AN INTRODUCTORY COURSE ON VOEVODSKY'S MOTIVIC COMPLEXES

FRÉDÉRIC DÉGLISE, WITH EXERCICES BY NIELS FELD

Abstract. These are the notes of a course given at the 2021 PCMI summer school "Motivic homotopy theory", organized by Marc Levine, Oliver Röndigs, Sasha Vishik and Kirsten Wickelgren. I thank Niels Feld and Emil Jacobsen for helping me polishing these notes.

Contents

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Introduction

Helpful readings for the first and second talks.

- basic algebraic geometry: [Har66], Chap. I, II, III, appendix A could help
- basic intersection theory: [Ful98], Chap. 1, 6, 7, 16, 19
- homological algebra: [GM03], Chap. 1-5
- basic algebraic topology:[Swi02], Chap. 0-5

For further helpful readings:

- Serre's Tor formula: [Ser00]
- general introduction to motivic homotopy theory: [DLOsr$^+$07]
- the introduction of [CD19] could help.

Conventions

We fix a base scheme S, assumed to be regular and noetherian for explicit definitions.[1]

By convention, smooth S-schemes will mean smooth separated of finite type S-schemes. We let Sm_S be the category of such smooth S-schemes. An essentially smooth morphism of schemes is a projective limit of smooth morphisms where the transition maps are affine and étale.

A closed pair (X, Z) is a pair of schemes such that Z is a closed subset of X. Such a pair is said smooth (resp. essentially smooth) over a base scheme S if both X and Z are so. The codimension of (X, Z) is the codimension of Z in X. A morphism of closed pairs $p : (Y, T) \to (X, Z)$ is a morphism of schemes $p : Y \to X$ such that $T = p^{-1}(Z)$ as topological spaces. The morphism p is said to be *cartesian* if $T = p^{-1}(Z)$ as schemes, and *excisive* if in addition the induced map $p|_T^Z : T \to Z$ is an isomorphism.

Given S-schemes X, Y, Z, ... we sometime denote

$$XYZ... = X \times_S Y \times_S Z...$$

their fiber product.

For a scheme X, and an integer n, $X^{(n)}$ denotes the set of points $x \in X$ of codimension n: $\dim(\mathcal{O}_{X,x}) = n$.

We do not base category theory on the Bernays-Gödel class axiomatic. Instead, we use ZFCU and assume that our categories are sets in a fixed universe.

We use the model of quasi-categories for our ∞-categories, and use [Lur09] as a reference book (see for example [Gro20] as an introduction).

[1] One can work with non-regular schemes: see [CD19, Part III]. Note however that the theory gets much more powerful when S is the spectrum of a perfect field due to Theorem 2.12.

1. Homotopy sheaves and transfers

1.1. The aim of Voevodsky's theory is to get a good notion of "motivic coefficients", based on the formalism of torsion and ℓ-adic sheaves. Voevodsky's theory is based on three main variations compared to SGA4:

- (algebraic cycles) sheaves admit **transfers**
- (topos theoretic) one uses the (big) **smooth Nisnevich site**
- (algebraic topology inspiration) one uses the $\mathbb{A}^1$**-homotopy relation**

The material in this part is based on the expository text [Dég07]. The whole theory and overall strategy of proof is due to Voevodsky, with some simplifications introduced in *loc. cit.*

1.1. Finite correspondences.

The next definition is inspired by the classical notion of *algebraic correspondences*, abundantly used by the mathematicians of the italian school as a way of enlarging morphisms of varieties modulo rational equivalence, and at the base of the theory of pure motives — modulo an adequate equivalence relation: see [And04, 3.1].

Definition 1.2. Let X, Y be smooth S-schemes. A *finite correspondence* from X to Y is an algebraic cycle $\alpha = \sum_i n_i.[Z_i]_{XY}$ in $X \times_S Y$ whose irreducible components Z_i are finite and dominant over a connected component of X.[2]

These finite correspondences form an abelian group that we denote by $\mathrm{c}(X, Y)$.

Example 1.3.

(1) Let $f : Y \to X$ be a morphism in Sm_S. Define Γ_f the graph of f defined by the pullback square:

$$\begin{array}{ccc} \Gamma_f & \longrightarrow & Y \times_S X \\ \downarrow & & \downarrow{\scriptstyle f \times_S 1_X} \\ X & \xrightarrow{\delta} & X \times_S X, \end{array}$$

where δ is the diagonal immersion of X/S, which is a closed immersion as X/S is separated. Thus Γ_f is a closed subscheme of YX and the associated algebraic cycle $[\Gamma_f]_{YX}$ defines a finite correspondence from Y to X

(2) Consider the notation of the previous point. Assume in addition that f is finite equidimensional.[3] Let $\epsilon : YX \to XY$ be the automorphism permuting the factors. Then $\epsilon_*([\Gamma_f])$ defines a finite correspondence from X to Y denoted by ${}^{\mathrm{t}}f$: the *transpose* of f.

The interest of finite correspondences is that they can be composed, without requiring an equivalence relation on algebraic cycles (see [Dég07, 1.15, 1.16]).

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[2]One can also say that the support of α, $\mathrm{Supp}(\alpha) = \cup_i Z_i$ is finite and equidimensional overt X.

[3]For example, this rules out the case of closed immersion.

Proposition 1.4. *Let X, Y, Z be smooth S-schemes, and $(\alpha, \beta) \in \mathrm{c}(X, Y) \times \mathrm{c}(Y, Z)$ be a pair of finite correspondences. We consider $p^{XY}_{XYZ} : XYZ \to XY$, and so on, the canonical projection.*

1. *the algebraic cycles $p^{XY*}_{XYZ}(\alpha)$ and $p^{YZ*}_{XYZ}(\beta)$ intersects properly in XYZ, so that their intersection product γ is well-defined using Serre's Tor-formula.*
2. *The support T of γ is finite over X, so that the induced morphism*
$$q = p^{XZ}_{XYZ}|_T : T \to p^{XZ}_{XYZ}(T)$$
is finite and the algebraic cycle
$$q_*\big(p^{XY*}_{XYZ}(\alpha).p^{YZ*}_{XYZ}(\beta)\big)$$
defines a finite correspondence, denoted by $\beta \circ \alpha$.

Example 1.5. Let $f : Y \to X$ be a finite surjective morphism between connected smooth S-schemes. We let d be the degree of the induced morphism on function fields.[4] Then one gets the *degree formula*:
$$f \circ {}^{\mathrm{t}} f = \deg(f). \mathrm{Id}_X,$$
where Id_X is the finite correspondence corresponding to the diagonal of X/S.

One can check that the bilinear operator $(- \circ -)$ on finite correspondences defined above is associative, and that the graph of the identity is a neutral element: see [Dég07, 1.18]. Therefore:

Definition 1.6. We let $\mathrm{Sm}^{\mathrm{cor}}_S$ be the category whose objects are smooth S-schemes and whose morphisms are finite correspondences. We call it the category of smooth correspondences over S.

It is easy to check that $\mathrm{Sm}^{\mathrm{cor}}_S$ is *additive*. The direct sum, or equivalently product, of objects being given by the coproduct of the underlying S-schemes.

1.7. *Graph functor.* Example 1.3(1) allows to define a map:
$$\mathrm{Hom}_{\mathrm{Sm}_S}(X, Y) \to \mathrm{c}(X, Y), F \mapsto [\Gamma_f]_{XY}$$
which can be checked to be compatible with composition. Therefore, this defines a faithful functor $\gamma : \mathrm{Sm} \to \mathrm{Sm}^{\mathrm{cor}}$ which is the identity on objects.

Remark 1.8. It is possible to extend all the definitions of this section to arbitrary (noetherian) bases. In the case of a possibly singular base[5] S, one can define finite correspondences from X to Y (and actually, there is no need to assume X and Y smooth anymore) as algebraic cycles α in XY whose support is finite equidimensional over X, but one has to consider further restriction on these algebraic cycles: namely α must be *special* and *universally integral* over X in the terminology of [CD19, III, 8.1.28, 8.1.49]. Then, all the theory goes through: see [CD19, Sec. 9].

[4] This is also the cardinal of any fiber of f. Remember f is finite equidimensional.

[5] actually the problems occur more precisely for non-geometrically unibranch schemes

1.9. *Further operations.* One can define three more operations on the categories of smooth correspondences.

- *Tensor product.* $\mathrm{Sm}_S^{\mathrm{cor}}$ is symmetric monoidal. The tensor product on object is the cartesian product over S, and on finite correspondences, it is induced by exterior product of algebraic cycles.
- Let $f : T \to S$ be an arbitrary morphism. One defines a base change functor: $f^* : \mathrm{Sm}_S^{\mathrm{cor}} \to \mathrm{Sm}_T^{\mathrm{cor}}, X \mapsto X \times_S T$, using a pullback operation on finite correspondences.
- Let $p : T \to S$ be a smooth morphism. One defines a forgetting the base functor: $p_\sharp : \mathrm{Sm}_T^{\mathrm{cor}} \to \mathrm{Sm}_S^{\mathrm{cor}}, Y/T \mapsto Y/S$, using a direct image operation on finite correspondences.

This is a blueprint of the six operations ! We refer the reader to [Dég07, Sections 4.1.3 and 4.1.5] for details.

1.2. **Transfers.**

Definition 1.10. A *presheaf with transfers* over S is an additive functor $F : (\mathrm{Sm}^{\mathrm{cor}})^{\mathrm{op}} \to \mathscr{A}b$. We denote by $\mathrm{PSh}^{\mathrm{tr}}(S)$ the category of presheaves with transfers, with natural transformations of additive functors as morphisms.

