AN INTODUCTORY COURSE ON MOTIVIC HOMOTOPY THEORY AND RATIONAL MIXED MOTIVES

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CONTENTS

Introduction 2
Conventions 2
1. Unstable $\mathbb{A}^1$-homotopy theory 2
1.1. The ∞-categorical definition 2
1.2. Definition via Nisnevich sheaves, $\mathbb{A}^1$-local objects 3
1.3. Topological realization and motivic spheres 4
1.4. A glimpse of six operations 5
2. Stable $\mathbb{A}^1$-homotopy theory 7
2.1. The stabilization procedure and spectra 7
2.2. A (brief) summary of the six functors formalism 10
3. Algebraic K-theory and Beilinson motives 11
3.1. Representability algebraic K-theory 11
3.2. Beilinson motivic cohomology 13
3.3. Beilinson motives 15
4. Conclusion 17
References 17

- basic algebraic geometry: [Har66], Chap. I, II, III, appendix A could help
- basic algebraic topology: [Swi02], Chap. 0-5
- the ∞-categorical language: [Gro20] (for example).
- basic algebraic K-theory: [Wei13], chap. IV

For further helpful readings:

- general introduction to motivic homotopy theory: [DLOsr$^+$07], [AE17]
- the full ∞-categorical language: [Lur09]

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Introduction

Voevodsky's initial program: building the *Eilenberg MacLane motivic spectrum*. Motivic homotopy towards Milnor conjecture. Morel-Voevodsky's motivic homotopy theory.

Recall on rational homotopy theory and program for rational motivic complexes/spectra.

Conventions

We fix a base scheme S, which is only assumed to be noetherian finite dimensional.

By convention, smooth S-schemes will mean smooth separated of finite type S-schemes. We let Sm_S be the category of such smooth S-schemes.

1. Unstable $\mathbb{A}^1$-homotopy theory

1.1. The ∞-categorical definition.

1.1. We first start from a universal, ∞-categorical, construction of the unstable $\mathbb{A}^1$-homotopy category.

It is strictly parallel to that of motivic complexes except one forgets about transfers, and one uses the ∞-category of *pointed* spaces $\mathscr{S}_*$ instead of the derived category of abelian groups.[1]

Definition 1.2. The $\mathbb{A}^1$-homotopy (or motivic homotopy) ∞-category $\mathscr{H}_*(S)$ of spaces over S is the ∞-category of functors

$$\mathcal{X} : (\mathrm{Sm}_S)^{\mathrm{op}} \to \mathscr{S}_*,$$

simply called spaces over S, which satisfies the following properties:

(1) *Excision.* For any smooth S-schemes X, Y and any excisive morphism $p : (Y,T) \to (X,Z)$ of closed pairs, the induced map $p_* : \mathcal{X}(Y,T) \to \mathcal{X}(X,Z)$ is a weak equivalence in $\mathscr{S}_*$.

(2) $\mathbb{A}^1$-*invariance.* for any smooth S-scheme X, the map $\mathcal{X}(X) \to \mathcal{X}(\mathbb{A}^1_X)$ induced by the canonical projection is a weak equivalence in $\mathscr{S}_*$.

For a closed pair (X,Z), we have denoted by $\mathcal{X}(X,Z)$ the (homotopy) fiber of $\mathcal{X}(X) \to \mathcal{X}(X-Z)$ in $\mathscr{S}_*$.

Note that an advantage of this definition is that it is clear that $\mathscr{H}_*(S)$ admits products. These are the smash products, usually denoted by $\mathcal{X} \wedge \mathcal{Y}$.

Example 1.3. Here are two examples of objects of the motivic homotopy category.

[1] We have used pointed spaces in order to get a formulation of excision closed to that of motives. It is of course possible to work with unpointed spaces!

(1) Take a pointed simplicial set K, that we view as an object of $\mathscr{S}_*$. Then the constant functor with value K defines an object of $\mathscr{H}_*(S)$ that we will still denote by K. This actually defines an embedding of $\mathscr{S}_*$ in $\mathscr{H}_*(S)$!
(2) Let X/S be a smooth scheme, and $x : S \to X$ an S-point (*i.e.* a section of the canonical projection). Then the Yoneda ∞-categorical embedding allows to view (X, x) as a (simplicially constant) object of the motivic homotopy category. If there is no given S-point of X/S (of course it can happen there is no S-point at all !), one put $X_+ = X \sqcup S$, seen as a pointed S-scheme and therefore as an object of $\mathscr{H}_*(S)$.

The magic of $\mathbb{A}^1$-homotopy theory is to mix non-trivially these two examples: the simplicial direction and the (algebraic) geometric direction!

Remarque 1.4. A useful (and more classical) way of formulating the excision property of the above definition is that for any Nisnevich distinguished square Δ in Sm_S, the square

$$\begin{tikzcd} W_+ \arrow[r] \arrow[d] & V_+ \arrow[d] \\ U_+ \arrow[r] & X_+ \end{tikzcd}$$

is homotopy cartesian in $\mathscr{H}_*(S)$.

1.2. Definition via Nisnevich sheaves, $\mathbb{A}^1$-local objects.

1.5. The previous construction is nice as it reveals the universal property of the pointed $\mathbb{A}^1$-homotopy category. As in the case of motivic complexes, there is a more down-to-earth construction, the original one of Morel and Voevodsky. Its main interest is to allow basic arguments at the 1-categorical level, *i.e.* that of sheaves.

A pointed simplicial presheaf over Sm_S is a contravariant functor:

$$\mathcal{X} : (\mathrm{Sm}_S)^{\mathrm{op}} \to \Delta^{\mathrm{op}}\mathscr{S}et_*.$$

It is a Nisnevich sheaf if it satisfies the sheaf condition with respect to Nisnevich covers. Equivalently, one asks that for any Nisnevich distinguished square Δ (see lecture one), the square $\mathcal{X}(\Delta)$ is cocartesian in the category of simplicial sets.

