X) \xrightarrow{\epsilon} \bigoplus_{i\in I} \Pi_S(Z_i, \langle N_i\rangle) \rightrightarrows \bigoplus_{J\subset I, \sharp J=2} \Pi_S(Z_J, \langle N_J\rangle)\ldots \to \Pi_S(Z_I, \langle N_I\rangle)$$

given by the maps

$$\epsilon = \sum_{i\in I} \nu_i^!$$

$$\delta_k^n = \sum_{K=\{i_0<\ldots<i_n\}, J=\{i_0<\ldots<\cancel{i_k}<\ldots<i_n\}} (\nu_K^J)^!$$

Similarly, the motive $M_S(U)$ is the limit of the augmented semi-simplicial diagram

$$(5.1.15.b) \qquad M_S(X) \xrightarrow{\epsilon} \bigoplus_{i \in I} M_S(Z_i)\langle 1 \rangle \rightrightarrows \bigoplus_{J \subset I, \sharp J = 2} M_S(Z_J)\langle 2 \rangle \ldots \to M_S(Z_I)\langle c \rangle$$

with the same formulas as in (5.1.15.a) *for the augmentation ϵ and the coface maps δ_k^n.*

Proof. Similarly to Corollary 5.1.13, inserting $p^!(\mathbb{1}_S)$ in the localization distinguished triangle for the closed immersion i and applying $\bar{p}_!$ yields the exact homotopy sequence

$$M_S(U) = \bar{p}_! j_! j^! \bar{p}^!(\mathbb{1}_S) \to M_S(X) = \bar{p}_! \bar{p}^!(\mathbb{1}_S) \to \bar{p}_! i_* i^* \bar{p}^!(\mathbb{1}_S)$$

Theorem 5.1.7 identifies the right-hand term with

$$\lim_{n \in \Delta^{\mathrm{inj}}} \sum_{J \subset I, \sharp J = n+1} \bar{p}_! i_{J!} i_J^* \bar{p}^!(\mathbb{1}_S)$$

Under the assumptions (1), (2) we have $\bar{p}^!(\mathbb{1}_S) \simeq \mathrm{Th}(T_{X/S})$. Owing to purity, absolute or relative, for the regular closed immersion i_J we deduce the isomorphism

$$\bar{p}_! i_{J!} i_J^* \bar{p}^!(\mathbb{1}_S) \simeq \bar{p}_! i_{J!}\big(\mathrm{Th}(N_J) \otimes i_J^! \bar{p}^!(\mathbb{1}_S) \big) = \Pi_S(Z_J, \langle N_J \rangle)$$

The computation for $M_S(U)$ follows because motives are oriented. $\qquad \square$

Remark 5.1.16. Suppose X/S is smooth and proper. The formula for $M_S(X - D)$, the motive of the complement of a normal crossing divisor D of X/S, is a relative motivic analog of the de Rham complex with logarithmic poles that Deligne used to define mixed Hodge structures. The motive of the relative non-compact family $X - D$ over S is expressed as the "complex" (5.1.15.b) whose terms are pure of weight 0 for Bondarko's motivic weight structure. In particular, it gives a canonical and functorial weight filtration for the motive $M_S(X - D)$ (recall that a pure object of weight 0 shifted n times has weight n). This is a motivic analog of the fact that the weight filtration of the mixed Hodge structure on $X - D$, at least when $S = \mathrm{Spec}(\mathbb{C})$, is obtained by the naive filtration of the de Rham complex with logarithmic poles associated with (X, D).

Dually, we can identify the Chow motive $h_S(X) = p_*(\mathbb{1}_S)$ with the colimit of the diagram

$$(5.1.16.a) \qquad h_S(D_I)\langle -c \rangle \to \ldots \bigoplus_{J \subset I, \sharp J = 2} h_S(D_J)\langle -2 \rangle \rightrightarrows \bigoplus_{i \in I} h_S(D_i)\langle -1 \rangle \xrightarrow{\epsilon} h_S(X)$$

When $S = \mathrm{Spec}(\mathbb{C})$, the de Rham realization of (5.1.16.a), see [CD12, §3.1], can be canonically identified with the de Rham complex with logarithmic poles associated with (X, D).[16]

5.2. **Strong duality.**

We deduce some applications of the computations in the previous section towards strong duality results.

Theorem 5.2.1. *(1) Let X be a proper scheme over S with smooth reduced crossings, and v be a virtual bundle over X. Then $\Pi_S(X, v)$ is rigid with dual $\mathrm{H}_S(X, -v)$.*

(2) Let X/S be scheme which admits a smooth proper compactification $\bar{X}$ whose complement ∂X is a smooth reduced crossing S-scheme. Then $\Pi_S(X, v)$ is rigid with dual $\Pi_S^c(X, -v - T_X)$, where T_X is the tangent bundle of X/S.

Proof. For point (1): according to Equation (5.1.10.b), $\Pi_S(X, v)$ is a colimit of a finite diagram whose components are rigid spectra (as they are given by spectra of smooth proper schemes). This implies $\Pi_S(X, v)$ is rigid. The fact that its dual is $\mathrm{H}_S(X, -v)$ follows from Lemma 4.2.1(1). For point (2), one applies Corollary 5.1.15 and the first assertion of Theorem 4.2.3. $\qquad \square$

---

[16]As explained in [D́18, Example 5.4.2(1)], this follows from the identification of the orientation of the motivic spectrum representing algebraic de Rham cohomology.

Remark 5.2.2. In case (1), the rigid spectrum $\Pi_S(X, v)$ is isomorphic to the homotopy colimit of the diagram

$$\Pi_S(X'_I, v_I) \longrightarrow \ldots \Longrightarrow \bigoplus_{J \subset I, \sharp J=2} \Pi_S(X'_J, v_J) \Longrightarrow \bigoplus_{i \in I} \Pi_S(X'_i, v_i)$$

Moreover, its dual $H_S(X, -v)$ is isomorphic to the homotopy limit of the diagram

$$\bigoplus_{i \in I} \Pi_S(X'_i, -v_i - T_i) \Longrightarrow \bigoplus_{J \subset I, \sharp J=2} \Pi_S(X'_J, -v_J - T_J) \Longrightarrow \ldots \longrightarrow \Pi_S(X'_I, -v_I - T_I)$$

Here, for $J \subset I$, T_J (resp. v_J) is the tangent bundle of (resp. pullback of v over) X'_J/S. A similar computation is valid in case (2) using Corollary 5.1.15.

Example 5.2.3. Combining (ii) with the theorem of embedded resolution of singularities shows that for any smooth separated scheme X over a field k of characteristic 0, the spectrum $\Pi_k(X, v)$ is rigid with dual $\Pi_k^c(X, -v - T_X)$. So the above theorem vastly generalizes the case of geometric motives from [VSF00, Chapter 5, 4.3.7].

Remark 5.2.4. When S is of positive dimension, one cannot expect all constructible spectra to be rigid (compare with [NSØ09, Rio05]). Properness or smoothness does not ensure rigidity. For example, let $U \subset S$ be an open subscheme with non-empty complement Z. Assume $i : Z \to S$ is h-smooth (for example Z and S are smooth over a field k). We claim that $\Pi_S(U) = j_!(\mathbb{1}_U)$ is not rigid. Assuming the contrary, its dual would be isomorphic to $j_*(\mathbb{1}_U)$ according to Lemma 4.2.1. Since i^* is monoidal, it would follow that $i^* j_!(\mathbb{1}_U)$ is rigid with dual $i^* j_*(\mathbb{1}_U)$. The first spectrum is trivial, whereas purity identifies the second one with $\mathbb{1}_Z \oplus \text{Th}(N_Z S)[1]$ which is nontrivial. An identical (dual) argument shows that $\Pi_S(Z)$ is not rigid.