Note in particular, that such a presheaf is equiped with a specific operation, called the *transfer*: for a finite equidimensional morphism $f : Y \to X$ in Sm_S, one gets:

$$f_* = F({}^{\mathrm{t}} f) : F(Y) \to F(X).$$

Example 1.11. (1) The multiplicative group $\mathbb{G}_m$ defines a presheaf with transfers over S.
(2) Let A be an abelian variety over a field k. Then $\underline{A} : X \mapsto \mathrm{Hom}(X, A)$ defines a presheaf with transfers over k (see [Org04]).
(3) Let k be a field, and H^* be a mixed Weil cohomology: Betti cohomology, De Rham cohomology in characteristic 0, rigid cohomology in positive characteristic, ℓ-adic étale cohomology in any characteristic $p \neq l$.[6] Then for any smooth k-scheme S, the presheaf $X \mapsto H^n(X)$ defines a presheaf with transfers over S. This follows from the existence of the cycle class map.
(4) Let ℓ be a prime number and S a regular $\mathbb{Z}[1/\ell]$-scheme, and n an integer. For any scheme X, put $H^n(X) = H^{2n}_{\text{ét}}(X, \Lambda(n))$ where $\Lambda = \mathbb{Z}_\ell$ in the pro-étale topology of $\Lambda = \mathbb{Z}/l^n$ in the étale topology. Then $X \mapsto H^n(X)$ defines a presheaf with transfers over S, with the same justification as in the previous point.[7]

[6] See [CD12] for an axiomatic definition.
[7] More conceptually, one can use the arguments of [CD16, Section 7.2].

Remark 1.12. Unlike in point (3) and (4) of the above example, K-theory does not define a presheaf with transfers over a regular base. However, one can show that the associated Nisnevich sheaf, the so-called *unramified K-theory*, does define a presheaf with transfers (at least over a field).

1.13. *Structures on presheaves with transfers.* The advantage of using presheaves, as abundantly showed by topos theory, is to enlarge the category of smooth correspondences via the Yoneda embedding:

$$\mathrm{Sm}^{\mathrm{cor}} \to \mathrm{PSh}^{\mathrm{tr}}(S), X/S \mapsto \mathrm{c}(-, X) := \mathbb{Z}_S^{\mathrm{tr}}(X)$$

which is fully faithful. On the other hand, $\mathrm{PSh}^{\mathrm{tr}}(S)$ is abelian (even Grothendieck abelian), complete and cocomplete. Moreover, it is possible to extend the operations of 1.9 - and the functors automatically aquire left adjoints. We will make an explicit statement for sheaves with transfers.

1.14. *Nisnevich topology.* Recall that a *Nisnevich cover* of a scheme X is a family $(p_i : W_i \to X)_{i \in I}$ of étale morphism such that for any $x \in X$, there exists $i \in I$ and $w \in W_i$ such that $p_i(w) = x$ and the induced residual extension $\kappa(w)/\kappa(x)$ is trivial.

An important property of the Nisnevich topology is that it is *finitely generated.* More precisely, one defines a Nisnevich distinguished square as a cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{q} & V \\ {\scriptstyle k}\downarrow & \Delta & \downarrow{\scriptstyle p} \\ U & \xrightarrow[j]{} & X \end{array}$$

in Sm_S such that j is an open immersion say with reduced closed complement Z, p is étale and induces an isomorphism $p^{-1}(Z) \to Z$; *i.e.* the map p induces an *excisive* morphism $(V, p^{-1}Z) \to (X, Z)$ of closed pairs.

Lemma 1.15. *Let F be an abelian presheaf over Sm_S. Then F is a sheaf for the Nisnevich topology if and only if for any Nisnevich distinguished square Δ, the square of abelian groups $F(\Delta)$ is cartesian.*

See [MV99, Prop. 1.4].

Remark 1.16. The lemma holds for sheaves of sets. From the cohomological point of view, one gets a stronger property which can be summarized by saying that the simplicial objects associated with the semi-simplicial smooth S-scheme

$$W \underset{k}{\overset{q}{\rightrightarrows}} U \sqcup V \underset{p}{\overset{j}{\rightrightarrows}} X$$

generates Nisnevich hyper-covers. This is expressed by the Brown-Gersten property and theorem. See [MV99, Def. 1.13, Prop. 1.16 p. 100], or for a more elegant statement [AE17, 3.53].

Definition 1.17. A *sheaf with transfers* is a presheaf with transfers F such that $F \circ \gamma$ is a Nisnevich sheaf. We denote by $\mathrm{Sh}^{\mathrm{tr}}(S)$ the full category of $\mathrm{PSh}^{\mathrm{tr}}(S)$ made by sheaves with transfers.

Example 1.18. (1) Let X be a smooth S-scheme. Then $\mathbb{Z}_S^{\mathrm{tr}}(X)$ is in fact a sheaf with transfers. In particular, one gets a Nisnevich-local Yoneda (fully faithful) embedding:

$$\mathrm{Sm}^{\mathrm{cor}} \to \mathrm{Sh}^{\mathrm{tr}}(S), X \mapsto \mathbb{Z}_S^{\mathrm{tr}}(X).$$

(2) Considering the notations of 1.11, $\mathbb{G}_m$ (resp. $\underline{A}$) is a sheaf with transfers over S (resp. k). Examples (3) and (4) do not provides sheaves: the associated Nisnevich sheaf is 0 except if $n = 0$, in which case one in fact get the sheaf with transfers $\mathbb{Z}_S^{\mathrm{tr}}(S)$ — equivalently the constant sheaf of abelian groups associated with $\mathbb{Z}$.

1.19. *Link between small and big site.* The small Nisnevich site X_{Nis} over a scheme X is given by the category of étale X-scheme (which we can take separated of finite type). The smooth site Sm_S is a "big" site in the following sense. Give a Nisnevich sheaf F on Sm_S is equivalent to give for each smooth S-scheme X a sheaf F_X over X_{Nis} and for each morphism $f : Y \to X$ in Sm_S a map

$$\tau_f : f^*(F_Y) \to F_X$$

which is not an isomorphism in general, and measure the "defect of base change" — this situation is customary in the theory of *crystalline sheaves.* In particular, the category of sheaves on Sm_S is much bigger than the category of sheaves on S_{Nis}.

Example 1.20. Let F be an étale sheaf on $S_{\text{ét}} = S_{\mathrm{Nis}}$. Following the above interpretation, one can extend F to an étale, and therefore Nisnevich, sheaf $\underline{F}$ on Sm_S by taking the identities for τ_f. Then $\underline{F}$ is automatically and canonically a sheaf with transfers (see [CD16, Cor. 2.1.9, 2.1.12]).[8]

Remark 1.21. It is interesting to consider other topologies, and in particular the étale one. The theory one gets is closer to SGA4 étale coefficients but further from algebraic cycles. Fore singular base schemes, the cdh-topology is better behaved than the Nisnevich one (see [CD15]).

The main result to use sheaves with transfers is the following one.

Theorem 1.22. *The forgetful functor* $\mathrm{Sh}^{\mathrm{tr}}(S) \to \mathrm{PSh}^{\mathrm{tr}}(S)$ *admits a right adjoint* $a^{tr} : \mathrm{PSh}^{\mathrm{tr}}(S) \to \mathrm{Sh}^{\mathrm{tr}}(S)$ *such that for any presheaves with transfers F over S, $a^{tr}(F)$ restricted to* Sm_S *via the graph functor is the associated Nisnevich sheaves to $F \circ \gamma$.*

[8]This works even for a singular base scheme S.

The corollary of this theorem are numerous, and can be summarized by saying that we have a good theory of *Nisnevich-local coefficients*:

Proposition 1.23. *The category* $\mathrm{Sh}^{\mathrm{tr}}(S)$ *is Grothendieck abelian, complete and cocomplete. It admits the following operations:*

- *a closed symmetric monoidal structure* $(\otimes^{\mathrm{tr}}, \underline{\mathrm{Hom}})$
- *for any morphism of schemes* $f : T \to S$*, adjoint functors* (f^*, f_*) *base change/direct image.*
- *for any smooth morphism of schemes* $p : T \to S$*, adjoint functors* $(p_\sharp, p^*)$ *forget the base/base change. In particular, if* $p = j$ *is an open immersion,* $j_\sharp = j_!$ *is the usual exceptional direct image functor.*

These operations are uniquely determined by saying that the Yoneda embedding:

$$\mathrm{Sm}^{\mathrm{cor}} \to \mathrm{Sh}^{\mathrm{tr}}(S)$$

commutes with the functors of 1.9 and the above left adjoints (in the obvious manner).

Moreover, the graph functor induces an adjunction of categories:

$$\gamma^* : \mathrm{Sh}(S) \to \mathrm{Sh}^{\mathrm{tr}}(S) : \gamma_*$$

where the left hand side is the category of sheaves on the smooth Nisnevich site, adding/forgetting transfers in such a way that the above six operations are compatible with the analogous theory on sheaves without transfers.[9]

Remark 1.24. The operations of the above proposition are in fact a blueprint of the six operations, and allows to build the whole formalism in favorable cases: this is precisely described the *cross functors' theorem* of Ayoub and Voevodsky. This theorem has been extended in the axiomatic of premotivic category in [CD19], and the preceding proposition can be stated by saying that $\mathrm{Sh}^{\mathrm{tr}}(-)$ is a premotivic abelian category: see [CD19, 10.3.11, 10.4.2].[10].

Note also that that the operations described in 1.9 can be stated by saying that $\mathrm{Sm}^{\mathrm{cor}}_?$ is a smooth-fibered category over the category regular schemes. It is essential in order to apply Ayoub-Voevodsky cross functors' theorem to have a category fibred over singular bases (in order to consider so-called the localization property).

1.3. **Homotopy invariance.**

It remains to introduce the last property in Voevodsky's theory, which allows to use techniques from algebraic topology.

Definition 1.25. A sheaf (resp. presheaf) with transfers F over S will be said $\mathbb{A}^1$-*invariant* if for any smooth S-scheme X, the map $p^* : F(X) \to F(\mathbb{A}^1_X)$ induced by the canonical projection p is an isomorphism.

[9] Briefly, put, it means all left (resp. right) adjoint commutes.

[10] In fact, the latter assertion is stronger as it also comprises smooth base change and smooth projection formulas

Such sheaves will be called *homotopy sheaves (with transfers)* and the corresponding subcategory opf $\mathrm{Sh}^{\mathrm{tr}}(S)$ will be denoited by $\mathrm{HI}^{\mathrm{tr}}(S)$.

Remark 1.26. In motivic homotopy, the notion of homotopy sheaves without transfers is central (see [Mor12b]) and transfers have been weakened in several formalisms. See [Fel20a] for a comparison of these variants. As we will only use Voevodsky's transfers here, we will not indicate them in our terminology.