Definition 1.6. A *pointed space over S* will be a pointed simplicial Nisnevich sheaf over S.

Example 1.7. It is obvious how to make the previous examples spaces in that sense. For a pointed simplicial set K, one considers K_{ct} the constant Nisnevich sheaf with values K over Sm_S — that is, the Nisnevich sheaves associated with K. For a pointed smooth S-scheme X, this is the classical Yoneda lemma: $X(-) = \mathrm{Hom}_{\mathrm{Sm}_S}(-, X)$.

As above, we will abusively denote this spaces respectively as K and X.

1.8. We can view the category of spaces over S as an ∞-category. A good way to do it is to define a well suited *mapping space*. This can be done by using model categories, and that was the solution used by Morel and Voevodsky in [MV99]. In brief, one can take the so-called *injective* model structure, where weak equivalence are defined fibrewise and cofibrations are monomorphisms. Morel and Voevodsky showed that this model category is simplicial, therefore provided the hoped-for mapping space.

Then the category of pointed spaces over S becomes a *simplicial category*, and this is a model for ∞-categories.[2]

Here is an presentation, closer to Morel and Voevodsky's original definition, for the pointed motivic homotopy category.

Proposition 1.9. *The ∞-category $\mathscr{H}_*(S)$ is equivalent to the localization of the ∞-category of spaces over S with respect to $\mathbb{A}^1$-homotopy* i.e. *maps of the form $\mathbb{A}^1_S \times_S \mathcal{X} \to \mathcal{X}$ for a space $\mathcal{X}$.*

Remarque 1.10. Note that the ∞-category of spaces over S is presentable. Therefore, one can use the localization theory for presentable ∞-category. In particular, one defines $\mathbb{A}^1$-*local spaces* as those space $\mathcal{X}$ such that for any smooth scheme X/S[3], the canonical map

$$\mathrm{Map}(X_+, \mathcal{X}) \to \mathrm{Map}\left((\mathbb{A}^1_X)_+, \mathcal{X}\right)$$

is a weak equivalence. Then the ∞-category $\mathscr{H}_*(S)$ is equivalent to the subcategory of spaces over S made by $\mathbb{A}^1$-local spaces.

Formally, the inclusion of $\mathbb{A}^1$-local spaces into spaces admits a left adjoint. Concretely, for any space $\mathcal{X}$ over S, there exists an $\mathbb{A}^1$-local space $L_{\mathbb{A}^1}(\mathcal{X})$ and a natural map

$$\mathcal{X} \to L_{\mathbb{A}^1}(\mathcal{X})$$

which is a weak equivalence. Everything can be made functorial in the space $\mathcal{X}$. This called is the $\mathbb{A}^1$-*localization functor*. This is the analogue for spaces of the Suslin complex for sheaves with transfers. Some of the main theorems of $\mathbb{A}^1$-homotopy theory rely on the ability to find a nice construction (model) of this functor: see [Mor12], with $L_{\mathbb{A}^1} = Ex_{\mathbb{A}^1}$ defined p. 107, or [AE17, Sec. 4.3].

1.3. **Topological realization and motivic spheres.**

1.11. Let us work over the field $\mathbb{C}$ of complex numbers. Then any smooth complex scheme X gives a topological space by taking the algebraic variety of its complex points $X(\mathbb{C})$, and the topology coming from the fact it is an analytic variety.

Of course $\mathbb{A}^1(\mathbb{C}) = \mathbb{C}$ is contractible. Using moreover excision for topological spaces, one can prove:

[2]But of course, the natural world to define the ∞-category of Nisnevich (pointed) simplicial sheaves is Lurie's theory of ∞-topos. See [Lur09].

[3]Note it is enough to restrict to smooth S-schemes for this definition

Proposition 1.12. *The functor $X \mapsto X(\mathbb{C})$ extends (in fact Kan-extends) to a realization functor*
$$\mathscr{H}_*(\mathbb{C}) \to \mathscr{S}_*.$$

The existence of this functor allows to guess the $\mathbb{A}^1$-homotopy type of complex varieties, as this functors respects weak equivalences !

Example 1.13. All the following complex schemes realize to spheres: $(\mathbb{G}_m, 1)$, $(\mathbb{P}^1, \infty)$, $(\mathbb{A}^n - \{0\}, 1, ..., 1)$. In fact S^n seen as a space is sent to S^n by the above functor (this is true for any pointed simplicial set!).

In fact, all this schemes, as well as there smash products, are what we call *motivic spheres.*

In fact, one has the following weak $\mathbb{A}^1$-equivalences that relate these spheres:
$$\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$$
$$\mathbb{A}^n - \{0\} \simeq S^{n-1} \wedge \mathbb{G}_m^{\wedge,n}$$
A last example of sphere is given by Thom spaces. One defines $\mathrm{Th}(\mathbb{A}^n)$ as the homotopy cofiber of $\mathbb{A}^n - \{0\} \to \mathbb{A}^n$. Then:
$$\mathrm{Th}(\mathbb{A}^n) \simeq S^n \wedge \mathbb{G}_m^{\wedge,n}.$$
These weak $\mathbb{A}^1$-equivalences exist over any base !

As a conclusion, one can remark that all these sphere can be expressed as a smash product of S^1 and $\mathbb{G}_m$. In fact, as we have seen in the case of motivic complexes, $\mathbb{G}_m$ corresponds to the Tate twist (up to 1 suspension!): we have obtained in the previous lecture that the reduced motivic complex of the pointed smooth S-scheme $\mathbb{G}_m$ is $\mathbb{1}_S(1)[1]$.