Next we show an improvement of Theorem 4.1.12.

Theorem 5.2.5. *Consider a separated S-scheme X which admits a smooth compactification $\bar{p} : \bar{X} \to S$ with tangent bundle $\bar{T}$ and boundary $\nu : \partial X \to \bar{X}$ a normal crossing S-scheme.*
    Then there exists a canonical isomorphism

$$\Pi_S(\bar{X}/X) \simeq \Pi_S(\partial X, -\nu^* \bar{T})^\vee$$

Here, $\Pi_S(\partial X, -\nu^* \bar{T})^\vee$ is the dual of the rigid motive $\Pi_S(\partial X, -\nu^* \bar{T})$ (according to Theorem 5.2.1).
    Moreover, the map β' obtained by adjunction from

$$\beta : \Pi_S(\partial X) \xrightarrow{\nu_*} \Pi_S(\bar{X}) \to \Pi_S(\bar{X}/X) \simeq \Pi_S(\partial X, -\nu^* \bar{T})^\vee$$

fits into the commutative diagram

$$
\begin{array}{ccc}
\Pi_S(\partial X) \otimes \Pi_S(\partial X, -\nu^* \bar{T}) & \xrightarrow{\hspace{3cm} \beta' \hspace{3cm}} & \mathbb{1}_S \\
{\scriptstyle Id \otimes \nu_*} \downarrow & & \uparrow {\scriptstyle q_*} \\
\Pi_S(\partial X) \otimes \Pi_S(\bar{X}, -\bar{T}) \xrightarrow[\simeq]{(*)} \Pi_S(\partial X \times_S \bar{X}, -p_2^* \bar{T}) & \xrightarrow{\gamma_\nu^!} & \Pi_S(\partial X)
\end{array}
$$

where γ_ν is the graph of the closed immersion $\nu : \partial X \to \bar{X}$.

Proof. The first assertion is a combination of Theorems 4.1.12, 5.2.1. For the second assertion, let us note that γ_ν is a section of the smooth separated morphism $Id \times_S \bar{p} : \partial X \times_X \bar{X} \to \partial X$. So it is a regular closed immersion whose normal bundle is isomorphic to the tangent bundle of $Id \times_S \bar{p}$, that is $p_2^* \bar{T}$. This justifies the existence of the Gysin map $\gamma_\nu^!$ in 4.1.6. Secondly, the isomorphism $(*)$ follows from the computation of $\pi_S(\partial X)$ performed in Example 5.1.10 and the Künneth isomorphism. A routine check using the definitions of the maps show that the diagram commutes. □

5.2.6. Theorem 5.2.5 gives a new tool for computing stable motivic homotopy types at infinity. According to Proposition 3.2.2, $\Pi_S^\infty(X)$ is isomorphic to the homotopy fiber of the map

$$\beta : \Pi_S(\partial X) \to \Pi_S(\bar{X}/X)$$

Using the isomorphism in the above theorem, the commutative diagram tells us that β can be computed as the class

$$(\mathrm{Id} \times_S \nu)^*([\Gamma_\nu]) \in \Pi^{2d,d}(\partial X \times_S \partial X, p_2^* \bar{T})$$

Here d is the dimension of $\bar{X}$, and $[\Gamma_\nu]$ is the fundamental class of the graph of ν in

$$\Pi^{2d,d}(\partial X \times_S \bar{X}, p_2^* \bar{T})$$

via the canonical isomorphism

$$[\Pi_S(\partial X), \Pi_S(\bar{X}/X)] \simeq [\Pi_S(\partial X), \Pi_S(\partial X, -\nu^*\bar{T})^\vee]$$
$$\simeq [\Pi_S(\partial X) \otimes \Pi_S(\partial X, -\nu^*\bar{T}), \mathbb{1}_S] = \Pi^{2d,d}(\partial X \times_S \partial X, p_2^*\bar{T})$$

The latter isomorphism is a consequence of the Künneth formula

$$\Pi_S(\partial X) \otimes \Pi_S(\partial X, -\nu^*\bar{T}) \simeq \Pi_S(\partial X \times \partial X, -p_2^*\bar{T})$$

which follows from the smooth case and Example 5.1.10.

5.3. **Application to stable motivic homotopy types at infinity.**

5.3.1. We are ready for our main computation, following Theorem 5.1.7 and Proposition 3.2.2. Let X/S be separable with compactification $(\bar{X}, \partial X)$ over S. In addition, we assume that $\bar{X}$ is proper and smooth over S and ∂X is a normal crossing divisor on $\bar{X}$, see Definition 5.1.1. We write $\partial X = \cup_{i \in I} \partial_i X$ for the decomposition of ∂X into its irreducible components, and we consider $\partial_i X$ as a reduced subscheme of $\bar{X}$. There is a canonical closed immersion $\nu_i : \partial_i X \to \bar{X}$. For a subset $J \subset I$, we equip $\partial_J X = \cap_{j \in J} \partial_j X$ with its reduced subscheme structure. There is a canonical closed immersion $\nu_K^J : \partial_K X \to \partial_J X$ for subsets $J \subset K \subset I$.

Theorem 5.3.2. *Consider the above assumptions and notations. The stable homotopy type at infinity $\Pi_S^\infty(X)$ of X/S is canonically isomorphic to the homotopy fiber of the map*

$$\mathrm{colim}_{n \in (\Delta^{\mathrm{inj}})^{op}} \left(\bigoplus_{J \subset I, \sharp J = n+1} \Pi_S(\partial_J X) \right) \xrightarrow{\mu} \lim_{n \in \Delta^{\mathrm{inj}}} \left(\bigoplus_{J \subset I, \sharp J = m+1} \Pi_S(\partial_J X, \langle N_J \rangle) \right)$$

Here, the direct images define the face maps in the source

$$\delta_n^k = \sum_{K = \{i_0 < ... < i_n\}, J = \{i_0 < ... < i_k < ... < i_n\}} (\nu_K^J)_*$$

and the Gysin maps define the coface maps in the target

$$\tilde{\delta}_l^m = \sum_{K = \{i_0 < ... < i_m\}, J = \{i_0 < ... < i_l < ... < i_m\}} (\nu_K^J)^!$$

Moreover, μ is induced by the canonical map in degree zero

$$(\nu_j^! \nu_{i*})_{i,j \in I} : \bigoplus_{i \in I} \Pi_S(\partial_i X) \longrightarrow \bigoplus_{j \in I} \Pi_S(\partial_j X, \langle N_j \rangle)$$

Proof. According to Proposition 3.2.2, the spectrum $\Pi_S^\infty(X)$ is the homotopy fiber of the map $\beta_X : \Pi_S(\partial X) \to \Pi_S(\bar{X}, X)$. Theorem 5.1.7 (see Example 5.1.10) identifies μ's source with the desired colimit. An application of Theorems 5.2.5, 5.1.7, (see also Remark 5.2.2) identifies μ's target with the desired spectrum on account of the identity

$$\langle \bar{T} \rangle - \langle T_{\partial_i X} \rangle = \langle N_i \rangle$$

between (virtual) vector bundles over $\partial_i X$. The computation of the (co)face maps and μ follows from the definition of the Gysin maps. $\qquad\square$

One can suggestively summarize the computation in Theorem 5.3.2 with the diagram

$$\left(\begin{array}{c} \vdots \\ \bigoplus_{i_1 < i_2} \Pi_S(\partial_{i_1 i_2} X) \\ \downarrow\downarrow \\ \bigoplus_{i \in I} \Pi_S(\partial_i X) \xrightarrow{\ \ \mu\ \ } \bigoplus_{j \in I} \Pi_S(\partial_j X, \langle N_j \rangle) \\ \downarrow\downarrow \\ \bigoplus_{j_1 < j_2} \Pi_S(\partial_{j_1 j_2} X, \langle N_{j_1 j_2} \rangle) \\ \vdots \end{array} \right)$$

6. A MOTIVIC ANALOG OF MUMFORD'S PLUMBING GAME

6.0.1. In this section, we consider (relative) compactified smooth surfaces whose boundary is a union of (relative) rational curves. Inspired by the technique of "plumbing" disk bundles over manifolds, this case was first studied by Mumford [Mum61] in the context of normal surface singularities, e.g., rational double points.