Example 1.27. The sheaf with transfers $\mathbb{G}_m$ (resp. $\underline{A}$) of examples 1.11 and 1.18 are $\mathbb{A}^1$-invariant. In particular, over a field k, there exists a fully faithful embedding from the category of semi-abelian schemes G over k into $\mathrm{HI}^{\mathrm{tr}}(k)$, which maps G to $\underline{G} = \mathrm{Hom}_{\mathrm{Sm}_k}(-, G)$.

1.28. $\mathbb{A}^1$-*homotopy relation.* Let X and Y be smooth S-schemes, and $\alpha, \beta \in \mathrm{c}(X, Y)$ be finite correspondences. One says that α and β are $\mathbb{A}^1$-homotopical is there exists a finite correspondence $H \in \mathrm{c}(\mathbb{A}^1 \times X, Y)$ such that $\alpha = H \circ s_0$ and $\beta = H \circ s_1$ where s_0, s_1 are respectively the zero and unit sections of the ring X-scheme $\mathbb{A}^1_X$.

This is obviously a reflexive and symmetric relation, and one can take the associated equivalence relation as the $\mathbb{A}^1$-homotopy relation $\sim$ on finite correspondences. We put $\pi_S(X, Y) = \mathrm{c}(X, Y)/\sim$.

This $\mathbb{A}^1$-homotopy relation is compatible with composition of finite correspondences, and we therefore get the $\mathbb{A}^1$-homotopy category of smooth correspondences over S, which we denote by $\pi\mathrm{Sm}^{\mathrm{cor}}_S$.

A sheaf with transfers if $\mathbb{A}^1$-invariant if and only if it factors through the canonical projection map $\mathrm{Sm}^{\mathrm{cor}}_S \to \pi\mathrm{Sm}^{\mathrm{cor}}_S$.

These considerations, for finite correspondences over regular bases, were introduced in [Dég07] as a way of expressing some of the main results in Voevodsky's theory. This relies on the following essential computation due to Suslin and Voevodsky.

Theorem 1.29. *Let S be a regular affine scheme, C be a smooth quasi-affine relative curve over S which admits a* good compactification*:*

- *$\bar{C}/S$ is proper, $\bar{C}$ is normal and contains C as a dense open subset.*
- *The complementary $C_\infty = \bar{C} - C$ admits an affine open nieghborhood in $\bar{C}$.*

Then for any smooth affine S-scheme X, there exists a canonical isomorphism

$$\pi_S(X, C) \to \mathrm{Pic}(X \times_S \bar{C} \times_S C_\infty)$$

which to a finite correspondence $\alpha \in \mathrm{c}(X, C)$ associates the class of the line bundle $\mathcal{O}(\alpha)$ associated with α seen as a Cartier divisor in $X \times_S \bar{C}$.

Under the above formulation, this is proved in [Dég07, 4.3.16]. There is a more general computation in terms of Suslin homology in [SV96, th. 3.1].

2. Homotopy sheaves with transfers over a perfect field

In this section we fix a field k. For the main results, we need that k is perfect. However, we prefer to reserve this assertion when it is really needed.

2.1. Function fields as fiber functors.

Points for the smooth Nisnvich site a priori corresponds to essentially smooth henselian local S-schemes.[11] The first result of the theory is that points for homotopy sheaves are much smaller.

Definition 2.1. A *function field* over k will be a field extension E/k finitely generated and separable (*i.e.* essentially smooth.)

A (smooth) model of E/k will be a smooth finitely generated sub-k-algebra $A \subset E$. Ordered by inclusion this form a directed poset (equivalently filtered category).

One define the fiber of a homotopy sheaf F at the function field E/k as the filtered colimit:

$$F(E) = \varinjlim_{A \subset E} F(\mathrm{Spec}(A)).$$

Theorem 2.2. *The category* $\mathrm{HI}^{\mathrm{tr}}(k)$ *is Grothendieck abelian, complete and cocomplete. The forgetful functor* $\mathrm{HI}^{\mathrm{tr}}(k) \to \mathrm{Sh}^{\mathrm{tr}}(k)$ *is exact and admits a left adjoint* h_0 *such that*

$$\Gamma(X, h_0(F)) = \mathrm{coKer}\left(F(\mathbb{A}^1 \times X) \xrightarrow{s_0^* - s_1^*} F(X)\right).$$

For each function field E/k*, the functor*

$$\mathrm{HI}^{\mathrm{tr}}(k) \to \mathscr{A}b, F \mapsto F(E)$$

is a fiber functor (exact and commutes with coproducts), and the family of such functors for all function fields E/k *is conservative.*

More precisely, for any smooth k*-scheme* X*, the canonical map*

$$F(X) \to \oplus_{x \in X^{(0)}} F(\kappa(x)),$$

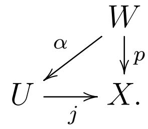
where $X^{(0)}$ *is the set of generic points, is injective.*

The main ingredient to prove the last result consists in building relevant finite correspondences up to homotopy:

Lemma 2.3. *Let* $j : U \to X$ *be an open immersion, into a smooth* k*-scheme* X*. Then there exists a Zariski cover* $p : W \to X$ *and a finite correspondence*

[11] i.e. spectrum of the henzelisation of the local ring of a smooth scheme X at a given point $x \in X$. These schemes can also be described as the projective limit of the system of Nisneivch neighborhoods of x in X. This is equivalent to a pro-object of smooth S-schemes with affine étale transition maps, hence the nam "essentially smooth S-scheme".

$\alpha \in \mathrm{c}(W, U)$ *such that the following diagram commutes up to* $\mathbb{A}^1$*-homotopy:*



The lemma uses in an essential way Theorem 1.29. One argues locally on X. If k is infinite, one can fiber X as a smooth curve over a smooth affine scheme S. Now, finite correspondences with values in X can be expressed as certain line bundles using the previous theorem. As we argue locally on the smooth scheme X, and because the Picard group of a local regular ring has trivial Picard group, one can build the finite correspondence α with the required property by finding an appropriate trivialization. The case of a finite field reduces to that of an infinite field by using transfers. See [Dég07, 4.3.22].

2.4. *Idea of proof of the Theorem.* The last assertion of the theorem follows from the lemma: indeed, one obtains that $j^* : F(X) \to F(X)$ is a split monomorphism. The second assertion formally follows.
For the first one, the essential point is to prove that the Nisnevich (resp. Zariski) sheaf associated with an $\mathbb{A}^1$-invariant presheaf with transfers not only has transfers (Proposition 1.23) but is also $\mathbb{A}^1$-invariant. See [Dég07, 4.4.15].

2.2. **The minus 1 construction.** One can use the theorem of the previous section to get the following result.

Proposition 2.5. *Let* X *be a smooth* k*-scheme. Put* $h_0(X) = h_0(\mathbb{Z}_k^{\mathrm{tr}}(X))$. *Then the homotopy sheaves* $h_0(X)$ *for a smooth* S*-scheme* X *form a generating family in the abelian category* $\mathrm{HI}^{\mathrm{tr}}(k)$. *Moreover, this category is closed symmetric monoidal with tensor product* $\otimes^{\mathrm{H}}$ *such that*

$$h_0(X) \otimes^{\mathrm{H}} h_0(X) = h_0(X \times_k Y).$$

Example 2.6.
(1) The unit for the above tensor product is the sheaf the constant sheaf $h_0(k) = \underline{\mathbb{Z}}$, such that $\Gamma(X, \underline{\mathbb{Z}}) = \mathbb{Z}^{\pi_0(X)}$.
(2) One can deduce from Theorem 1.29 that $h_0(\mathbb{G}_m) = \underline{\mathbb{Z}} \oplus \mathbb{G}_m$.

Definition 2.7. Let F be a homotopy sheaf. One defines a new homotopy sheaf F_{-1} as follows:

$$F_{-1}(X) = \mathrm{Ker}\left(F(\mathbb{G}_m \times X) \xrightarrow{s_1^*} F(X)\right).$$

As s_1 is a split monomorphism, this indeed defines a homotopy sheaf with transfers. Moreover, one derives from the previous example that $F_{-1} = \underline{\mathrm{Hom}}_{\mathrm{HI}^{\mathrm{tr}}(k)}(\mathbb{G}_m, F)$.

Example 2.8. As an exercise, one gets:

- $(\mathbb{G}_m)_{-1} = \underline{\mathbb{Z}}$.
- For any abelian variety A over k, $A_{-1} = 0$.

The following result, a consequence of Voevodsky's cancellation theorem, is much more difficult to get:

Theorem 2.9 (Cancellation). *Let F be homotopy sheaf over a (perfect) field k. Then the canonical map*

$$F \mapsto (\mathbb{G}_m \otimes^{\mathrm{H}} F)_{-1}$$

is an isomorphism.

One can find a direct proof (without using the assumption k perfect) of that result in the thesis [Dég02, 6.3.2].

One of the main lemma of Voevodsky for what will follow is the following purity result.

Proposition 2.10. *Let F be a homotopy sheaf and $i : Z \to X$ be a codimension 1 closed immersion of smooth k-schemes. Put $U = X - Z$ and let $j : U \to X$ be the open immersion.*

Then there exists a short exact sequence of Nisnevich sheaves over X_{Nis}:

$$0 \to F_X \xrightarrow{\tau'_j} j_*(F_U) \to i_*(F_{-1,Z}) \to 0.$$

where we use the notation of Paragraph 1.19, τ'_j being obtained by adjunction from $\tau_j : j^*(F_X) \to F_U$.

The fact τ'_j is a monomorphism follow from the last assertion of Theorem 2.2. The computation of the cokernel of τ'_j then uses several ingredients, such as Nisnevich excision and the fact that Nisnevich locally, (X, Z) looks like $(\mathbb{A}^1_Z, Z)$. This last assertion uses in an essential way the fact we work over a base field. We refer the reader to [Dég07, Section 4.5.3].