Before going to the next point, we must state the following theorem of Morel, computing some stable $\mathbb{A}^1$-homotopy groups of one particular motivic spheres, of fundamental importance:

Theorem 1.14 (Morel, [Mor12])**.** *Let k be a field, and $\mathrm{GW}(k)$ be the Grothendieck group of quadratic forms of k. Then for any $n \geq 2$, one has the following computation:*
$$\pi_{n-1}^{\mathbb{A}^1}(\mathbb{A}^n - \{0\}, 1) := [S^{n-1}, \mathbb{A}^n - \{0\}]_*^{\mathbb{A}^1} \simeq \mathrm{GW}(k)$$
where $[-,-]_*^{\mathbb{A}^1}$ denotes the homotopy classes of maps in the ∞-category $\mathscr{H}_*(k)$.

1.4. **A glimpse of six operations.**

1.15. As we have seen in the case of motivic complexes, it is possible to build a basic functoriality for the motivic homotopy category:

- The smash product admits a right adjoint, the internal Hom functor.[4] Note that as left adjoint, $(- \wedge -)$ commutes with colimits. Moreover, it is

[4]The only way I am aware of to get this right adjoint is to use Blander's model structure as one can prove it is monoidal.

characterized by the property:

$$X_+ \wedge Y_+ = (X \times_S Y)_+.$$

- For $f : T \to S$ any map, one obtains a pair of adjoint ∞-functor:

$$f^* : \mathscr{H}_*(S) \leftrightarrows \mathscr{H}_*(T) : f_*$$

  such that f^* is obtained by left Kan extension of the functor f^* on smooth schemes. In particular, for X/S smooth, $f^*(X_+) = (Y \times_S T)_+$ and this characterizes the adjunction.
- for $p : T \to S$ smooth, one an adjunction of ∞-functors:

$$p_\sharp : \mathscr{H}_*(S) \leftrightarrows \mathscr{H}_*(T) : p^*$$

  such that $p_\sharp$ is obtained by left Kan extension of the functor $p_\sharp$ at the level of smooth schemes. In particular, for Y/T smooth, $p_\sharp(Y_+) = (Y \to T \to S)_+$ and this characterizes the adjunction.

Let us go further. The characterizing properties of the left adjoints yield:

Proposition 1.16. *With the above notations, one gets:*

(1) *Smooth base change: for any cartesian square of arbitrary schemes*

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ T \arrow[r, "p"] & S \end{tikzcd}$$

there exists a canonical equivalence of ∞-functors: $q_\sharp g^* \xrightarrow{\sim} f^* p_\sharp$.
By adjunction: $p^* f_* \xrightarrow{\sim} g_* q^*$.

(2) *Smooth projection formula: let $p : T \to S$ be a smooth morphism. For any pointed space $\mathcal{X}$ (resp. $\mathcal{Y}$) over S (resp. T), one has an equivalence:*

$$p_\sharp(\mathcal{Y} \wedge p^* \mathcal{X}) \xrightarrow{\sim} p_\sharp(\mathcal{Y}) \wedge \mathcal{X}.$$

In fact, this is an equivalence of bifunctor: $p_\sharp\big(- \wedge p^*(-)\big) \xrightarrow{\sim} \big(p_\sharp(-) \wedge -\big)$.
By adjunction, one gets equivalences:

$$p^* \underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{X}') \xrightarrow{\sim} \underline{\mathrm{Hom}}(p^*\mathcal{X}, p^*\mathcal{X}')$$
$$\underline{\mathrm{Hom}}(p_\sharp \mathcal{Y}, \mathcal{X}) \xrightarrow{\sim} p_* \underline{\mathrm{Hom}}(\mathcal{Y}, p^*\mathcal{X})$$

In fact, the smooth base change formula (resp. projection formula) is obviously true when the functors are evaluated at smooth schemes (exercice!). Then the result follows by uniqueness of Kan extensions.

The following result is a crucial theorem when dealing with the functoriality

Theorem 1.17 (Localization, Morel-Voevodsky)**.** *Let $i : Z \to S$ be a closed immersion with complementary open immersion j. Then for any pointed space $\mathcal{X}$, the following is a homotopy cofiber (right exact) sequence:*

$$j_\sharp j^*(\mathcal{X}) \xrightarrow{ad'_j} \mathcal{X} \xrightarrow{ad_i} i_* i^*(\mathcal{X})$$

Here ad'_j (resp. ad_i) is the counit map for the adjunction $(j_\sharp, j^*)$ (resp. (i^*, i_*)).

The original proof in [MV99, 2.21, p. 114] is fundamentally correct, but with inaccuracies. It was written correctly in [Ayo07b, 4.5.36]. One can restate the theorem more geometrically by saying that for any smooth S-scheme X, $X_Z = X \times_S Z$, the canonical map

$$X/(X - X_Z) \to i_*(X_{Z+})$$

is a weak $\mathbb{A}^1$-equivalence. This statement uses both Excision and the $\mathbb{A}^1$-homotopy relation.

2. Stable $\mathbb{A}^1$-homotopy theory

2.1. **The stabilization procedure and spectra.**

2.1. We will now pass from the unstable category to the stable one. Historically, this models the passage from effective pure (Chow, etc...) motives to pure motives. The reason to make this passage was to get a *rigid* category, *i.e.* one were an object (pure motive here) admits (strong) dual in the monoidal sense. The same procedure can be applied to motivic complexes, with respect to the *Tate twist*.

We will present the construction for in the $\mathbb{A}^1$-homotopical setting. Then another motivation comes from algebraic topology: it is (much) simpler to study stable homotopy groups than homotopy groups. The companion theory to stable homotopy groups is that of S^1-spectra, such as the infinite suspension spectrum of a space. The passage from space to spectra can be understood as a simple stabilization procedure in the sense of ∞-categories. A similar procedure works for the $\mathbb{A}^1$-homotopical setting, but we now have a choice of spheres to do, say between S^1 and $\mathbb{G}_m$ given the discussion in Example 1.13. Actually, based on the (pure and mixed) motivic picture, we choose to invert both. That is done by taking the sphere $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$ as our sphere for spectra.