We work in the setting of 5.3.1, where S is a base scheme, X/S is a smooth family of surfaces with a smooth proper compactification $\bar{X}/S$, whose boundary $\partial X = \cup_{i \in I} \partial_i X$ is a normal crossing divisor on $\bar{X}$ (see Definition 5.1.1). We also assume that each branch $\partial_i X$, with its reduced schematic structure, is a rational curve over S. For each $i \in I$, we fix an S-isomorphism $\epsilon_i : \partial_i X \to \mathbf{P}_S^1$. The choice of isomorphism determines 3 distinct S-points on $\partial_i X$, say $0, 1, \infty$. For $i < j$ in I, we let $\partial_{ij} X$ be the reduction of $\partial_i X \times_S \partial_j X$, which is a finite étale S-scheme. We consider the immersion $\nu_{ij}^k : \partial_{ij} X \to \partial_k X$, for $k = i, j$. Up to changing the isomorphism ϵ_i we may assume:

(6.0.1.a)   the image of $\epsilon_i \circ \nu_{ij}^k$ does not meet the point at infinity

6.1. **Quadratic intersection degrees.**

6.1.1. Recall from [DJK18] that to a regularly embedded closed subscheme $i : Z \to X$ one can associate a fundamental class $\eta_X(Z)$; it takes value in cohomotopy with support[17]

$$\eta_X(Z) \in \Pi_Z^0(X, \langle N_Z X \rangle) := [i_*(\mathrm{Th}(-N_Z X)), \mathbb{1}_X] (\simeq H_0(Z/X, -N_Z X))$$

If X and Z are smooth over the base scheme S, the said class comes via Morel-Voevodsky's homotopy purity theorem

$$\mathfrak{p}_{Z/X} : \Pi^0(Z) \simeq \Pi_Z^0(X, \langle N_Z X \rangle), 1 \mapsto \eta_X(Z)$$

---

[17]This is the map (4.1.6.a) for $f = i$, evaluated at $\mathbb{1}_X$.

Definition 6.1.2. Let S be a base scheme, and X a smooth (proper) S-scheme. Let $D \subset X$ be an effective Cartier divisor (*i.e.*, in our case a closed subscheme of X of pure codimension 1) and C/S a smooth relative curve with an embedding $i : C \to X$. We set $\bar{Z} = D \times_X C$, $Z = \bar{Z}_{red}$, and assume the following holds:

   (1) The intersection of D and C in X is proper: $\bar{Z}$ is finite over S
   (2) The projection map $f : Z \to S$ is étale[18]

The (unoriented) quadratic intersection degree of D and C in X is the class

$$(D, C)_{quad} = \widetilde{\deg} \left(i^*(\eta_X(D)) \right) \in \Pi^0(S)$$

Here, $i^* : \Pi^0_D(X, \langle N_D X \rangle) \to \Pi^0_Z(C, \langle N_Z C \rangle)$ is the pullback map[19], $f_* : \Pi^0(Z) \to \Pi^0(S)$ is the Gysin map associated with the finite étale morphism f, see 4.1.6, and the (quadratic) degree map is the composite

$$\widetilde{\deg} := f_* \circ \mathfrak{p}_{C/Z}^{-1}$$

In this definition's generality, the quadratic intersection degree can be interpreted as a family of quadratic forms. Indeed, by a pullback, one can specialize $(D, C)_{quad} \in \Pi^0(S)$ to any point $x \in S$ of the base. This yields an element of $\Pi^0(x) \simeq \mathrm{GW}(\kappa(x))$, the Grothendieck-Witt ring of the field $\kappa(x)$ according to [Mor12].[20]

For certain base schemes, we can give a more concrete formula for the quadratic intersection degree. Let us first note the following consequence of Morel's proof of the Gersten resolution for homotopy sheaves (and modules, see [Mor12]).

Lemma 6.1.3. *If $S = \mathrm{Spec}(\mathcal{O})$ is an essentially smooth semi-local k-scheme, then there is a canonical isomorphism[21]*

$$\mathrm{GW}(\mathcal{O}) = \Gamma(S, \underline{\mathbf{K}}^{MW}) \xrightarrow{\simeq} \Pi^{0,0}(S)$$

The left-hand-side is the Grothendieck-Witt ring of $\mathcal{O}$.

Proposition 6.1.4. *Let $S = \mathrm{Spec}(\mathcal{O})$ be an essentially smooth semi-local scheme over a field k. Under the assumptions in Definition 6.1.1, let x be a generic point of Z. We write $Z_x = \mathrm{Spec}(B_x)$ for the connected component of Z containing x and moreover:*

   - *$m_x = \mathrm{lg}(\mathcal{O}_{\bar{Z},x})$ for the geometric multiplicity of $\bar{Z}$ at x, which is the intersection multiplicity of Z and C at x*
   - *τ_x for the class in the Grothendieck-Witt ring $\mathrm{GW}(\mathcal{O})$ of the non-degenerate bilinear form*

$$\varphi_x : B_x \otimes_\mathcal{O} B_x \to \mathcal{O}, x \otimes y \mapsto \mathrm{Tr}_{B_x/\mathcal{O}}(xy)$$

Here, $\mathrm{Tr}_{B_x/\mathcal{O}}$ is the trace form of the finite flat $\mathcal{O}$-algebra B_x[22]

Then the quadratic intersection degree of D and C in X equals

$$(D, C)_{quad} = \sum_{x \in Z^{(0)}} (m_x)_\epsilon \tau_x \in \Pi^{0,0}(S) \simeq \mathrm{GW}(\mathcal{O})$$

Here, for an integer $n > 0$, we set $n_\epsilon = \sum_{i=1}^n \langle (-1)^{i+1} \rangle$.

---

[18]This is automatic if S is the spectrum of a perfect field.

[19]The proper intersection assumption gives a canonical isomorphism: $N_Z C \simeq i^*(N_D X)$.

[20]When bad reduction can occur, specializing at a point does not preserve the normal crossing divisor. Nevertheless, for good reduction, one can show that our definition of quadratic intersection degree specializes correctly.

[21]This is generalized to $\mathbb{Z}[1/2]$ and other number rings in [BØ21].

[22]Recall the trace form φ is non-degenerate if and only if B_x is étale over $\mathcal{O}$.

Proof. We note that [Fel21, Theorem 2.2.2] implies the formula

$$i^*\eta_X(D) = \sum_{x \in Z^{(0)}} (m_x)_\epsilon \eta_X(Z_x)$$

Indeed, by additivity and localisation around x, one reduces to the case where Z has a single connected component. The conditions of *loc. cit.* are fulfilled with $e = m_x$, according to our assumptions in Definition 6.1.1. Let $\nu^{(m_x)} : N_Z C \to i^* N_D X$ be the map given by sending a local parameter to its m_x-th power. The induced map $\nu_*^{(m_x)} : \mathrm{Th}(N_Z C) \to \mathrm{Th}(i^* N_D X)$ corresponds to the quadratic form e_ϵ under the isomorphism

$$[\mathrm{Th}(N_Z C), \mathrm{Th}(i^* N_D X)] \simeq [\mathbb{1}_S, \mathrm{Th}(i^* N_D X - N_Z C)] \simeq [\mathbb{1}_S, \mathbb{1}_S] \simeq \mathrm{GW}(\mathcal{O})$$

Since $\widetilde{\deg}$ commutes with multiplication by $(m_x)_\epsilon$, it suffices to check that $\widetilde{\deg}(\eta_C(Z_x)) = \tau_x$. As recalled in 6.1.1, $\mathfrak{p}_{Z_x/C}(\eta_C(Z_x)) = 1$; thus we are reduced to showing $f_*(1) = \tau_x$, which follows using the computation of transfer maps in [Fel20]. $\qquad\square$

Remark 6.1.5. In particular, we have enhanced the classical notion of intersection degree to a quadratic version; one recovers the former from the latter by taking ranks of non-degenerate symmetric bilinear forms.