2.3. **Main theorem.** We now study Nisnevich cohomology of homotopy sheaves. In all this subsection, cohomology is always computed for the Nisenvich topology. As a prelude, we have:

Proposition 2.11. *Let k be any field. Then one has the following computations:*

$$H^n_{\mathrm{Nis}}(\mathbb{A}^1_k, F) = \begin{cases} F(k) & n = 0, \\ 0 & n > 0. \end{cases}$$

$$H^n_{\mathrm{Nis}}(\mathbb{G}_{m,k}, F) = \begin{cases} F(k) \oplus F_{-1}(k) & n = 0, \\ 0 & n > 0. \end{cases}$$

Differently put, we want to prove that the smooth curves $C = \mathbb{A}^1_k, \mathbb{G}_{m,k}$ is F-acyclic. The proof actually works for any smooth curves over k such that

(N) for any finite extension L/k, $\mathrm{Pic}(C_L) = 0$.

It consists in constructing an explicit contracting homotopy of the complex computing Nisnevich Čech cohomology. This contracting homotopy is defined by suitably constructed finite correspondences thanks to Theorem 1.29 and the property (N). See [Dég07, Th. 4.3.24] for the construction of contractions and [Dég07, Cor. 4.4.11] for the final computation.

The main result, on which the theory of motivic complexes over a perfect field is based, is the following:

Theorem 2.12 (Voevodsky)**.** *Let F be a homotopy sheaf over a perfect field k. Then for any integer $n \geq 0$, the presheaf $H^n_{\mathrm{Nis}}(-, F)$ over Sm_S is homotopy invariant: for any X/k smooth,*

$$p^* : H^n_{\mathrm{Nis}}(X, F) \to H^n_{\mathrm{Nis}}(\mathbb{A}^1_X, F)$$

is an isomorphism.

Remark 2.13. The cohomology groups considered are strictly speaking the cohomology of the sheaf $\gamma_*(F) = F \circ \gamma$ obtained after forgetting the transfers. However, because γ_* is exact (due to Proposition 1.23) and the existence of the associated sheaf with transfers), one gets for any smooth k-scheme Y:

$$\begin{aligned} H^n_{\mathrm{Nis}}(Y, \gamma_* F) &= \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}(Sm_k, \mathbb{Z}))}(\mathbb{Z}(Y), \gamma_* F[n]) \simeq \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}(k))}(\mathbf{L}\gamma^*(\mathbb{Z}(Y)), F[n]) \\ &= \mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}(k))}(\mathbb{Z}^{\mathrm{tr}}(Y), F[n]). \end{aligned}$$

Idea of proof. The full proof is given in [Dég07, Section 4.5.4].
We will denote by H^* the Nisnevich cohomology. Note that we can extend the homotopy sheaf F, and its cohomology presheaves, to essentially smooth k-schemes. We will do that in this summary, as it simplifies the arguments of *loc. cit.* As a preliminary, one deduces from the Leray spectral sequence applied to the projection $p : \mathbb{A}^1_X \to X$ and to the sheaf F_X on X_{Nis}, that it suffices to prove that for any henselian local essentially smooth k-scheme X — we will say k-point — and any $n > 0$,

$$\text{(V(X,n))} \qquad H^n(\mathbb{A}^1_X, F) = 0.$$

The case $n = 1$. One deduce from Proposition 2.10 that for an essentially smooth closed pair (X, Z) of codimension one,

$$H^1_Z(X, F) \simeq F_{-1}(Z).$$

Let now (X, Z) be a closed pair such that X and Z are k-points. According to the above result, and the fact $H^1(X, F) = 0$, one gets an exact sequence:

$$0 \to F(X) \to F(X - Z) \to F_{-1}(Z) \to 0.$$

If we now use (essentially smooth) closed pair $(\mathbb{A}^1_X, \mathbb{A}^1_Z)$, one deduces a long exact sequence of the form with $U = X - Z$:

$$0 \to F(\mathbb{A}^1_X) \to F(\mathbb{A}^1_{X-Z}) \to F_{-1}(\mathbb{A}^1_Z) \to H^1(\mathbb{A}^1_X, F) \xrightarrow{(*)} H^1(\mathbb{A}^1_{X-Z}, F)$$

Applying the homotopy invariance for F and F_{-1}, and the preceding short exact sequence, one obtains that the map $(*)$ is injective.

If X is a k-popint of dimension 1, its closed point Z is essentially smooth as k **is perfect**. Then $(X - Z) = \mathrm{Spec}(E)$, where E is the function field of X. In this case the map $(*)$ gives an injection of $H^1(\mathbb{A}^1_X, F)$ into $H^1(\mathbb{A}^1_E, F)$. Therefore Proposition 2.11 implies that $V(X, 1)$ is true.

The argument for a k-point X of arbitrary dimension is similar but one has first to prove that for any open subscheme $U \subset X$, the map $H^1(\mathbb{A}^1_X, F) \xrightarrow{(*)} H^1(\mathbb{A}^1_U, F)$ is a monomorphims. This can be deduced from the case $(*)$ has $X - U$ can always be included into a normal crossing divisors, and then one can apply the case of smooth divisors. One concludes by taking the limit over all open subschemes U that $H^1(\mathbb{A}^1_X, F)$ injects into $H^1(\mathbb{A}^1_{\kappa(X)}, F)$ and then one concludes by applying again Proposition 2.11.

The general case proceeds by induction on n showing in the inductive step the following property:

$$\text{(J(n))} \qquad \begin{aligned} &\text{For any smooth closed pair } (X, Z) \text{ of codimension 1,} \\ &\text{with complementary open immersion } j : U \to X, \\ &\forall 0 < m < n,\ \mathbf{R}^m j_*(F_U) = 0. \end{aligned}$$

□

2.4. **Gersten resolution.** As a corollary of Theorem 2.12, we get the following result.

Corollary 2.14. *Let F be a homotopy sheaf over a perfect field k. Then for any smooth closed pair (X, Z) of codimension c, there exists a canonical isomorphism:*

$$H^n_Z(X, F) \simeq H^{n-c}_{\mathrm{Nis}}(Z, F_{-c})$$

where the left hand-side is the Nisnevich with support.

In fact, the property $(J(n))$ in the proof of Theorem 2.12 together with Proposition 2.10 implies the codimension 1 case. The general case is obtained by induction on the codimension, or via Morel-Voevodsky relative purity theorem and a detour via orientation theory.[12]

2.15. Let F be as above. Let X be a smooth k-scheme, and $X^{(p)}$ the set of codimension p points of X.

Another way of stating the above corollary, in term of local Nisnevich cohomology is:

$$H^n_x(X_{(x)}, F) = \begin{cases} F_{-p}(\kappa(x)) & n = p, \\ 0 & n \neq p. \end{cases}$$

One deduces from the coniveau spectral sequence for Nisnevich cohomology.

[12] Sheaves with transfers are oriented on the nose given the isomorphism $H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X)$.

Proposition 2.16 (Gersten resolution). *The E_1-term of the coniveau spectral sequence for the Nisnevich cohomology of X with coefficients in X gives a complex of the form:*

$$C^p(X,F) = \oplus_{x \in X^{(p)}} F_{-p}(\kappa(x)),$$

called the Gersten complex of X with coefficients in F. There exists a canonical isomorphism

$$H^n_{\text{Nis}}(X,F) \simeq H^n(C^*(X,F)).$$

The Gersten complex and the isomorphism are functorial in X with respect to étale morphisms.

In particular, if we denote by $C^*(-,F)_X$ the sheaf on X_{Nis} which to V/X assocaited $C^*(V,F)$, we get a canonical augmentation map $F_X \to C^*(-,F)_X$ of sheaves on X_{Nis} and the preceding result tells us that this is a quasi-isomorphism. As $C^*(-,F)_X$ is a complex of Zariski flasque sheaves, we have in fact obtained.

Corollary 2.17. *Under the notations of the preceding proposition, the comparison map*

$$H^n_{\text{Zar}}(X,F) \to H^n_{\text{Nis}}(X,F)$$

is an isomorphism.

Moreover, F_X seen as a sheaf on X_{Zar} is Cohen-Macaulay in the sense of [Har66, Definition p. 238]*, and the complex $C^*(-,F)_X$ on X_{Zar} is the Cousin complex associated with F_X.*

The last two assertions follow, by the definitions in [Har66], from the comparison between Nisnevich and Zariski cohomology and the computation in Paragraph 2.15.

2.18. Let E be a field. Recall that one defines the Milnor K-theory $K^M_*(E)$ of E as the $\mathbb{N}$-graded algebra obtained as the quotient of the tensor algebra $T^*_{\mathbb{Z}}(E^\times)$ modulo the ideal generated by $x \otimes (1-x)$ for $x \in K - \{0,1\}$.

Let now E be a function field. The fiber of the homotopy sheaf $\mathbb{G}_m$ over E is obviously $E^\times$. By definition of the tensor product of homotopy sheaves, one deduces a canonical map

$$\lambda^E_n : T^n_{\mathbb{Z}}(E^\times) = \mathbb{G}_m(E) \otimes_{\mathbb{Z}} \ldots \otimes_{\mathbb{Z}} \mathbb{G}_m(E) \to \mathbb{G}_m^{\otimes^{\text{H}},n}(E)$$

The following result is a consequence of a computation of Suslin and Voevodsky in [SV00, Th. 3.4] but can be obtained within the theory of homotopy sheaves (see [Dég02, 5.5.10]).