We now present a summarized construction, using the language of ∞-category. This is due to Robalo [Rob15, Section 2]. The idea is to construct a category whose objects are "infinitely divisible" by $\mathbb{P}^1$. More topologicaly, objects are ∞-loop spaces with respect to the sphere $\mathbb{P}^1$: $\Omega(\mathcal{X}) = \underline{\mathrm{Hom}}(\mathbb{P}^1, \mathcal{X})$.

Definition 2.2. One defines the stable homotopy category $\mathrm{SH}(S)$ over a scheme S as the limit of presentable ∞-categories of the $\mathbb{N}$-tower:

$$\dots \xrightarrow{\Omega} \mathscr{H}_*(S) \xrightarrow{\Omega} \mathscr{H}_*(S).$$

In particular, one gets a canonical ∞-functor, the infinite suspension functor

$$\Sigma^\infty : \mathscr{H}_*(S) \to \mathrm{SH}(S)$$

and one can check the following properties (here the real work starts!):

Proposition 2.3. *There exists a symmetric monoidal structure $\otimes$ on the ∞-category $\mathrm{SH}(S)$, the functor Σ^∞ is monoidal and maps the motivic sphere $\mathbb{P}^1$ to a $\otimes$-invertible object.*

The ∞-category $\mathrm{SH}(S)$ is stable.[5]

Note that, with the above definition, the existence of the symmetric monoidal structure is not formal. It follows from the fact that the cyclic permutation on $(\mathbb{P}^1)^{\wedge,3}$ is equivalent to the identity (see again *loc. cit.*).

2.4. To understand concretely this definition, it is useful to come back to an explicit model of $\mathrm{SH}(S)$. By the above definition, one is naturally led to Ω-spectra: these are sequence of spaces $(\mathcal{X}_n)_{n\in\mathbb{N}}$ together with a given map of spaces:

$$\mathcal{X}_n \to \mathbf{R}\,\underline{\mathrm{Hom}}(\mathbb{P}^1, \mathcal{X}_{n+1})$$

which is a weak equivalence. In fact, to get a good model category structure, one is led to drop the latter condition. These are called the $\mathbb{P}^1$-spectra.[6] The standard example is the *infinite suspension spectrum* associated with a pointed space $\mathcal{X}$, defined by

$$(\Sigma^\infty \mathcal{X})_n = (\mathbb{P}^1)^{\wedge,n} \wedge \mathcal{X},$$

The structural maps being obvious.

Example 2.5. As announced, the spectrum construction amount to invert for the tensor product $\mathbb{P}^1$, and therefore also S^1 and $\mathbb{G}_m$ taking into account Example 1.13. Similarly, all Thom spaces $\mathrm{Th}(\mathbb{A}^n)$ becomes $\otimes$-invertible in $\mathrm{SH}(S)$. By an easy argument using Excision (in fact a Zariski-local argument suffices), we get further that for any vector bundle V/S, the (stable) Thom space $\mathrm{Th}(V) = \Sigma^\infty V/V^\times$ is $\otimes$-invertible.

Remarque 2.6. (1) Using the above definition with the Tate twist $\mathbb{Z}_S(1)$, one obtains the stable/non effective category $\mathrm{DM}(S)$, which is closer to the category of pure motives. Indeed, one obtains along the lines of the previous course a fully faithful embedding: $\mathrm{CHM}(k)^{op} \to \mathrm{DM}(k)$, mapping a smooth projective k-scheme to the (infinite suspension) of the motivic complex $M(X)$.

[5] This means that a square is a homotopy pullback if and only if it is a homotopy pushout. It implies that the associated homotopy category is triangulated in the sense of Verdier.

[6] It is then possible to get a model structure (out of a good model monoidal structure on spaces) along classical lines, and to compare the associated monoidal ∞-category with the above definition. See [Rob15, 2.3].

(2) It is possible to consider the simplicial sphere S^1 in $\mathscr{H}_*(S)$ in the above construction. One gets an interesting intermediate category, $\mathrm{SH}^{\mathrm{eff}}(S)$, which is still stable, and a factorisation:

$$\mathscr{H}_*(S) \to \mathrm{SH}^{\mathrm{eff}}(S) \to \mathrm{SH}(S).$$

This allows to break the study of the $\mathbb{P}^1$-stabilisation procedure into two steps.

The interest of this construction is also to fit in the following link between $\mathbb{A}^1$-homotopy and motivic complexes:

$$\begin{tikzcd}
\mathrm{Sm}_S \arrow[r, "(-)_+"] \arrow[d, "\gamma"'] & \mathscr{H}_*(S) \arrow[r] & \mathrm{SH}^{\mathrm{eff}}(S) \arrow[r] \arrow[d, "\gamma^*"'] & \mathrm{SH}(S) \arrow[d, "\gamma^*"] \\
\mathrm{Sm}_S^{\mathrm{cor}} \arrow[rr, "M"] & & \mathrm{DM}^{\mathrm{eff}}(S) \arrow[r] & \mathrm{DM}(S)
\end{tikzcd}$$

the two left vertical functors are obtained as a mixture of the Dold-Kan correspondence (which abeliaized the (stable) homotopy types) and the functor "adding transfers" γ^* seen in the preceding course. In fact, it is useful to further factorized the vertical maps through the so-called $\mathbb{A}^1$-derived and stable $\mathbb{A}^1$-derived category, built out of (complexes of) Nisnevich sheaves of abelian groups on Sm_S, that is without transfers.

Note that all functors in this diagram are monoidal, and have a right adjoint. Moreover, γ^* send the motivic sphere $\mathbb{G}_m$ to $\mathbb{Z}_S(1)[1]$.