6.1.6. A merit of Definition 6.1.1 is that it dispenses with orientation conditions. To get a more general definition, we will need such conditions. Suppose S is defined over a field k. We write $\mathbb{H}_{MW}\mathbb{Z}_S$ for ring spectrum representing Milnor-Witt cohomology over S, see [BCD$^+$20]. Recall that a virtual bundle v over a scheme X is orientable if the line bundle $\det(v)$ is divisible by 2 in $\mathrm{Pic}(X)$. An orientation of v is an isomorphism $\tau : \det(v) \to L^{\otimes,2}$ for some line bundle L/X. Further, if $p : X \to S$ is an lci morphism with virtual tangent bundle τ_p, we say that p is orientable if τ_p is orientable; an orientation of p is an isomorphism $\tau : \omega_p = \det(\tau_p) \to L^{\otimes,2}$.

When p is projective of relative dimension d, one can define the Milnor-Witt degree as the composite

$$\widetilde{\deg}^\tau_{X/S} : \mathbb{H}^{2d,d}_{MW}(X) \xrightarrow{\tau_*} \mathbb{H}^{0,0}_{MW}(X, \langle L_f \rangle) \xrightarrow{p_*} \mathbb{H}^{0,0}_{MW}(S)$$

Here, the map τ_* is the isomorphism induced by τ, since $\mathbb{H}_{MW}\mathbb{Z}$ is $\mathbf{SL}^c$-orientable.

If now $p = i : Z \to X$ is a regular closed immersion of codimension n, an orientation θ of i is an orientation of the normal bundle $N_Z X$. In that case, one defines the oriented cycle class $[Z]^\Theta_X$ of Z in X as the image of $\eta_X(Z)$ under the composite[23]

$$\Pi^0_Z(X, \{N_Z X\}) \to \mathbb{H}^0_{MW,Z}(X, \langle N_Z X \rangle) \xrightarrow{\theta_*} \mathbb{H}^0_{MW,Z}(X, \langle n \rangle) \to \mathbb{H}^{2n,n}_{MW}(X)$$

Here, the first map is induced by the unit of the ring spectrum $\mathbb{H}_{MW}\mathbb{Z}_S$ and the second map forgets the support.

Definition 6.1.7. Suppose S is a k-scheme and X a finite type S-scheme. We consider a regular closed immersion $i : Z \to X$ of codimension n such that $N_Z X$ is orientable, and a morphism $f : T \to X$ such that the projection $p : T \to S$ is lci projective, orientable and of relative dimension n. Let θ (resp. τ) be an orientation of i (resp. p).

The oriented quadratic intersection degree of Z along f over S is the class

$$(Z, T)^{\theta,\tau}_{quad} = \widetilde{\deg}_{T/S}(f^*[Z]^\Theta_X) \in \mathbb{H}^{0,0}_{MW}(S).$$

---

[23]Equivalently, when X is smooth it is the image of 1 under the composite map

$$\widetilde{\mathrm{CH}}^0(Z) \xrightarrow{\Theta_*} \widetilde{\mathrm{CH}}^0(Z, \det(N_Z X)) \xrightarrow{i_*} \widetilde{\mathrm{CH}}^0(X)$$

Though all the previous notions depends on the chosen orientations, we will sometimes simply write $[Z]_X^{or}$, $\widetilde{\deg}_{X/S}^{or}$, $(Z, T)_{quad}^{or}$, the chosen orientation being intended.

In the case where $S = \mathrm{Spec}(\mathcal{O})$ is semi-local essentially smooth over k, as explained in Lemma 6.1.3, the class $(Z, T)_{quad}^{or}$ leaves in $\mathrm{GW}(\mathcal{O})$.

Example 6.1.8. We consider the notations of the previous definition.

(1) We assume that $f : X \to S$ is smooth projective and oriented, of dimension $2n$, Z and T are closed regular subschemes of X of codimension n, whose normal bundles are oriented. In this case, the oriented cycle class defined in 6.1.6 are elements $[Z]_X^{or}, [T]_X^{or}$ in the Chow-Witt group $\widetilde{\mathrm{CH}}^n(X)$. Then the projection formula gives the following more symmetric formula for the oriented quadratic intersection degree:

$$(Z, T)_{quad}^{or} = \widetilde{\deg}_{X/S}^{or}([Z]_X^{or}.[T]_X^{or})$$

using the intersection product of $\widetilde{\mathrm{CH}}^n(X)$ (hint: use the projection formula).

(2) Assume that $Z = T$. Then according to the excess intersection formula [DJK18, Proposition 3.3.4], we get:

$$(Z, Z)_{quad}^{or} = \widetilde{\deg} \, Z/S\big(e^{or}(N_Z X)\big)$$

where $e^{or}(N_Z X) \in \mathbb{H}_{MW}^{2n,n}(Z)$ is the Euler class of the oriented vector bundle $N_Z X$.

6.2. **A universal homotopical formula.**

6.2.1. We follow the setting of 6.0.1. Theorem 5.3.2 shows $\Pi_S(\partial X)$ is the homotopy cofiber of the canonical map

$$d = \sum_{i<j} \nu_{ij*}^i - \nu_{ij*}^j : \bigoplus_{i<j} \Pi_S(\partial_{ij} X) \to \bigoplus_{i \in I} \Pi_S(\partial_i X)$$

Moreover, we consider the isomorphism

(6.2.1.a) $$\epsilon = \sum_{i \in I} \epsilon_{i*} : \bigoplus_{i \in I} \Pi_S(\partial_i X) \to \bigoplus_{i \in I} \mathbb{1}_S \oplus \bigoplus_{i \in I} \mathbb{1}_S(1)[2]$$

The following lemma is immediate.

Lemma 6.2.2. *The composite $\epsilon \circ d$ factors through the inclusion of the direct summand $\bigoplus_{i \in I} \mathbb{1}_S$ on the right-hand-side of* (6.2.1.a). *Moreover, the factorization coincides with the map*

(6.2.2.a) $$p = \sum_{i<j} p_{ij*}^i - p_{ij*}^j : \bigoplus_{i<j} \Pi_S(\partial_{ij} X) \to \bigoplus_{i \in I} \mathbb{1}_S$$

Here, $p_{ij} : \partial_{ij} X \to S$ is the canonical projection (a finite étale map), and for $k = i, j$, we have written p_{ij*}^k for the projection on the k-th factor of the right hand-side. In particular, if $\mathcal{D}_X$ is the homotopy cofiber of p, there is a canonical isomorphism

$$\Pi_S(\partial X) \simeq \mathcal{D}_X \oplus \bigoplus_{i \in I} \mathbb{1}_S(1)[2]$$

One can interpret the decomposition by saying that $\Pi_S(\partial X)$ is a sum of the combinatorial part $\mathcal{D}_X$, which is a smooth Artin motivic spectrum[24] and the "geometric" part $\oplus_{i \in I} \mathbb{1}_S(1)[2]$.

---

[24]By analogy with the case of motives, it is the smallest ∞-category containing $\Pi_S(V)$ for V/S finite étale, and stable under suspensions, homotopy (co)fibers.