Theorem 2.19. *The map λ^E_n induces an isomorphism of $\mathbb{N}$-graded algebra:*

$$K^M_*(E) \to \mathbb{G}_m^{\otimes^{\text{H}},*}(E).$$

Example 2.20. Let us consider the Gersten complex $C^*(X, \mathbb{G}_m^{\otimes^{\text{H}},n})$. According to the above theorem and the cancellation theorem, one gets:

(1) assuming X is connected for simplicity, with function field K: it starts as follows:

$$\begin{array}{ccccccc} 0 \longrightarrow & C^0(X, \mathbb{G}_m^{\otimes^{\mathrm{H}},n}) & \longrightarrow & C^1(X, \mathbb{G}_m^{\otimes^{\mathrm{H}},n}) & \longrightarrow 0 \ldots \\ & \| & & \| & \\ & \mathbb{G}_m(K) & \xrightarrow{d_X^1} & \oplus_{x \in X^{(n-1)}} K_{n-1}(\kappa(x)). & \end{array}$$

One denotes by $\underline{\mathrm{K}}_n^M(X)$ the kernel of the map d_X^1. This is classically called the n-th *unramified Milnor K-theory* of X. Then the Gersten resolution for $\mathbb{G}_m^{\otimes^{\mathrm{H}},n}$ implies that, as sheaves, one gets:

$$\mathbb{G}_m^{\otimes^{\mathrm{H}},n} \simeq \underline{\mathrm{K}}_n^M.$$

(2) the Gersten complex ends as follows:

$$\begin{array}{ccccc} \ldots \longrightarrow & C^{n-1}(X, \mathbb{G}_m^{\otimes^{\mathrm{H}},n}) & \longrightarrow & C^n(X, \mathbb{G}_m^{\otimes^{\mathrm{H}},n}) & \longrightarrow 0 \ldots \\ & \| & & \| & \\ & \oplus_{y \in X^{(n-1)}} \mathbb{G}_m(\kappa(y)) & \xrightarrow{\mathrm{div}} & \oplus_{x \in X^{(n)}} \mathbb{Z}, & \end{array}$$

where div associates to a meromorphic function $f \in \mathbb{G}_m(\kappa(y))$, its codimension n divisor (made by the zeros and poles of f with their multiplicity as coefficient!) In particular, one deduces:

$$H^n_{\mathrm{Nis}}\big(X, \mathbb{G}_m^{\otimes^{\mathrm{H}},n}\big) = \mathrm{CH}^n(X),$$

the group of codimension n cycles modulo rational equivalence. This is in fact *Bloch's formula* using the identification of $\mathbb{G}_m^{\otimes^{\mathrm{H}},n}$ with the n-th unramified Milnor K-theory.

3. Voevodsky's motivic complexes

3.1. **Definition.**

3.1. In the preceding section, we have built a good theory of "motivic" coefficients, homotopy sheaves. Following classical perspective, it would be natural to consider the derived category of homotopy sheaves. However, this procedure is too coarse for $\mathbb{A}^1$-homotopy, and will not capture "higher" invariants such as higher Chow groups. In particular, we have to go with a more evolved definition of derived $\mathbb{A}^1$-homotopy, which still uses the three ingredients put into light in 1.1: transfers, Nisnevich topology, $\mathbb{A}^1$-homotopy.

To give a definition, we will use the framework of ∞-categories. In particular, $\mathrm{D}(\mathscr{A}b)$ denotes the derived ∞-category of abelian groups. It is additive in the sense of [GGN15, Def. 2.6].

Definition 3.2. Let S be a (regular) scheme. The ∞-category of motivic complexes $\mathrm{DM}^{\mathrm{eff}}(S,\mathbb{Z})$ is the ∞-category of additive functors[13]

$$K : (\mathrm{Sm}_S^{\mathrm{cor}})^{\mathrm{op}} \to \mathrm{D}(\mathscr{A}b),$$

called for today ∞-presheaves with transfer over S, which satisfy the following properties:

(1) *Excision.* For any smooth S-schemes X, Y and any excisive morphism $p : (Y,T) \to (X,Z)$ of closed pairs, the induced map $p_* : K(Y,T) \to K(X,Z)$ is a weak equivalence in $\mathrm{D}(\mathscr{A}b)$.

(2) $\mathbb{A}^1$-*invariance.* for any smooth S-scheme X, the map $K(X) \to K(\mathbb{A}^1_X)$ is weak equivalence in $\mathrm{D}(\mathscr{A}b)$.

For a closed pair (X,Z), we have denoted by $K(X,Z)$ the (homotopy) kernel of $K(X) \to K(X-Z)$ in $\mathrm{D}(\mathscr{A}b)$.

The ∞-category $\mathrm{DM}^{\mathrm{eff}}(S,\mathbb{Z})$ is stable and presentable. We will see below an equivalent definition which makes this statement more obvious. Note that any (abelian) presheaf with transfers gives in particular an ∞-presheaf with transfers. This includes in particular objects $\mathbb{Z}_S^{\mathrm{tr}}(X)$ for X/S smooth — one can also use the ∞-categorical Yoneda embedding.

Remark 3.3. In fact, the excision property is equivalent to say that for any distinguished square Δ as in Paragraph 1.14, $K(\Delta)$ is homotopy cartesian in $\mathrm{D}(\mathscr{A}b)$. According to Remark 1.16, this is also equivalent to say that K satisfies Nisnevich descent (with respect to all Nisnevich hyper-covers).

3.4. The advantage of this definition is to insist on the universal property of the ∞-category of motivic complexes $\mathrm{DM}^{\mathrm{eff}}(S,\mathbb{Z})$. One can also easily derive some of its basic properties. Recall that $\mathrm{Sm}^{\mathrm{cor}}$ has a premotivic structure (Paragraph 1.9). One can extend some of these operations as follows.

Let $f : T \to S$ be a morphism of schemes. Given an ∞-presheaf with transfers K over T, one defines $f_*(K)$ as the ∞-presheaves with transfers:

$$(\mathrm{Sm}_T^{\mathrm{cor}})^{\mathrm{op}} \xrightarrow{(f^*)^{\mathrm{op}}} (\mathrm{Sm}_T^{\mathrm{cor}})^{\mathrm{op}} \to \mathrm{D}(\mathscr{A}b).$$

Indeed, one checks easily that $f_*(K)$ satisfies excision and $\mathbb{A}^1$-invariance. One indeed gets an ∞-functor:

$$f_* : \mathrm{DM}^{\mathrm{eff}}(T,\mathbb{Z}) \to \mathrm{DM}^{\mathrm{eff}}(S,\mathbb{Z}).$$

This functor obviously commutes with products, and as the ∞-categories involved are presentable, this functor automatically admits a left adjoint f^*. This left

[13] *i.e.* functors commuting with finite products, [GGN15, Def. 2.6]. In fact, this property is readily implied by Excision ! However, one can restrict to functors whose morphisms on objects are really presheaves with transfers in the sense of Definition 1.10.

adjoint is characterized by the fact it commutes with (homotopy) colimits and the property:[14]

$$f^*(\mathbb{Z}_S^{\mathrm{tr}}(X)) = \mathbb{Z}_T^{\mathrm{tr}}(X \times_S T).$$

Let now $p : T \to S$ be smooth. Given an ∞-presheaf K over S, one defines $p^\sharp(K)$ as the infty-presheaf with transfers:

$$(\mathrm{Sm}_S^{\mathrm{cor}})^{\mathrm{op}} \xrightarrow{(p_\sharp)^{\mathrm{op}}} (\mathrm{Sm}_S^{\mathrm{cor}})^{\mathrm{op}} \to \mathrm{D}(\mathscr{A}b).$$

Again we obtain a functor

$$p^\sharp : \mathrm{DM}^{\mathrm{eff}}(S, \mathbb{Z}) \to \mathrm{DM}^{\mathrm{eff}}(T, \mathbb{Z}).$$

But one can check it satisfies the characterizing properties of p^*. Therefore, one gets an equivalence of functors: $p^* \simeq p^\sharp$. Finally, this functor commutes with co-products, and therefore admits a left adjoint $p_\sharp$ which is characterized by the fact it commutes with (homotopy) colimits and the property:

$$p_\sharp(\mathbb{Z}_T^{\mathrm{tr}}(Y)) = \mathbb{Z}_S^{\mathrm{tr}}(Y \to T \xrightarrow{p} S).$$

The tensor product, and internal Hom, can be defined along the same lines, but it is better to use the description of $\mathrm{DM}^{\mathrm{eff}}(S, \mathbb{Z})$ by localization as it allows to get that the stable ∞-category $\mathrm{DM}^{\mathrm{eff}}(S, \mathbb{Z})$ is generated under suspensions and colimits by the objects $\mathbb{Z}_S^{\mathrm{tr}}(X)$ for X/S smooth.

Admitting this construction, for the moment, we have therefore formally obtained six ∞-functors, organized by adjoint pairs (f^*, f_*), $(p_\sharp, p^*)$ and $\otimes, \underline{\mathrm{Hom}}$ which will serve as a basis for the six functors formalism via Ayoub-Voevodsky cross functors theorem. These functors do satisfy some basic properties which are summarized in the axiomatic of premotivic ∞-categories (derived from [CD19, Section 1.3]).

3.5. The category $\mathrm{PSh}^{\sqcup}(\mathrm{Sm}^{\mathrm{cor}}, \mathrm{D}(\mathscr{A}b))$ of additive ∞-presheaves with transfers is presentable and stable. Therefore, one can use localization theory of such ∞-categories. This allows to obtain the description of $\mathrm{DM}^{\mathrm{eff}}(S, \mathbb{Z})$ as the localization of $\mathrm{PSh}^{\sqcup}(\mathrm{Sm}^{\mathrm{cor}}, \mathrm{D}(\mathscr{A}b))$ with respect to the following maps:

(1) $\mathbb{Z}_S^{\mathrm{tr}}(X_\bullet) \xrightarrow{\pi_*} \mathbb{Z}_S^{\mathrm{tr}}(X)$, for any Nisnevich hyper-cover $X_\bullet \xrightarrow{\pi} X$ of a smooth X/S, where $\mathbb{Z}_S^{\mathrm{tr}}(X_\bullet))$ means the complex associated with the obvious simplicial presheaf with transfers.

(2) $\mathbb{Z}_S^{\mathrm{tr}}(\mathbb{A}_X^1) \xrightarrow{p_*} \mathbb{Z}_S^{\mathrm{tr}}(X)$ for a smooth X/S

In classical Voevodsky's theory, one hides the first localization into the theory of sheaves. More precisely one uses the following description of motivic complexes over S.

Proposition 3.6. *The ∞-category $\mathrm{DM}^{\mathrm{eff}}(S, \mathbb{Z})$ as defined above is canonically equivalent to the localization of the ∞-category $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}(S))$ with respect $\mathbb{A}$-homotopy, that is maps in point (2).*

[14] This follows from the definitions and the (additive) Yoneda embedding.

As a consequence, one obtains the following classical definition.

Definition 3.7. Let S be a (regular) scheme. A motivic complex over S is a complex K of sheaves with transfers whose Nisnevich cohomology $H^*_{\mathrm{Nis}}(-,K)$ of K is $\mathbb{A}^1$-homotopy invariant: for any smooth X/S, and any integer n,

$$p^* : H^n_{\mathrm{Nis}}(X,K) \to H^n_{\mathrm{Nis}}(\mathbb{A}^1_X,K)$$

is an isomorphism.