Of course, stable homotopy and stable motives are quite different. Morel's computating gives a first hint of this difference.

Theorem 2.7 (Morel, [Mor12])**.** *Let k be a perfect field and $n \in \mathbb{Z}$ an integer. Then one gets isomorphisms and a commutative diagram:*

$$\begin{tikzcd}
\pi_n^{st}(\mathbb{G}_m^{\wedge,n}) = [\mathbb{1}_k, \mathbb{G}_m^{\wedge,n}]_{\mathrm{SH}(k)} \arrow[r, "\gamma^*"] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathrm{DM}(k)}(\mathbb{1}_k, \mathbb{1}_k(n)[n]) = H_M^{n,n}(k) \arrow[d, "\sim"] \\
K_n^{MW}(k) \arrow[r, "\mathrm{mod}\ \eta"] & K_n^M(k)
\end{tikzcd}$$

where $K_n^{MW}(k)$ is the Milnor-Witt K-theory of the field k and the lower vertical maps is the canonical morphism of rings sending η to 0.

Remarque 2.8. (1) The identification of Homs in DM with Milnor K-theory uses the cancellation theorem mentioned in the first course. In fact, this theorem can be stated as the fact that the natural stabilization functor $\mathrm{DM}^{\mathrm{eff}}(k) \to \mathrm{DM}(k)$ is fully faithful.

(2) Recall that unlike Milnor K-theory, Milnor-Witt K-theory is negatively graded and one has:

$$K_n^{MW}(k) = \begin{cases} GW(k) & n = 0 \\ W(k) & n < 0. \end{cases}$$

2.2. **A (brief) summary of the six functors formalism.** Based on the operations Proposition 1.16 and thir obvious extension to the stable homotopy category, and most notably on Theorem 1.17, we now have all the ingredients to state the following theorem, due to Ayoub and Voevodsky:

Theorem 2.9. *For any separated morphism of finite type $f : X \to S$, there exists a pair of adjoint triangulated functors*

$$f_! : \mathrm{SH}(X) \leftrightarrows \mathrm{SH}(S) : f^!$$

which is uniquely characterized by the properties:

- $(f \circ g)_! \simeq f_* \circ g_!$.
- *Proper support. there exists a canonical natural transformation $\alpha_f : f_! \to f_*$ compatible with composition, which is an isomorphism if f is proper.*
- *Smooth purity. for f smooth with tangent bundle T_f, there exists a natural* purity *isomorphism:*

$$\mathfrak{p}_f : f_\sharp \to f_!(\mathrm{Th}(T_f) \otimes -)$$

  compatible with composition. Here $\mathrm{Th}(T_f) = \Sigma^\infty T_f / T_f^\times$ is the Thom space, relative to X, associated with the vector bundle T_f/X.

Moreover, this exceptional functoriality satisfies the required property of the so called Grothendieck six functors formalism*:*

- *Base change: $f^! p_* \simeq q_* g^!$.*
- *Projection formula $f_!(\mathbf{E} \otimes f^*(\mathbf{F})) \simeq f_!(\mathbf{E}) \otimes \mathbf{F}$.*

Remarque 2.10. This theorem was first stated by Voevodsky, but he never published his proof. It was then proved by Ayoub in his Ph. D., published in [Ayo07a, 1.4.2] for all the assertions except the projection formula, which was proved in [Ayo07a, Section 2.3]. Actually, the theorem of Ayoub-Voevodsky is proved axiomatically: via the *cross functors formalism* in the terminology of Voevodsky, *monoidal stable homotopy functor* in that of Ayoub. In [CD19], we give another axiomatic framework, that of premotivic/motivic categories, in an attempt to generalize this statement to motivic complexes. We also give a full proof of the theorem,[7] and we refer the reader to [CD19, 2.4.50] for a precise account of the theorem.

Remarque 2.11. The formalism used in [Ayo07a] is that of triangulated categories, which is sufficient for many purposes. In [CD19], we have argued axiomatically, and uses both the formalism of triangulated and model categories. It is formal to extend the language of *loc. cit.* to the formalism of ∞-categories (see for example [Dre18]). However, the above theorem is formulated in terms of triangulated category in

[7] Actually two proofs of the theorem: one reduces to the smooth purity property for $\mathbb{P}^n_S/S$, which was proved by J. Ayoub in [Ayo07a, 1.7.9], and the other one is a direct proof in the particular case where the underlying motivic category is oriented in the sense of [CD19, 2.4.38].

[CD19]. The extension to the formalism of (presentable monoidal and stable) ∞-categories is not obvious at all: the problem is to build $f_!$ (or equivalently $f^!$) as an ∞-functor and to state the compatibility with composition correctly. There is a long history on that problem with many contributions: Liu-Zheng, Gaitsgory-Rosenblyum, Blanc-Robalo-Toën-Vezzosi, Khan.

The fact that f_* is not a derived functor was a known problem in the étale formalism. This is not a problem in general, but it makes some reasoning literally impossible, in particular functoriality statements involving cones.

Example 2.12. Let $f : X \to S$ be a smooth morphism. Inspired by the notations of Voevodsky's homological motives, let us put $\Pi_S(X) = f_! f^!(\mathbb{1}_S)$. Then it follows from the purity isomorphism (and its dual) that:

$$\Pi_S(X) = \Sigma^\infty X_+.$$

To go further, for any vector bundle V over X, let us put:

$$\Pi_S(X, -V) = f_*\big(\operatorname{Th}_X(V)^{\otimes,-1} \otimes f^!(\mathbb{1}_S)\big).$$

Then if f is proper and smooth, with tangent bundle T_f, it follows formally from the above theorem that the object $\Pi_S(X)$ is strongly dualizable with dual $\Pi_S(X, -T_f)$ (see [CD19, 2.4.31]).