6.2.3. Owing to Theorem 5.3.2, we can identify $\Pi_S(\bar{X}/X)$ with the homotopy fiber of the map

$$d_2 = \sum_{i<j} \nu_{ij}^{i!} - \nu_{ij}^{j!} : \bigoplus_{j \in I} \Pi_S(\partial_j X, \langle N_j \rangle) \to \bigoplus_{j<k} \Pi_S(\partial_{jk} X, \langle N_{jk} \rangle)$$

Here, N_j (resp. N_{jk}) denotes the normal bundle of $\partial_j X$ (resp. $\partial_{jk} X$) in $\bar{X}$. The isomorphism $\partial_j X \simeq \mathbf{P}_S^1$ reduces to considering the homotopy type of $\mathbf{P}_S^1$ twisted by the line bundle on $\mathbf{P}_S^1$ corresponding to N_j (which is determined by its degree).

Proposition 6.2.4. *Let w be a virtual vector bundle on $\mathbf{P}_S^1$ and write $w' = j^*(w)$ for the canonical open immersion $j : \mathbf{A}_S^1 \to \mathbf{P}_S^1$. Suppose w' is constant, i.e., there exists a virtual vector bundle w^0 on S such that $w' = \pi^*(w^0)$, where $\pi : \mathbf{A}_S^1 \to S$ is the projection.*[25]
    Then the exact homotopy sequence

$$\Pi_S(\mathbf{A}_S^1, w') \xrightarrow{j_*} \Pi_S(\mathbf{P}_S^1, w) \to \Pi_S(\mathbf{P}_S^1/\mathbf{A}_S^1, w)$$

is canonically isomorphic to

(6.2.4.a) $$\mathrm{Th}(w^0) \xrightarrow{s_{0*}} \Pi_S(\mathbf{P}_S^1, w) \xrightarrow{s_\infty^!} \mathrm{Th}(w^\infty)(1)[2]$$

Here, w^∞ is the restriction of w along the section s_∞ of $\mathbf{P}_S^1$ at ∞, and $s_\infty^!$ denotes the associated Gysin morphism.
    Assume also that $S = \mathrm{Spec}(\mathcal{O})$ is a semi-local essentially smooth scheme over a field k. Then the unit $\eta : \mathbb{1}_S \to \mathrm{K}_*^{MW}$ of the homotopy module representing Milnor-Witt K-theory over S induces an isomorphism

$$\eta_* : [\Pi_S(\mathbf{P}_S^1, w), \mathrm{Th}(w^0)] = \Pi^0(\mathbf{P}_S^1, w - \bar{\pi}^*(w^0)) \xrightarrow{\simeq} \widetilde{\mathrm{CH}}^0(\mathbf{P}_S^1, L)$$

The right-hand-side is the Chow-Witt group of $\mathbf{P}_S^1$ twisted by the line bundle

$$L = \det(w) \otimes \bar{\pi}^*(\det(w^0))^{-1}$$

It follows that (6.2.4.a) splits if and only if w is oriented (see 6.1.6), which amounts to assuming w is an even line bundle on $\mathbf{P}_S^1$.[26] *Moreover, any choice of orientation of w yields an isomorphism*

$$[\Pi_S(\mathbf{P}_S^1, w), \mathrm{Th}(w^0)] \simeq \mathrm{GW}(\mathcal{O})$$

and therefore a splitting of (6.2.4.a) so that

$$\Pi_S(\mathbf{P}_S^1, w) \simeq \mathrm{Th}(w^0) \oplus \mathrm{Th}(w^\infty)(1)[2]$$

Proof. The first statement follows by combining our generalized form of the homotopy purity theorem, see Theorem 5.2.5, and the observation that the isomorphisms

$$\mathrm{Th}(w^0) = \Pi_S(S, w^0) \xrightarrow{\sigma_{0*}} \Pi_S(\mathbf{A}_S^1, w')$$

$$\Pi_S(\mathbf{A}_S^1, w') \simeq \Pi_S(\mathbf{A}_S^1, \pi^*(w^0)) \xrightarrow{\pi_*} \Pi_S(S, w^0) = \mathrm{Th}(w^0)$$

are mutually inverse to each other (due to our assumption on w').

    For the second statement we apply the natural transformation $[-, \eta]$ to the exact homotopy sequence (6.2.4.a). This yields a commutative diagram of exact sequences of abelian groups

---

[25]This holds if S is regular by homotopy invariance of K-theory of regular schemes. Note that w^0 is always isomorphic to the restriction of w over the S-point zero.

[26]That is, w is isomorphic to $\mathcal{O}(2n)$.

(see [Fel20, 2.2.3] for Nisnevich cohomology of the twisted Milnor-Witt sheaf in degree -1)

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \Pi^0(\mathbf{P}^1_S, w - \bar{\pi}^* w^0) & \xrightarrow{s_0^*} & \Pi^0(S, 0) & \xrightarrow{s_{\infty *}} & \Pi^{-1,-1}(S, w^\infty - w^0) \\
& & \eta_* \downarrow & & \downarrow \simeq & & \downarrow \simeq \\
0 & \longrightarrow & \widetilde{\mathrm{CH}}^1(\mathbf{P}^1_S, L) & \xrightarrow{s_0^*} & \widetilde{\mathrm{CH}}^0(S) & \xrightarrow{s_{\infty *}} & H^0(S, \mathrm{K}^{MW}_{-1}\{L_S\})
\end{array}
$$

The middle isomorphism follows since both groups are isomorphic to $\mathrm{GW}(\mathcal{O})$ by Lemma 6.1.3, and the right-most isomorphism between copies of $W(\mathcal{O})$ follows the same argument (Morel's proof of Gersten's conjecture for homotopy sheaves). Therefore, η_* is an isomorphism.

Now, if w is $\bar{\pi}$-oriented, L being a square and $\widetilde{\mathrm{CH}}^*$ being $\mathbf{SL}^c$-oriented, one gets for a choice of an $\mathbf{SL}^c$-orientation of L the required canonical isomorphism

$$
\widetilde{\mathrm{CH}}^1(\mathbf{P}^1_S, L) \simeq \widetilde{\mathrm{CH}}^1(\mathbf{P}^1_S) \simeq \mathrm{GW}(\mathcal{O})
$$

according to [Fas13, 11.7] for the second isomorphism. If L is odd, then, again by *loc. cit.*, there is an isomorphism $\widetilde{\mathrm{CH}}^1(\mathbf{P}^1_S, L) \simeq W(\mathcal{O})$, and $s_{\infty *}$ is isomorphic to the canonical projection $h : \mathrm{GW}(\mathcal{O}) \to W(\mathcal{O})$. Thus the quadratic form $\langle 1 \rangle$ cannot be in the image of s_0^*.[27] $\qquad\square$

6.2.5. In the setting of 6.0.1 we assume in addition:
  (1) $S = \mathrm{Spec}(\mathcal{O})$ is essentially smooth semi-local over a field k
  (2) For each $j \in I$, the normal bundle N_j is relatively orientable over $\partial_j X/S$, and we choose such an isomorphism $o_j : N_j \otimes \omega_{\partial_j X/S} \simeq L^{\otimes 2}$

Owing to Proposition 6.2.4 we deduce a canonical isomorphism

$$
\epsilon_2^{-1} : \bigoplus_{j \in I} \Pi_S(\partial_j X, \langle N_j \rangle) \to \bigoplus_{j \in I} \mathrm{Th}(N_j^0) \oplus \bigoplus_{j \in I} \mathrm{Th}(N_j^\infty)(1)[2]
$$

We use this to show the following analog of Lemma 6.2.2.