One can reformulate that property, in more ∞-categorical terms, by saying that for any smooth X/S, the canonical map:

$$p^* : \mathbf{R}\operatorname{Hom}(\mathbb{Z}^{\mathrm{tr}}_S(X),K) \to \mathbf{R}\operatorname{Hom}(\mathbb{Z}^{\mathrm{tr}}_S(\mathbb{A}^1_X),K)$$

is a weak equivalence in the ∞-category $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}(X))$. In other words, K is $\mathbb{A}^1$-local with respect to the localization described in the preceding proposition. In other words, $\mathrm{DM}^{\mathrm{eff}}(S,\mathbb{Z})$ can be identified as the sub-∞-category of $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}(S))$ made by the $\mathbb{A}^1$-local complexes.

The main problem of this abstract definition is to be able to understand $\mathbb{A}^1$-local complexes. This is where Voevodsky's main theorem enter into play.

For the rest of this notes, unless stated otherwise, we will now work over a perfect field k.

3.2. Motivic complexes over a perfect field and Suslin singular complex.

3.8. As the category $\mathrm{Sh}^{\mathrm{tr}}(S)$ is abelian, we can define the cohomology $\underline{\mathrm{H}}^q(K)$ of a complex of sheaves with transfers K. Concretely, it is obtained by first computing the cohomology of K in the category of presheaves with transfers, and then applying the associated Nisnevich sheaf with transfers (see Theorem 1.22).

It is easy to obtain, using the hypercohomology spectral sequence

$$E_2^{p,q} = H^p(X,\underline{\mathrm{H}}^q(K)) \Rightarrow H^{p+q}(X,K)$$

and the fact it converges[15], that K is $\mathbb{A}^1$-local if and only if the sheaves with transfers $\underline{\mathrm{H}}^q(K)$ are $\mathbb{A}^1$-local. Therefore, as a corollary of Theorem 2.12, one gets.

Theorem 3.9. *A complex of sheaves with transfers K over k is $\mathbb{A}^1$-local if and only if for all $n \in \mathbb{Z}$, $\underline{\mathrm{H}}^n(K)$ is a homotopy sheaf.*

3.10. Recall that a t-structure on a stable ∞-category $\mathscr{T}$ is the data of *cohomologically non-negative* (resp. *negative*) objects $\mathscr{T}^{\leq 0}$ (resp. $\mathscr{T}^{>0}$) such that:

- $\mathscr{T}^{\leq 0}$ (resp. $\mathscr{T}^{>0}$) is stable under suspension $-[1]$ (resp. desuspension $-[-1]$)
- $\operatorname{Hom}_{\mathrm{Ho}(\mathscr{T})}(\mathscr{T}^{\leq 0},\mathscr{T}^{>0}) = 0$

[15] Recall that Ninsevich cohomology is bounded: for a scheme X of finite dimension d, and any Nisnevich sheaf F over X_{Nis}, $H^n(X,F)=0$ if $n \notin [0,d]$; in particular, $E_2^{p,q}$ is concentrated in degree $p \in [0,\dim(X)]$.

- for any object K of $\mathscr{T}$, there exists $(K', K'') \in \mathscr{T}^{\leq 0} \times \mathscr{T}^{>0}$ and a homotopy exact sequence:

$$K' \to K \to K'' \xrightarrow{+1}$$

One defines: $\mathscr{T}^{\leq n} = \mathscr{T}^{\leq 0}[-n]$, $\mathscr{T}^{\geq n} = \mathscr{T}^{>0}[-n+1]$. Then, the *heart* $\mathscr{T}^{\heartsuit} = \mathscr{T}^{\leq 0} \cap \mathscr{T}^{\geq 0}$ is automatically abelian category.

As a corollary of the preceding theorem, the natural t-structure on $\mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}(S))$, induces a t-structure on $\mathrm{DM}^{\mathrm{eff}}(S)$: an $\mathbb{A}^1$-local complex is cohomologically non-negative (resp. *negative*) if for all $i < 0$ (resp. $i \geq 0$), $\underline{\mathrm{H}}^i(K) = 0$.

Definition 3.11. The t-structure on $\mathrm{DM}^{\mathrm{eff}}(k)$ described above is called the *homotopy t-structure.* In particular, $\mathrm{DM}^{\mathrm{eff}}(k)^{\heartsuit} = \mathrm{HI}^{\mathrm{tr}}(k)$.

Remark 3.12. In motivic homotopy theory, it is more customary to use homological conventions. This means we put: $\mathscr{T}_{\geq 0} := \mathscr{T}^{\leq 0}$ (resp. $\mathscr{T}_{<0} := \mathscr{T}^{>0}$) and call them the homologically non-positive (resp. positive) objects.

For motivic complexes, we also put $\underline{\mathrm{H}}_i(K) = \underline{\mathrm{H}}^{-i}(K)$.

3.13. *Suslin (singular) complex.* Recall one defines the standard cosimplicial k-scheme which in degree n is:

$$\Delta^n = \mathrm{Spec}(k[t_0, \dots, t_n]/(t_0 + \dots + t_n))$$

Abstractly, $\Delta^n \simeq \mathbb{A}^n_k$, but the above presentation immediately gives a cosimplicial structure.

Let K be a complex of sheaves with transfers over k. One defines a new complex $\mathrm{C}^{\mathrm{S}}_*(K)$ whose global sections on a smooth X/S are given by the coproduct total complex of the bicomplex:

$$K(\Delta^\bullet \times_k X).$$

One formally obtains that the cohomology presheaves of $\mathrm{C}^{\mathrm{S}}_*(K)$ are $\mathbb{A}^1$-invariant. Thanks to the last point of 2.4, this implies that the cohomology sheaves of $\mathrm{C}^{\mathrm{S}}_*(K)$ are homotopy sheaves; in other words, $\mathrm{C}^{\mathrm{S}}_*(K)$ is $\mathbb{A}^1$-local. Further:

Proposition 3.14. *The Suslin complex functor induces an ∞-functor:*

$$L_{\mathbb{A}^1} : \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}(S)) \to \mathrm{DM}^{\mathrm{eff}}(S), K \mapsto \mathbf{R}\,\mathrm{Hom}(\mathbb{Z}^{\mathrm{tr}}_S(\Delta^\bullet), K)$$

which is left adjoint to the inclusion $\mathrm{DM}^{\mathrm{eff}}(S) \to \mathrm{D}(\mathrm{Sh}^{\mathrm{tr}}(S))$.

One calls $L_{\mathbb{A}^1}$ the $\mathbb{A}^1$-*localization functor* (over k).

Remark 3.15. Note that as a left adjoint to the inclusion of $\mathbb{A}^1$-local objects, there always exists an $\mathbb{A}^1$-localization functor (even over any base). All the interest of Voevodsky's theory is to get a simple construction of the latter.

3.3. **Voevodsky (homological) motives.** Thanks to Proposition 3.14, we have now an easy way to produce motivic complexes over k.

Definition 3.16. One defines the homological motives of a smooth k-scheme as the motivic complex $M(X) = \mathrm{C}^{\mathrm{S}}_* \mathbb{Z}^{\mathrm{tr}}_k(X)$.

In particular, $M(X)$ is covariantly functorial in X, with respect to morphisms and even finite correspondences. Moreover, $M(X)$ is concentrated in homological positive degree for the homotopy t-structure. Its homotopy sheaves are determined by their fibers over function field E/k and one has the formula:

$$\underline{\mathrm{H}}^n(M(X))(E) = H_n\big(\,\mathrm{c}_E(\Delta^*_E, X_E)\big)$$

The latter group is the Suslin homology of X_E/E.

Remark 3.17. A further link with the previous section is that one gets:

$$\underline{\mathrm{H}}_0(M(X)) = h_0(X)$$

using notations of Proposition 2.5 (see also Theorem 2.2).

Extending the computation of Theorem 1.29, one gets:

Proposition 3.18. *Let $\bar{C}/k$ be a smooth projective curve, and $C \subsetneq \bar{C}$ an open subscheme. Put $C_\infty = \bar{C} - C$. Then*

$$\underline{\mathrm{H}}_n(M(C)) = \begin{cases} \mathrm{Pic}(\bar{C}, C_\infty) & n = 0 \\ 0 & n \neq 0. \end{cases}$$

In other words, $M(C)$ is concentrated in homotopy degree 0.

Example 3.19. (1) The motive $M(\mathbb{G}_m)$ is concentrated in degree 0 and from Example 2.6, one gets:

$$M(\mathbb{G}_m) = \underline{\mathrm{H}}_0(\mathbb{G}_m) = \underline{\mathbb{Z}} \oplus \mathbb{G}_m.$$

(2) More generally, after the choice of a base point $x \in C(k)$, one can consider the *generalized albanese* variety A associated with C/k, which is the dual of Rosenlicht-Serre's generalized jacobian associated with $(\bar{C}, C_\infty)$ — and the base point x.[16] In particular, A is a semi-abelian variety and there is a universal map $C \to A$ mapping x to 0. Using Example 1.27, one associates to the semi-abelian variety A a homotopy sheaf $\underline{A}$ and one can recast the previous computation as follows:

$$M(C) = \underline{\mathbb{Z}} \oplus \underline{A}.$$

[16]We refer the interested reader to [SS03].

3.4. **Tate twists and tensor products.**

3.20. As mentioned previously, one can extend the monoidal structure of $\mathrm{Sm}_k^{\mathrm{cor}}$ to the category of motivic complexes. The current construction uses an explicit monoidal model structure on the category of complexes of sheaves with transfers (see [CD09, Ex. 3.3] for the regular case and [CD19, 11.1.1,] for the general one). In particular $\mathrm{DM}^{\mathrm{eff}}(k,\mathbb{Z})$ is a monoidal ∞-category. Moreover, the closed symmetric monoidal structure $\otimes$ is uniquely characterized by the property:

$$M(X) \otimes M(Y) = M(X \times_k Y).$$

Remark 3.21. Note that the monoidal structure on $\mathrm{DM}^{\mathrm{eff}}(k,\mathbb{Z})$ is compatible with the homotopy t-structure, in the sense that a tensor product of homlologically positive objects is homologically positive. This implies that the monoidal structure descends on the heart. In fact, on obtains for non-negative homological motivic complexes K and L:

$$\underline{\mathrm{H}}_0(K \otimes L) = \underline{\mathrm{H}}_0(K) \otimes^{\mathrm{H}} \underline{\mathrm{H}}_0(L).$$

Definition 3.22. One define the Tate twist (for motivic complexes) by the formula:

$$\mathbb{Z}(1) = \mathrm{coKer}\big(M\{1\} \to M(\mathbb{G}_m)\big)[-1].$$

For any integer $n \geq 0$, we put $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes,n}$.