This is the $\mathbb{A}^1$-homotopical analogue of the Poincaré duality.

2.13. Because we do not know the localization theorem 1.17 for motivic complexes, we do not have the preceding theorem in general for them. It is known however in several particular cases.

On the other hand, the situation for motives (and torsion étale sheaves prime to the characteristics, Saito's mixed Hodge modules, etc...) Thom spaces always reduces to Tate twists: $\operatorname{Th}(V) = \mathbb{1}_S(r)[2r]$ where r is the rank of V/S. One can still prove the preceding duality result in general for motivic complexes. It has the following more usual form: for X/S smooth proper of dimension d, $M_S(X)$ is rigid in $\mathrm{DM}(S)$ with dual $M_S(X)(-d)[-2d]$ where d is the dimension of X/S

Again, this explains the interest of inverting Tate twists!

3. Algebraic K-theory and Beilinson motives

As mentioned in the first course, Beilinson conjectures explicitly referred to algebraic rational K-theory as the motivic cohomology with rational coefficients for regular bases. This is the idea that we will exploit here.

3.1. **Representability algebraic K-theory.**

3.1. Recall that algebraic K-theory of a ring A has been constructed by Quillen from the classifying space $\mathrm{BGL}(A)$ of the infinite group of invertible matrices with coefficients in A:

$$\mathrm{GL}(A) = \varinjlim_{n>0} \mathrm{GL}_n(A),$$

and the plus construction of a space X which roughly abelianize the first homotopy group of X without changing its homology. Then one puts for $n > 0$:

$$K_n(A) = \pi_n(\mathrm{BGL}(A)^+).$$

Of course many other constructions of algebraic K-theory have been proposed. Morel and Voveodsky come after this rich story and propose the following result, in plain analogy with the representability theorem of complex K-theory in the (usual) homotopy category.

Theorem 3.2 ([MV99], Th. 3.13, p. 140)**.** *Let S be a regular scheme. Then there exists a classifying space BGL_S in $\mathscr{H}_*(S)$ which is a commutative H-monoid and such that there exists a canonical isomorphism of abelian groups:*

$$[S^n \wedge \mathbb{G}_m^{\wedge,m}, \mathbb{Z} \times \mathrm{BGL}_S]_{\mathscr{H}_*(S)} = \begin{cases} K_{n-m}(S) & n \geq m \\ 0 & n < m. \end{cases}$$

where $\mathbb{Z} \times \mathrm{BGL}$ is pointed by $(0, *)$, and is an H-group in $\mathscr{H}_*(S)$.

The main ingredients for this theorem are the following properties of algebraic K-theory: Thomason-Trobaugh's Nisnevich descent theorem, $\mathbb{A}^1$-homotopy invariance regular bases (which easily comes from the existence of K'-theory). However, it also relies on the Morel-Voevodsky's theory of classifying spaces of sheaves of algebraic groups and the ability to give good models from these. As a byproduct, they also obtain a canonical isomorphism in $\mathscr{H}_*(S)$ with the infinite Grassmanian (over S):

$$\mathrm{BGL}_S \simeq \mathrm{Gr}_{\infty,\infty}$$

where $\mathrm{Gr}_{n,m}$ is the Grassmannian scheme, classifying n dimensional sub-vector spaces in $\mathbb{A}_S^{n+m}$.

Remarque 3.3. The isomorphism is functorial in S with respect to pullbacks. Moreover, it follows formally that for any smooth S-scheme X,

$$[S^n \wedge \mathbb{G}_m^{\wedge,i} \wedge X_+, \mathbb{Z} \times \mathrm{BGL}_S]_{\mathscr{H}_*(S)} = \begin{cases} K_{n-i}(X) & n \geq i \\ 0 & \mathrm{n} < m. \end{cases}$$

3.4. Recall that $\mathbb{P}^1 = S^1 \wedge \mathbb{G}_m$. It follows essentially from the above theorem that one has a weak $\mathbb{A}^1$-equivalence:

$$\tau : \mathbb{Z} \times \mathrm{BGL} \to \mathbf{R}\,\underline{\mathrm{Hom}}(\mathbb{P}^1, \mathbb{Z} \times \mathrm{BGL}) = \Omega(\mathbb{Z} \times \mathrm{BGL}).$$

This is the algebraic analog of the Bott periodicity theorem! One can therefore deduce from that map an Ω-spectrum of the form:

$$(\mathbb{Z} \times \mathrm{BGL}, \mathbb{Z} \times \mathrm{BGL}, \dots)$$

where the structural maps are given by the chosen map τ. Elaborating this construction, one gets:

Theorem 3.5 ([Rio10], [PPR09]). *Let S be a regular scheme.*

The following definition does not depend on the choice of τ and gives a well-defined $\mathbb{P}^1$-spectrum $\mathbf{KGL}_S$ such that for any pair $(n,m) \in \mathbb{Z}^2$, there exists an isomorphism of abelian groups:

$$\epsilon_{n,i}^S : [\mathbb{1}_S(i)[n], \mathbf{KGL}_S]_{\mathrm{SH}(S)} \simeq \begin{cases} K_{n-2i}(X) & n \geq 2i \\ 0 & n < 2i. \end{cases}$$

Moreover, there exists a commutative ring structure on $\mathbf{KGL}$ *such that the preceding isomorphism is compatible with products on K-theory as defined Waldhausen.*

More precisely, the existence of the spectrum was obtained in [Rio10], and the construction of the product with the good properties, done in [PPR09].[8]

Remarque 3.6. One can define $\mathbf{KGL}_S$ over an arbitrary scheme S. Actually, we can even put $\mathbf{KGL}_S = f^*(\mathbf{KGL}_\mathbb{Z})$ where $f : S \to \mathrm{Spec}(\mathbb{Z})$ is the unique map. However, as algebraic K-theory is **not** $\mathbb{A}^1$-invariant over singular bases, this object cannot represent algebraic K-theory when S is singular. It was proved by Cisinski in [Cis13] that $\mathbf{KGL}_S$ actually represents Weibel *homotopy invariant K-theory.*