Corollary 6.2.6. *The composite map*

$$
\bigoplus_{j \in I} \mathrm{Th}(N_j^0) \oplus \bigoplus_{j \in I} \mathrm{Th}(N_j^\infty)(1)[2] \xrightarrow{\epsilon_2} \bigoplus_{j \in I} \Pi_S(\partial_j X, \langle N_j \rangle) \xrightarrow{d_2} \bigoplus_{j < k} \Pi_S(\partial_{jk} X, \langle N_{jk} \rangle)
$$

factors through the inclusion of the second factor of the left-hand-side. Moreover, via this factorization, it corresponds to the morphism

$$
p_2 = \sum_{j < k} p_{jk}^{j!} - p_{jk}^{k!} : \bigoplus_{j \in I} \mathrm{Th}(N_j^\infty)(1)[2] \to \bigoplus_{j < k} \Pi_S(\partial_{jk} X, \langle N_{jk} \rangle)
$$

Here, $p_{jk} : \partial_{jk} X \to S$ is the canonical projection (a finite étale map), for $l = j, k$, $p_{jk}^{l!}$ is the composition of the inclusion of the l-th factor of the left hand-side with the obvious Gysin morphim, and we have identified the twists $\langle p_{jk}^{-1} N_j^\infty \oplus \mathbf{A}^1 \rangle$ and $\langle N_{jk} \rangle$ (as rank two virtual bundles over a semi-local scheme). In particular, the motivic spectrum $\Pi_S(\bar{X}/X)$ is isomorphic to

$$
\mathcal{D}'_X \oplus \bigoplus_{j \in I} \mathrm{Th}(N_j^0)
$$

Here, $\mathcal{D}'_X$ is the homotopy fiber of p_2.

We can now refine Theorem 5.3.2 with the help of Lemma 6.2.2 and Corollary 6.2.6.

---

[27]If $\mathcal{O} = k$ is a quadratically closed field, the map h is multiplication by 2 on $\mathbb{Z}$.

Theorem 6.2.7. *In the setting of 6.0.1 and 6.2.5, the stable motivic homotopy type at infinity $\Pi_S^\infty(X)$ is the homotopy fiber of the map*

$$\beta : \mathcal{D}_X \oplus \bigoplus_{i \in I} \mathbb{1}_S(1)[2] \to \mathcal{D}'_X \oplus \bigoplus_{j \in I} \mathrm{Th}(N_j^0)$$

given by the matrix

$$\begin{pmatrix} a & b' \\ b & \mu \end{pmatrix}$$

The map $\mu : \bigoplus_{i \in I} \mathbb{1}_S(1)[2] \to \bigoplus_{j \in I} \mathrm{Th}(N_j^0)$ is computed by the "quadratic Mumford matrix"

$$(\mu_{ij})_{1 \leq i,j, \leq n}$$

defined by the composite

(6.2.7.a) $\qquad \mu_{ij} : \mathbb{1}_S(1)[2] \overset{\iota_i}{\hookrightarrow} \Pi_S(\partial_i X) \xrightarrow{\nu_{i*}} \Pi_S(\bar{X}) \xrightarrow{\nu_j^!} \Pi_S(\partial_j X, \langle N_j \rangle) \overset{\pi_j}{\twoheadrightarrow} \mathrm{Th}(N_j^0)$

Here, ι_i (resp. π_j) is the (canonical) split monomorphism (resp. epimorphism) in the isomorphism ϵ_1 (resp. ϵ_2) of Lemma 6.2.2 (resp. 6.2.5). Moreover, if one chooses a trivialization of N_j^0 over S, μ_{ij} becomes an element of $\mathrm{GW}(\mathcal{O})$ such that

$$\mu_{ij} = (\partial_j X, \partial_i X)_{quad}^{or}$$

Here, we use the natural orientation of the tangent space of $\partial_i X \overset{\epsilon_i}{\simeq} \mathbf{P}_k^1$ (that is, $\mathcal{O}_{\mathbf{P}_k^1}(2)$) and the chosen orientation of N_j to define the oriented quadratic intersection degree on the right hand-side (see Definition 6.1.7). Our identification of μ_{ij} follows from the definitions: $\nu_j^!$ is essentially determined by the fundamental class $\eta_{\bar{X}}(\partial_j X)$, and the composition with ν_{i*} is to be understood as the pullback by ν_i in the homotopy category.

6.3. **Triangulated and abelian mixed motives.**

6.3.1. The motive of a Thom space depends only on the rank of the vector bundle, i.e., for Λ-linear motives we do not need to assume normal bundles are relatively oriented. Thus the motive at infinity $M^\infty(X)$ – or Wildeshaus' boundary motive – is the homotopy fiber of the canonical map

$$M(\partial X) \to M(\partial X)^\vee(2)[4]$$

associated with the fundamental class of the diagonal of $\partial X/S$ (see Theorem 5.2.5). Let $\mathcal{D}_X^M$ be the smooth Artin motive over S defined as the homotopy cofiber of (6.2.2.a) seen as a rigid object in $\mathrm{DM}(S, \Lambda)$. Theorem 5.3.2 implies the next result.

Theorem 6.3.2. *In the setting of 6.0.1, the motive at infinity $M_S^\infty(X)$ of the relative surface X/S is the homotopy fiber of the map*

$$\beta : \mathcal{D}_X^M \oplus \bigoplus_{i \in I} \mathbb{1}_S(1)[2] \xrightarrow{\begin{pmatrix} a & b^\vee(2)[4] \\ b & \mu \end{pmatrix}} (\mathcal{D}_X^M)^\vee(2)[4] \oplus \bigoplus_{j \in I} \mathbb{1}_S(1)[2]$$

Moreover, μ is given by the matrix $(\mu_{ij})_{i,j \in I}$ where

$$\mu_{ij} = \deg(\nu_i^*(\eta_{\bar{X}}(\partial_j X)) \in H_M^{00}(S, \Lambda)$$

Here, the quadratic intersection degree is an integer, $\mu_{ij} \in \mathbb{Z}$, in the following cases:

- *S is regular connected and $\Lambda = \mathbb{Q}$ (use [CD19])*
- *S is smooth connected over a field or a Dedekind ring and $\Lambda = \mathbb{Z}$ (use [Spi18])*

Note that $M_S^\infty(X)$ is a triangulated mixed and smooth Artin-Tate motive. Next we relate the above to the motivic t-structure, which exists for rational coefficients $\Lambda = \mathbb{Q}$ when:

- $S = \operatorname{Spec}(K)$, K is a finite field, or a global field (see [Lev93])
- $S = \operatorname{Spec}(O_K)$, $\mathcal{O}_K$ is a number ring (see [Sch11])

The homology motive at infinity of X/S in the abelian category of Artin-Tate mixed motives $\operatorname{MM}^{\mathrm{AT}}(S, \mathbb{Q})$ is defined by

$$\underline{\mathrm{H}}_i^\infty(X) := \underline{\mathrm{H}}_i^\mu(M_S^\infty(X))$$

where $\underline{\mathrm{H}}_i^\mu$ is the i-th homology functor for the motivic t-structure (with standard homological conventions).

Corollary 6.3.3. *In the notation of Theorem 6.3.2, the homology motive at infinity $\underline{\mathrm{H}}_i^\infty(X)$ vanishes for $i \notin [0,3]$ and there is an exact sequence in the abelian category $\operatorname{MM}^{\mathrm{AT}}(S, \mathbb{Q})$*

$$0 \to \underline{\mathrm{H}}_3^\infty(X) \to \bigoplus_{i \in I} \mathbb{1}_S(2) \xrightarrow{\sum_{i<j} p_{ij}^{i!} - p_{ij}^{j!}} \bigoplus_{i<j} M_S(\partial_{ij}X)(2)$$

$$\to \underline{\mathrm{H}}_2^\infty(X) \to \bigoplus_{i \in I} \mathbb{1}_S(1) \xrightarrow{\mu} \bigoplus_{j \in I} \mathbb{1}_S(1)$$

$$\to \underline{\mathrm{H}}_1^\infty(X) \to \bigoplus_{i<j} M_S(\partial_{ij}X) \xrightarrow{\sum_{i<j} p_{ij*}^i - p_{ij*}^j} \bigoplus_{i \in I} \mathbb{1}_S \to \underline{\mathrm{H}}_0^\infty(X) \to 0$$

where μ is the Mumford matrix, and $M_S(\partial_{ij}X) = \underline{\mathrm{H}}_0^\mu(M_S(\partial_{ij}X))$ is seen as an abelian Artin-Tate motive.