As a corollary of the previous example, one obtains that $\mathbb{Z}(1) = \mathbb{G}_m[-1]$: $\mathbb{Z}(1)$ in concetrated in cohomological homotopy degree 1.

Remark 3.23. Using the above property of the tensor structure of motivic complexes, one easily obtains that $\mathbb{Z}(n)[n]$ is the cokernel of the map

$$\oplus_{i=1}^n M(\mathbb{G}_m^{n-1}) \xrightarrow{\sum_i \mathbb{1}_{i*}} M(\mathbb{G}_m^n)$$

where $\nu_i : \mathbb{G}_m^{n-1} \simeq \mathbb{G}_m^n$ is the closed immersion which equate the i-coordinate to 1. This implies in particular that $\mathbb{Z}(n)$ is concentrated in cohomological homotopy degree $]-\infty, n]$.

The (reinforced) Beilinson-Soulé conjecture asks the question if $\mathbb{Z}(n)$ is concentrated in cohomological homotopy degree $[1,n]$ for any $n > 0$. The only known case is that of $n = 1$, due to the previous computation. Due to Voevodsky's proof of the Bloch-Kato conjecture, the case of integral coefficients is equivalent to the case of rational coefficients.

Example 3.24. At least we know one homotopy sheaf of $\mathbb{Z}(n)$, the highest cohomological one. Due to Remark 3.21 and Example 2.20, one gets:

$$\underline{\mathrm{H}}^n(\mathbb{Z}(n)) \simeq \mathbb{G}_m^{\otimes^{\mathrm{H}},n} \simeq \underline{\mathrm{K}}_n^M$$

where the right hand-side is the n-th unramified Milnor K-theory (see *loc. cit.*).

Definition 3.25. One defines the motivic cohomology of a smooth k-scheme X in bidegree $(n,i) \in \mathbb{Z} \times \mathbb{N}$ as

$$H_M^{n,i}(X) = H^n_{\mathrm{Nis}}(X, \mathbb{Z}(i)).$$

We call n the degree and i the twist.

Note that according to Remark 2.13 (and the fact $\mathbb{Z}(i)$ is $\mathbb{A}^1$-local), one also gets:

$$H_M^{n,i}(X) = \mathrm{Hom}_{\mathrm{DM}^{\mathrm{eff}}(k,\mathbb{Z})}\big(M(X), \mathbb{Z}(i)[n]\big).$$

This was one of Beilinson's desired interpretation of motivic cohomology.

Summarizing the computations obtained so far, we get:

Proposition 3.26. *Let X be a smooth k-scheme. Then:*

(1) *in twist* 0*:*

$$H^{n,0}(X) = \begin{cases} \mathbb{Z}^{\pi_0(X)} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

(2) *in twist* 1*:*

$$H^{n,1}(X) = \begin{cases} \mathbb{G}_m(X) & n = 1 \\ \mathrm{Pic}(X) & n = 2 \\ 0 & n \neq 1, 2. \end{cases}$$

(3) *for* $i > 1$,

$$H_M^{n,i}(X) = \begin{cases} \mathrm{CH}^i(X) & n = 2i \\ 0 & (n > 2i) \text{ or } (n - i > \dim(X)) \end{cases}$$

Each of the computations follows from what we have obtained. For the last one, we can use either the Nisnevich hypercohomology spectral sequence for the complex $\mathbb{Z}(n)$, Bloch's formula and the computation of the above example or the coniveau spectral sequence.[17]

3.5. **Further results.**

3.27. In the next course, we will see how to define a stable (effective) version of the category of motivic complexes. This allows to obtain duality results for the motive of smooth projective varieties. We can still state results that reflect this embedding. From the computations of Proposition 3.18, one can derive:

Proposition 3.28. *Let C/k be a smooth projective curve, with a rational point $x \in C(k)$. Let A be the albanese variety of C based at x.*[18]

[17] In fact, the two spectral sequence coincides from E_2 on ! See [Bon10, Dég14]

[18] *i.e.* A is the univeral abelain variety with a given map $C \to A$ maping x to 0. This is also the dual of the Picard scheme of C based at x.

Then one gets:

$$\underline{\mathrm{H}}_i(M(C)) = \begin{cases} \underline{\mathbb{Z}} \oplus \underline{A} & n = 0 \\ \mathbb{G}_m & n = 1 \\ 0 & n \neq 0, 1. \end{cases}$$

Moreover, one gets a decomposition in the category of motivic complexes:

$$M(C) = \underline{\mathbb{Z}} \oplus \underline{A} \oplus \mathbb{Z}(1)[2].$$

This corresponds to the (Chow-)Künneth decomposition of the Chow motive associated with C (see [And04, 4.3.4])

In fact, we get the following comparison result, which hides a duality.

Proposition 3.29. *Let X and Y be smooth proper k-schemes, $d = \dim(Y)$. Then there exists an isomorphism:*

$$\mathrm{Hom}\big(M(X), M(Y)\big) \simeq \mathrm{Hom}(M(X \times_k Y), \mathbb{Z}(d)[2d]) \simeq \mathrm{CH}^d(X \times_k Y).$$

Moreover, we get a fully faithful contravariant functor:

$$\mathrm{CHM}^{\mathrm{eff}}(k, \mathbb{Z})^{op} \to \mathrm{DM}^{\mathrm{eff}}(k, \mathbb{Z}), h(X) \mapsto M(X)$$

where the left hand-side is the category of effective Chow motives with integral coefficients (see [And04, 4.1.1]*).*

For this comparison, and a stronger result in the relative case, we refer the reader to [Fan16].

Remark 3.30. Beware that we are sure that Chow-Künneth decompositions do not always exist for integral Chow motives, contrary to what we obtain in Proposition 3.28. This can already be seen for example as we had to assume that C has a rational point ! In general, if there is no 0-cycle of degree 0 in a smooth projective k-scheme X, it is not possible to construct the Künneth projector of the 0-th part.

3.31. It is possible to extend some of the results to non perfect fields. In particular, realizing a long-term project of Suslin[19], one gets the comparison with Bloch's higher Chow groups which were actually the first known definition of what should deserve the name of motivic cohomology according to Beilinson's conjectures.

Theorem 3.32 (Voevodsky, [Voe02])**.** *Let X be a smooth k-scheme. Then for any integers (n, i), there exists an isomorphism*

$$H^n\big(X, \mathbb{Z}(i)\big) \simeq \mathrm{CH}^i(X, 2i - n)$$

when one defines $\mathbb{Z}(i)$ by the formula of Remark 3.23.

Remark 3.33. (1) Actually, it was proved as a consequence that for singular k-schemes (of finite type), Bloch's higher Chow groups computs the so-called *Borel-Moore motivic homology*: see [Lev04].

[19]This was actually the motivation to introduce Suslin singular homology in the 1987 conference on K-theory in Luminy.

(2) The comparison of the above theorem provides the relation between motivic cohomology with algebraic K-theory: see the talk by Matthew Morrow. With rational coefficients, we will see another approach in the next course.

Appendix A. Exercices by Niels Feld

A.1. **Milnor K-theory.** (see [GS17]) The Milnor K-groups $K_n^M(k)$ attached to a field k is the quotient of the n-th tensor power $(k^\times)^{\otimes n}$ of the multiplicative group of k by the subgroup generated by those elements $a_1 \otimes \cdots \otimes a_n$ for which $a_i + a_j = 1$ for some $1 \leq i < j \leq n$. Thus $K_0^M(k) = \mathbb{Z}$ and $K_1^M(k) = k^\times$. Elements of $K_n^M(k)$ are called *symbols*; we write $[a_1, \dots, a_n]$ for the image of $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(k)$.

(1) Show that Milnor K-groups are functorial with respect to field extensions: given an inclusion $\phi : k \subset K$, there is a natural map $i_{K/k} : K_n^M(k) \to K_n^M(K)$ induced by ϕ.
   Given $\alpha \in K_n^M(K)$, we shall often abbreviate $i_{K/k}(\alpha)$ by α_K.
(2) Show that the product pairings
$$(k^\times)^{n\otimes} \times (k^\times)^{m\otimes}$$
induce a structure of graded ring on
$$K_*^M(k) = \bigoplus_{n\geq 0} K_n^M(k).$$
(3) (a) Prove that the group $K_2^M(k)$ satisfies the relations
$$[x, -x] = 0 \text{ and } [x, x] = [x, -1].$$
   (b) Prove that the product operation on $K_*^M(k)$ is graded-commutative, i.e. it satisfies
$$[\alpha, \beta] = (-1)^{nm}[\beta, \alpha]$$
   for $\alpha \in K_n^M(k)$ and $\beta \in K_m^M(k)$
(4) Let $\mathbf{F}$ be a finite field. Prove that, for all $n > 1$, the groups $K_n^M(\mathbf{F})$ are trivial.
(5) Let K be a field equipped with a discrete valuation $v : K^\times \to \mathbb{Z}$. Denote by $\mathcal{O}_v$ the associated valuation ring and by $\kappa(v)$ its residue field.
   (a) Fix π a local parameter (i.e. an element satisfying $v(\pi) = 1$). For n a natural number, show that $K_n^M(K)$ is generated by symbols of the form $[\pi, u_2, \dots, u_n]$ and $[u_1, \dots, u_n]$ where u_i are units in $\mathcal{O}_v$.
   (b) For each $n > 0$, there exists a unique morphism
$$\partial^M : K_n^M(K) \to K_{n-1}^M(\kappa(v))$$
   satisfying
$$\partial^M([\pi, u_2, \dots, u_n]) = [\bar{u_2}, \dots, \bar{u_n})$$
   for all local parameters π and all units u_i, where $\bar{u_i}$ denotes the image of u_i in $\kappa(v)$.
   Moreover, once a local parameter π is fixed, there is a unique morphism
$$s_\pi^M : K_n^M(K) \to K_n^M(\kappa(v))$$
   with the property
$$s_\pi^M([\pi^{i_1} u_1, \dots, \pi^{i_n} u_n]) = [\bar{u_1}, \dots, \bar{u_n}]$$
   for all integers i_j and units u_i of $\mathcal{O}_v$.