3.2. Beilinson motivic cohomology.

3.7. Let us go back to the origin of (algebraic) K-theory, used by Grothendieck to prove the Riemann-Roch formula in higher dimension, and in a relative form. Let X be a regular scheme. Given a closed (integral) subscheme $Z \subset S$, the structure sheaf $\mathcal{O}_Z$ can be seen as a coherent sheaf on S. As S is regular, this sheaf admits a well-defined class in the K-group of locally free $\mathcal{O}_S$-modules:

$$[\mathcal{O}_Z]_S \in K_0(S).$$

This extends to a morphism of groups from the group of algebraic cycles on S to $K_0(S)$:

$$Z(S) \to K_0(S), \alpha = \sum_i n_i.[Z_i] \mapsto \sum_i n_i.[\mathcal{O}_{Z_i}]_S.$$

In SGA6, Grothendieck proves that this extends to an isomorphism, modulo torsion:

$$\mathrm{CH}(S)_\mathbb{Q} \xrightarrow{\sim} K_0(S)_\mathbb{Q}.$$

The left hand-side, the group of algebraic cycles modulo rational equivalence, is natural graded by codimension. So this gives a natural grading on K_0:

$$\mathrm{Gr}^i K_0(S)_\mathbb{Q} = \mathrm{CH}^i(S)_\mathbb{Q}.$$

In SGA6, this graduation was explained from three different point of views:

- the filtration induced by the codimension of the support of coherent sheaves, and its associated graduation;

[8] In fact, the principle of this construction was singled out by Voevodsky, in his long road to prove the Milnor conjecture: see [Voe99].

- the filtration induced by the so-called λ-ring structure[9] on $K_0(S)$, and the associated γ-filtration;
- as some eigenspace for the so-called Adams operations ψ^k; more precisely, for a fixed $k > 0$:
$$Gr^i K_0(S)_\mathbb{Q} = \mathrm{Ker}\left(\psi^k - k^i.\mathrm{Id}\right).$$

This decomposition was extended, independently by Gillet and Soulé, on higher K-theory. More recently, Riou obtained the following lifting in the stable homotopy category:

Theorem 3.8 ([Rio10], Th. 5.3.10)**.** *Let S be a regular ring. Then there exists a decomposition in* $\mathrm{SH}(S)$

$$\mathbf{KGL}_S \otimes \mathbb{Q} \simeq \bigoplus_{i \in \mathbb{Z}} \mathbf{KGL}_S^{(i)} \tag{3.8.a}$$

such that for any integer i, the associated projector π_i of $\mathbf{KGL}_S$ corresponding to the factor $\mathbf{KGL}_S^{(i)}$ induces via the isomorphism $\epsilon_{s,t}^S$ of Theorem 3.5 the projection on $\mathrm{Gr}^i K_{s-2t}(S)_\mathbb{Q}$.

In other words:
$$[\mathbb{1}_S(s)[t], \mathbf{KGL}_S^{(i)}]_{\mathrm{SH}(S)} \simeq \mathrm{Gr}^{i-s} K_{t-2s}(S)_\mathbb{Q}.$$

Definition 3.9. With the preceding notations, one defines the (rational) Beilinson motivic cohomology spectrum over a regular scheme S as:
$$\mathbf{H}_{\mathrm{B},S} = \mathbf{KGL}_S^{(0)}.$$
We define the Beilinson motivic cohomology of S as:
$$\mathbf{H}_\mathrm{B}^{n,i}(S) := \left[\mathbb{1}_S, \mathbf{H}_{\mathrm{B},S}(i)[n]\right]_{\mathrm{SH}(S)}.$$

Note that the graduation on $\mathbf{KGL}_S \otimes \mathbb{Q}$ is compatible with products. This implies that the ring structure on $\mathbf{KGL}_S \otimes \mathbb{Q}$ induces a ring structure on $\mathbf{H}_{\mathrm{B},S}$. Therefore, Beilinson motivic cohomology is naturally equipped with products.

Example 3.10. From the preceding theorem, one deduces the following formula:
$$\mathbf{H}_\mathrm{B}^{n,i}(S) = Gr^i K_{2i-n}(S)_\mathbb{Q}. \tag{3.10.a}$$
In particular, from Grothendieck isomorphism, one gets
$$\mathbf{H}_\mathrm{B}^{2n,n}(S) = \left[\mathbb{1}_S, \mathbf{H}_{\mathrm{B},S}(n)[2n]\right]_{\mathrm{SH}(S)} \simeq \mathrm{CH}^n(X)_\mathbb{Q}.$$

Remarque 3.11.
(1) The definition of Belinson motivic cohomology through formula (3.10.a) goes back to 1985: see [Sou85].
(2) The $\mathbb{P}^1$-periodicity of K-theory can be seen in decomposiotion (3.8.a), as one deduce that $\mathbf{KGL}^{(i)} = \mathbf{H}_\mathrm{B}(i)[2i]$ — recall $\mathbb{P}^1_S = \mathbb{1}_S(1)[2]$.

[9] This corresponds to the λ-operations defined by the exterior power operators on locally free $\mathcal{O}_S$-modules.

In fact, the motivation for higher Chow groups was to extend the above example. As a consequence of the so-called localization long exact sequence for higher Chow groups, Levine obtained the following optimal result:

Theorem 3.12 ([Lev01])**.** *Let S be a smooth scheme over a field or a Dedekind scheme. Then there exists a canonical isomorphism:*

$$\mathbf{H}_{\mathrm{B}}^{n,i}(S) \simeq \mathrm{CH}^n(X, 2i-n) \otimes \mathbb{Q}.$$

where the right hand-side is Bloch's higher Chow grouups.