Note in particular that $\underline{\mathrm{H}}_0^\infty(X)$ and $\underline{\mathrm{H}}_3^\infty(X)$ are pure of respective weights 0 and -4, while $\underline{\mathrm{H}}_1^\infty(X)$ and $\underline{\mathrm{H}}_2^\infty(X)$ are in general mixed of weights $\{0, -2\}$ and $\{-2, -4\}$, respectively.

6.4. **Three general lines in projective plane.** Over a field k, let $L_1, L_2, L_3 \cong \mathbb{P}_k^1$ be three general lines in $\mathbb{P}_k^2$. Let $\sigma : \bar{X} \to \mathbb{P}_k^2$ be the surface obtained by blowing-up one k-rational point on each of the lines L_i distinct from the k-rational points $p_{12} = L_1 \cap L_2$, $p_{13} = L_1 \cap L_3$, $p_{2,3} = L_2 \cap L_3$. Then $\bar{X}$ is a del Pezzo surface of degree 6 and the proper transform ∂X of $L_1 \cup L_2 \cup L_3$ on $\bar{X}$ a strict normal crossing divisor; it is the support of an anti-canonical divisor on $\bar{X}$ and thus an ample divisor. We now compute the homology motive at infinity of the affine surface $X = \bar{X} - \partial X$. In the exact sequence of Corollary 6.3.3 the homomorphisms $\bigoplus_{i<j} M_k(\partial_{ij}(X)) \cong \bigoplus_{i=1}^3 \mathbb{1}_k \to \bigoplus_{i \in I} \mathbb{1}_k \cong \bigoplus_{i=1}^3 \mathbb{1}_k$ and $\bigoplus_{i \in I} \mathbb{1}_k(2) \cong \bigoplus_{i=1}^3 \mathbb{1}_k(2) \to \bigoplus_{i<j} M_k(\partial_{ij}(X))(2) \cong \bigoplus_{i=1}^3 \mathbb{1}_k(2)$ are given respectively by the matrix

$$N = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{pmatrix}$$

and its transpose. Moreover, since L_i has trivial normal bundle in $\bar{X}$ for $i = 1, 2, 3$, the quadratic Mumford intersection matrix equals

$$\mu = \begin{pmatrix} 0 & \langle 1 \rangle & \langle 1 \rangle \\ \langle 1 \rangle & 0 & \langle 1 \rangle \\ \langle 1 \rangle & \langle 1 \rangle & 0 \end{pmatrix}$$

The matrix $N \in \mathcal{M}_{3,3}(\mathbb{Z})$ is equivalent to the diagonal matrix $\mathrm{diag}(1,1,0)$ and the matrix μ in $\mathcal{M}_{3,3}(\mathrm{GW}(k))$ is equivalent to the diagonal matrix $\mathrm{diag}(1,1,2)$. In summary, the above yields

$$\underline{H}_i^\infty(X) = \begin{cases} \mathbb{1}_k & i = 0 \\ (\mathbb{1}_k/2)(1) \oplus \mathbb{1}_k & i = 1 \\ \mathbb{1}_k(2) & i = 2 \\ \mathbb{1}_k(2) & i = 3 \end{cases}$$

6.5. **Danielewski hypersurfaces.** For a field k and $n \geq 1$, the Danielewski hypersurface D_n is the smooth affine surface D_n in $\mathbf{A}_k^3$ cut out by the equation $x^n z = y(y-1)$. Owing to [Dan88], D_n becomes a Zariski locally trivial $\mathbf{G}_a$-bundle over the affine line with two origins $\check{\mathbf{A}}_k^1$ (using the factorization of the surjective projection $\pi_n = \mathrm{pr}_x : D_n \to \mathbf{A}_k^1$). Thus D_n is $\mathbf{A}^1$-equivalent to $\check{\mathbf{A}}_k^1$ and $\mathbf{P}_k^1$. The threefolds $D_n \times \mathbb{A}_k^1$ are isomorphic, but the surfaces D_n are pairwise non-isomorphic. Over $\mathbb{C}$, Danielewski [Dan88], Fieseler [Fie94] established this by showing the underlying complex analytic manifolds have non-isomorphic first singular homology groups at infinity. Our methods provide a base field independent argument which allows to distinguish between the D_n's via homology motives at infinity.

We begin by constructing explicit smooth projective completions $\bar{D}_n$ of the surfaces D_n, whose boundaries are strict normal crossing divisors. The morphism $\varphi_n = \mathrm{pr}_{x,y} : D_n \to \mathbf{A}_k^2$ expresses D_n as the affine modification of $\mathbf{A}_k^2$ with center at the closed subscheme Z_n with ideal $(x^n, y(y-1))$ and divisor $D_n = \mathrm{div}(x^n)$, cf. [DPØ19]. Furthermore, φ_n decomposes into a sequence of affine modifications

(6.5.0.a) $$\varphi_n = \varphi_1 \circ \psi_2 \cdots \circ \psi_n : D_n \to D_{n-1} \to \cdots D_2 \to D_1 \to \mathbf{A}_k^2$$

given by $\psi_\ell : D_\ell \to D_{\ell-1}; (x,y,z) \mapsto (x,y,xz)$, with center at the closed subscheme $Y_{\ell-1} = (x,z)$ and divisor $H_\ell = \mathrm{div}(x)$. That is, $\varphi_1 : D_1 \to \mathbf{A}_k^2$ is the birational morphism obtained by blowing-up the points $(0,0)$, $(0,1)$ in $\mathbf{A}_k^2$ and removing the proper transform of $\{0\} \times \mathbf{A}_k^1$, and $\psi_\ell : D_\ell \to D_{\ell-1}$ is the birational morphism obtained by blowing-up the points $(0,0,0)$, $(0,0,1)$ in $\pi_\ell^{-1}(0)$ and removing the proper transform of $\pi_{\ell-1}^{-1}(0)$.

Now consider the open embedding $\mathbf{A}_k^2 \hookrightarrow \mathbf{P}_k^1 \times \mathbf{P}_k^1; (x,y) \mapsto ([x:1],[y:1])$. Then $C_\infty = \mathbf{P}_k^1 \times [1:0]$ and $F_\infty = [1:0] \times \mathbf{P}_k^1$ are irreducible components of $\mathbf{P}_k^1 \times \mathbf{P}_k^1$ and we set $F_0 = [0:1] \times \mathbf{P}_k^1$. Let $\bar{\varphi}_1 : \bar{D}_1 \to \mathbf{P}_k^1 \times \mathbf{P}_k^1$ be the blow-up of the points $([0:1],[0:1])$, $([0:1],[1:1])$ in F_0, with respective exceptional divisors $E_{1,0}$, $E_{1,1}$. From now on the proper transform of F_0 in $\bar{D}_1$ is also denoted by F_0. With these definitions, there is a commutative diagram

$$\begin{array}{ccc} D_1 & \longrightarrow & \bar{D}_1 \\ {\scriptstyle \varphi_1} \downarrow & & \downarrow {\scriptstyle \bar{\varphi}_1} \\ \mathbf{A}^2 & \longrightarrow & \mathbf{P}^1 \times \mathbf{P}^1 \end{array}$$

Here, $D_1 \hookrightarrow \bar{D}_1$ is the open immersion given by the complement of the support of the strict normal crossing divisor $\partial D_1 = C_\infty \cup F_\infty \cup F_0$. The closures in $\bar{D}_1$ of the two irreducible components $\{x = y = 0\}$ and $\{x = y - 1 = 0\}$ of $\pi_1^{-1}(0)$ equal the exceptional divisors $E_{1,0}$ and $E_{1,1}$, respectively. We calculate the self-intersection numbers $C_\infty^2 = F_\infty^2 = 0$, $F_0^2 = -2$ in $\bar{D}_1$; that is, the usual degrees of the respective normal line bundles of these curves in $\bar{D}_1$, see e.g., [GH94, Chapter 5.6], [Sha13, Chapter IV].