(c) Prove that the tame symbol $\partial^M : K_1^M(K) \to K_0(\kappa(v))$ is the valuation map $v : K^\times \to \mathbb{Z}$, and that the tame symbol $\partial^M : K_2^M(K) \to K_1^M(\kappa(v))$ is given by the formula
$$\partial^M([a,b]) = (-1)^{v(a)v(b)}\overline{a^{v(b)}b^{-v(a)}}$$
where the lines denotes the image in $\kappa(v)$.

(d) Prove that, for $[a_1,\ldots,a_n] \in K_n^M(K)$, one has the formula
$$s_\pi^M([a_1,\ldots,a_n]) = \partial^M([-\pi, a_1,\ldots,a_n])$$
for all local parameters π.

(e) Let L/K be a field extension and b_L a discrete valuation of L extending v with residue field $\kappa(v_L)$ and ramification index e. Denoting the associated tame symbol by ∂_L^M, one has for all $\alpha \in K_n^M(K)$
$$\partial_L^M(\alpha_L) = e \cdot \partial^M(\alpha).$$

(f) Denote by U_n the subgroup of $K_n^M(K)$ generated by those symbols $[u_1,\ldots u_n]$ where all the u_i are units in $\mathcal{O}_v$, and $U_n^1 \subset K_n^M(K)$ the subgroup generated by symbols $[x_1,\ldots,x_n]$ with x_1 a unit in $\mathcal{O}_v$ satisfying $\overline{x}_1 = 1$.

(i) Prove that $U_n^1 \subset U_n$.

(ii) Prove that we have exact sequences
$$0 \longrightarrow U_n \longrightarrow K_n^M(K) \xrightarrow{\partial^M} K_{n-1}^M(\kappa(v)) \longrightarrow 0$$
and
$$0 \longrightarrow U_n^1 \longrightarrow K_n^M(K) \xrightarrow{(s_\pi^M,\partial^M)} K_n^M(\kappa(v)) \oplus K_{n-1}^M(\kappa(v)) \longrightarrow 0.$$

(g) Assume moreover that K is complete with respect to v, and let $m > 0$ be an integer invertible in $\kappa(v)$.
Prove that the pair (s_π^M, ∂^M) induces an isomorphism
$$K_n^M(K)/mK_n^M(K) \simeq$$
$$K_n^M(\kappa(v))/mK_n^M(\kappa(v)) \oplus K_{n-1}^M(\kappa(v))/mK_{n-1}^M(\kappa(v)).$$

(6) Recall that the discrete valuations of $k(t)$ trivial on k correspond to the local rings of closed points P on the projective line $\mathbb{P}_k^1$. As before, we denote by $\kappa(P)$ their residue fields and by v_P the associated valuations. At each closed point $P \neq \infty$ a local parameter is furnished by a monic irreducible polynomial $\pi_P \in k[t]$; at $P = \infty$ one may take $\pi_P = t^{-1}$. The degree of the field extension $[\kappa(P), k]$ is called the degree of the closed point P; it equals the degree of the polynomial π_P. Thus we obtain tame symbols
$$\partial_P^M : K_n^M(k(t)) \to K_{n-1}^M(\kappa(P))$$
and specialization maps
$$s_\pi^M : K_n^M(k(t)) \to K_n^M(\kappa(P)).$$

(a) Show that the image of the product map
$$\partial^M := (\partial_P^M) : K_n^M(k(t)) \to \prod_{P\in\mathbb{P}^1-\{\infty\}} K_{n-1}^M(\kappa(P))$$
lies in the direct sum.

(b) Denote by L_d the subgroup of $K_n^M(k(t))$ generated by those symbols $[f_1, \ldots, f_n]$ where f_i are polynomials in $k[t]$ of degree $\leq d$. For each $d > 0$, consider the map

$$\partial_d^M : K_n^M(k(t)) \to \bigoplus_{\deg(P)=d} K_{n-1}^M(\kappa(P))$$

defined as the direct sum of the maps ∂_P^M for all closed points P of degree d.

Prove that its restriction to L_d induces an isomorphism

$$\overline{\partial}_d^M : L_d/L_{d-1} \simeq \bigoplus_{\deg(P)=d} K_{n-1}^M(\kappa(P)).$$

(c) (*Homotopy invariance*) Prove that the sequence

$$0 \longrightarrow K_n^M(k) \longrightarrow K_n^M(k(t)) \xrightarrow{\partial^M} \bigoplus_{P \in \mathbb{P}^1 - \{\infty\}} K_{n-1}^M(\kappa(P)) \longrightarrow 0$$

is exact and split by the specialization map $s_{t^{-1}}^M$ at ∞.

A.2. **Milnor-Witt K-theory.** (see [Mor12a, Chapter 3])

(1) Generalize the previous results to the Milnor-Witt K-groups $\mathbf{K}_*^{\mathrm{MW}}(k)$.

A.3. **Smooth models.**

(1) Let E be a finitely generated field over the perfect field k. By definition, a *smooth model* of E is an affine smooth scheme $X = \operatorname{Spec} A$ of finite type such that A is a sub-k-algebra of E, with function field E.

Convince yourself that such a smooth model always exists.

(2) Let E/k and L/k be two extensions and $\phi : E \to L$ a morphism such that the extension L/E is finite. By definition, we call *k-model of L/E* any triplet $((X, x), (Y, y), f : Y \to X)$ such that (X, x) is a model of E/k, (Y, y) is a model of L/k and f is a dominant finite morphism making the following diagram commutative:

$$\begin{array}{ccc} \operatorname{Spec} L & \xrightarrow{\operatorname{Spec}\phi} & \operatorname{Spec} E \\ {\scriptstyle y}\downarrow & & \downarrow{\scriptstyle x} \\ Y & \xrightarrow{\quad f \quad} & X \end{array}$$

where the vertical maps are induced by the points x and y.

(a) Let $f : Y \to X$ be an equidimensional finite morphism of schemes. Assume that U is a dense open subscheme of Y.

Prove that the open subscheme $f^{-1}(X - f(Y - U))$ is dense containing U.

(b) Let E/k be an extension and E/L a finite extension of fields.

Prove that there exists a k-model of L/E.

(3) Let E/k be an extension and L/E a finite extension. Consider $f : Y \to X$ and $f' : Y' \to X'$ two k-models of L/E.

Prove that there is a k-model $f'' : Y'' \to X''$ of L/E such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ \uparrow & & \uparrow \\ Y'' & \xrightarrow{f''} & X'' \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{f'} & X' \end{array}$$

is commutative and compatible with the base points.

A.4. **Grothendieck-Witt groups.** (see [Fel20b])

(1) Let E be a field of characteristic $p > 0$. Let $\alpha \in \mathrm{GW}(E)$ be an element in the kernel of the rank morphism $\mathrm{GW}(E) \to \mathbb{Z}$.
Prove that α is nilpotent in $\mathrm{GW}(E)$.

A.5. **Enumerative geometry.**

A.5.1. *Apollonius circles.* (see [Che19])

(1) Show that the two following definitions are equivalent:
   (a) A circle in $\mathbb{P}^2$ is given by the equation
   $$(x - az)^2 + (y - bz)^2 = r^2z^2.$$
   (b) A circle in $\mathbb{P}^2$ is a conic given by $V(f)$ where $f \in (z, x^2 + y^2)$.
(2) Define
$$\Phi = \{(r, C) \in D \times \mathbb{P}^3 \,|\, C \text{ is tangent to } D \text{ at } r\}$$
where D is a smooth circle and $\mathbb{P}^3$ is viewed as the space of circles.
Prove that the correspondence Φ is 2-dimensional and irreducible.
(3) Denote by $\pi_2 : \Phi \to \mathbb{P}^3$ the second canonical projection and $Z_D = \pi_2(\Phi)$ its image.
Prove that Z_D has dimension 2.
(4) Consider a line L inside $\mathbb{P}^3$. Viewing $\mathbb{P}^3$ again as the space of circles, L parameterizes a family of circles $\{C_t\}_{t \in \mathbb{P}^3}$.
Assuming L is generic, prove that $L \cap Z_D$ consists of 2 points.
Conclude that Z_D is a quadric surface.
(5) Let C be a circle tangent to D. Prove that the line between C and D is in Z_D. Hence Z_D is a quadric cone with vertex in D.
(6) Given three circles in general position, how many circles are tangent to all three?

Appendix B. Table of notations

$\mathrm{CH}^n(X)$	Chow group (algebraic cycles of codimension n modulo rational equivalence)
$\mathrm{Pic}(X)$	Picard group (isomorphism classes of invertible vector bundles over X)
$\mathrm{Pic}(X, Z)$	relative Picard group (isomorphism classes of vector bundles over X with a given trivialization over Z)
$K_*^M(E)$	Milnor K-theory of the field E
X_t	small site for the topology $t = \mathrm{Zar}, \mathrm{Nis}, \text{ét}$
$X^{(n)}$	set of codimension n points x of X ($\dim(\mathcal{O}_{X,x} = n)$
$\mathrm{Sh}(-, \mathbb{Z})$	sheaves of abelian groups over some site
$\mathrm{Sh}(S, \mathbb{Z})$	sheaves of abelian groups over the smooth Nisnevich site over S
$\mathrm{Sm}_S^{\mathrm{cor}}$	category of smooth separated S-schemes of finite type
$\mathrm{Sm}_S^{\mathrm{cor}}$	category of smooth finite correspondences over S
$\mathrm{PSh}^{\mathrm{tr}}(S)$	presheaves with transfers (with $\mathbb{Z}$- coefficients)
$\mathrm{Sh}^{\mathrm{tr}}(S)$	sheaves with transfers (with $\mathbb{Z}$- coefficients)
$\mathbb{Z}_S^{\mathrm{tr}}(X)$	sheaf with transfers represented by the smooth S-scheme X
$\mathbf{R}\,\mathrm{Hom}(-, -)$	mapping space in a stable ∞-category

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