To construct $\bar{D}_n$, $n \geq 2$, we start with $\bar{D}_1$ and proceed inductively by performing the same sequence of blow-ups as for the affine modifications $\psi_\ell : D_l \to D_{\ell-1}$ in (6.5.0.a). This yields birational morphisms $\bar{\psi}_\ell : \bar{D}_\ell \to \bar{D}_{\ell-1}$ consisting of the blow-up of one point on $E_{\ell,0} - E_{\ell-1,0}$

and another point on $E_{\ell,1} - E_{\ell-1,1}$ with respective exceptional divisors $E_{\ell+1,0}$ and $E_{\ell+1,1}$ (by convention $E_{0,0} = E_{0,1} = F_0$). Moreover, D_ℓ embeds into $\bar{D}_\ell$ as the complement of the support of the strict normal crossing divisor $\partial D_\ell = C_\infty \cup F_\infty \cup F_0 \cup \bigcup_{i=1}^{\ell-1}(E_{i,0} \cup E_{i,1})$ in such a way that the closures of the two irreducible components $\{x = y = 0\}$ and $\{x = y - 1 = 0\}$ of $\pi_\ell^{-1}(0)$ coincide with the divisors $E_{\ell+1,0}$ and $E_{\ell+1,1}$, respectively. By construction, there is a commutative diagram

$$
\begin{array}{ccccccccc}
\bar{D}_\ell & \xrightarrow{\bar{\psi}_\ell} & \bar{D}_{\ell-1} & \longrightarrow & \cdots & \longrightarrow & \bar{D}_2 & \xrightarrow{\bar{\psi}_2} & \bar{D}_1 & \xrightarrow{\bar{\varphi}_1} & \mathbf{P}^1 \times \mathbf{P}^1 \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & & \uparrow \\
D_\ell & \xrightarrow{\psi_\ell} & D_{\ell-1} & \longrightarrow & \cdots & \longrightarrow & D_2 & \xrightarrow{\psi_2} & D_1 & \xrightarrow{\varphi_1} & \mathbf{A}^2
\end{array}
$$

For every $n \geq 2$, we may visualize the boundary divisor ∂D_n as a fork of $\mathbf{P}^1$'s

$$
\begin{array}{c}
(E_{1,0},-2) \rule[0.5ex]{1em}{0.4pt} \cdots \rule[0.5ex]{1em}{0.4pt} (E_{n-1,0},-2) \\
\\
(F_\infty,0) \rule[0.5ex]{1em}{0.4pt} (C_\infty,0) \rule[0.5ex]{1em}{0.4pt} (F_0,-2) \\
\\
(E_{1,1},-2) \rule[0.5ex]{1em}{0.4pt} \cdots \rule[0.5ex]{1em}{0.4pt} (E_{n-1,1},-2)
\end{array}
$$

with the indicated self-intersection numbers for each irreducible component. We may order the irreducible components of ∂D_n as follows

$$F_\infty < C_\infty < F_0 < E_{1,0} < \ldots < E_{n-1,0} < E_{1,1} < \ldots < E_{n-1,1}$$

The Euler class of $\mathcal{O}_{\mathbf{P}^1}(2)$ equals the hyperbolic plane $\mathbb{H} = \langle 1, -1 \rangle \in \mathrm{GW}(k)$. Thus the Euler class of $\mathcal{O}_{\mathbf{P}^1}(-2)$ is $-\mathbb{H}$. Moreover, by Proposition 6.1.4, the transversal intersection of two copies of $\mathbf{P}^1$ in a unique k-rational point yields the class $\langle 1 \rangle \in \mathrm{GW}(k)$. The quadratic Mumford intersection matrix thus takes the form (with zero entries mostly left out of the notation)

$$
\mu_n = \begin{pmatrix}
0 & \langle 1 \rangle & & & & & & & \\
\langle 1 \rangle & 0 & \langle 1 \rangle & & & & & & \\
& \langle 1 \rangle & -\mathbb{H} & \langle 1 \rangle & 0 & & \langle 1 \rangle & & \\
& & \langle 1 \rangle & -\mathbb{H} & \langle 1 \rangle & & & & \\
& & 0 & \langle 1 \rangle & \ddots & \langle 1 \rangle & & & \\
& & & & \langle 1 \rangle & -\mathbb{H} & 0 & & \\
& & \langle 1 \rangle & & & 0 & -\mathbb{H} & \langle 1 \rangle & \\
& & & & & & \langle 1 \rangle & \ddots & \langle 1 \rangle \\
& & & & & & & \langle 1 \rangle & -\mathbb{H}
\end{pmatrix}
$$

Proposition 6.5.1. *Over a field k and $n \geq 1$, the homology motives at infinity of the Danielewski surfaces D_n are given by*

$$
\underline{H}_i^\infty(D_n) = \begin{cases}
\mathbb{1}_k & i = 0 \\
(\mathbb{1}_k/2n)(1) & i = 1 \\
0 & i = 2 \\
\mathbb{1}_k(2) & i = 3
\end{cases}
$$

Proof. The boundary ∂D_n has $2n + 1$ irreducible components which intersect in $2n$ k-rational points. As in Example 6.4, one infers from the description of ∂D_n that the homomorphism

$$\bigoplus_{i<j} M_k(\partial_{ij}(D_n)) \cong \bigoplus_{i=1}^{2n} \mathbb{1}_k \to \bigoplus_{i\in I} \mathbb{1}_k \cong \bigoplus_{i=1}^{2n+1} \mathbb{1}_k$$

is given by an element N of $\mathcal{M}_{2n,2n+1}(\mathbb{Z})$ equivalent to I_{2n}. This implies that $\underline{H}_0^\infty(D_n) = \mathbb{1}_k$ and $\underline{H}_1^\infty(D_n) = \mathrm{Coker}(\mu_n)$. Since the homomorphism

$$\bigoplus_{i\in I} \mathbb{1}_k(2) \cong \bigoplus_{i=1}^{2n+1} \mathbb{1}_k(2) \to \bigoplus_{i<j} M_k(\partial_{ij}(D_n))(2) \cong \bigoplus_{i=1}^{2n} \mathbb{1}_k(2)$$

is given by the transpose of N, it follows that $\underline{H}_3^\infty(D_n) = \mathbb{1}_k(2)$ and $\underline{H}_2^\infty(D_n) = \mathrm{Ker}(\mu_n)$. Finally, one can check by elementary transformations on rows and columns that as an element of $\mathcal{M}_{2n+1,2n+1}(\mathrm{GW}(k))$, the quadratic Mumford matrix μ_n is equivalent to the diagonal matrix $\mathrm{diag}(1,\ldots,1,n\mathbb{H})$. This implies in turn that $\underline{H}_2^\infty(D_n) = 0$ and $\underline{H}_3^\infty(D_n) = (\mathbb{1}_k/2n)(1)$. $\qquad\square$

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