3.3. **Beilinson motives.**

3.13. One possible approach to mixed motives with rational coefficients, once we are convinced we have the good theory, is to use Bousfield localization of the rational stable homotopy category: this amount to look only at phenomena (isomorphisms!) that are detected by $\mathbf{H}_{\mathrm{B},S}$.

We really want to work over arbitrary bases now. In fact, one can check that all our constructions are functorial in the regular scheme S. It means that for any morphism of regular schemes $f: T \to S$, one has $\mathbf{H}_{\mathrm{B},T} \simeq f^*(\mathbf{H}_{\mathrm{B},S})$. So we simply put for a general scheme S, with canonical projection $f: S \to \mathrm{Spec}(\mathbb{Z})$, $\mathbf{H}_{\mathrm{B},S} = f^*\mathbf{H}_{\mathrm{B},\mathbb{Z}}$.

In [CD19, Def. 14.2.1], we proposed the following definition:

Definition 3.14. We define the category $\mathrm{DM}_{\mathrm{B}}(S)$ of *Beilinson motives* over S as the localization of the stable presentable ∞-category $\mathrm{SH}(S)_{\mathbb{Q}}$ with respect to morphisms $f: \mathbf{E} \to \mathbf{F}$ such that $f \otimes \mathbf{H}_{\mathrm{B},S}$ is an isomorphism.

Again, we can use the general theory of localization of ∞-category.[10] This means, surprisingly enough, the the category of Beilinson motives is a full subcategory of $\mathrm{SH}(S) \otimes \mathbb{Q}$, made by the $\mathbf{H}_{\mathrm{B}}$*-local spectra*.

The reason why this is a reasonable definition is the following general properties.

Theorem 3.15. *The $\mathbf{H}_{\mathrm{B}}$-localization of the unit spectrum $\mathbb{1}_S = \Sigma^\infty S_+$ is the spectrum $\mathbf{H}_{\mathrm{B}}$.*

Moreover, a spectrum $\mathbf{E}$ is $\mathbf{H}_{\mathrm{B}}$-local if and only if the map

$$u \otimes \mathbf{E} : \mathbb{1}_S \otimes \mathbf{E} \to \mathbf{H}_{\mathrm{B},S} \otimes \mathbf{E}$$

is an isomorphism, where u is the unit of the ring spectrum (commutative monoid) $\mathbf{E}$.

Finally, there exists an equivalence of $\mathbf{H}_{\mathrm{B}}$-local spectra and modules over the commutative monoid $\mathbf{H}_{\mathrm{B},S}$ (in any possible sense!):

$$\mathrm{DM}_{\mathrm{B}}(S) \simeq \mathbf{H}_{\mathrm{B},S} - \mathrm{mod}.$$

[10] or model categories as it was done originally by Bousfield!

As a consequence, one can apply Ayoub-Voevodsky's theorem to the category $\mathrm{DM}_{\mathrm{B}}(S)$. To summarize, we have obtained a stable ∞-category $\mathrm{DM}_{\mathrm{B}}(S)$ satisfying the six functors formalism, and such that if we denote by $\mathbb{1}_S$ the unit object with respct to the monoidal structure on $\mathrm{DM}_{\mathrm{B}}(S)$, the following formula holds for a regular scheme S:

$$\mathrm{Hom}_{\mathrm{DM}_{\mathrm{B}}(S)}(\mathbb{1}_S, \mathbb{1}_S(i)[n]) = Gr^i_\gamma K_{2i-n}(S)_{\mathbb{Q}}.$$

3.16. *Grothendieck-Verdier duality.* Let us mention the following application, which was actually the original motivation behind Grothendieck's formalism.

To fix terminology, and by analogy with the notion of *perfect* complexes, we will say that a rational motive M is *constructible* if it is compact in the triangulated sense: the functor $\mathrm{Hom}(M, -)$ commutes with arbitrary coproducts.

Let $f : X \to S$ be any separated morphism of finite type, such that S is regular. Then the object $D_X = f^!(\mathbb{1}_S)$ is dualizing for constructible motives. More precisely, if we let $\mathcal{D}_X = \mathrm{Hom}_{\mathrm{DM}(S)}(-, D_X)$ then $\mathcal{D}_X$ is an auto-anti-equivalence of the category of constructible rational motives M:

$$M \simeq \mathcal{D}_X(\mathcal{D}_X(M)).$$

This result was proved only when S itself is of finite type over a 2-dimensional excellent scheme in [CD19, Th. 15.2.4]. The general case was obtained in [Cis20, 2.3.2].

Finally, we can link the previous abstract definition with the more concrete one described in the previous course.

Theorem 3.17. *Let S be a regular scheme (or even a geometrically unibranch scheme). Then there the graph functor*

$$\gamma^* : \mathrm{SH}(S) \to \mathrm{DM}(S)$$

where the right hand-side is the stable category of motivic complexes — built out from sheaves with transfers — induces an equivalence of monoidal ∞-categories:

$$\mathrm{DM}_{\mathrm{B}}(S) \xrightarrow{\sim} \mathrm{DM}(S) \otimes \mathbb{Q}.$$

One deduces in particular that for regular schemes, there exists an isomorphism:

$$H^{n,i}_M(S) \otimes \mathbb{Q} \simeq Gr^i_\gamma K_{2n-i}(S)_{\mathbb{Q}}$$

where the left hand-side are the motivic cohomology groups as defined by Voevodsky in [Voe02]. Besides, this isomorphism is compatible with the product on both sides, whereas in SGA6, one deduced a product on Chow groups from the one on algebraic K-theory!

4. Conclusion

This is only the beginning of the story. For rational coefficients, the situation seems very well understood. The above definition can be compared with various other possible candidates, and satisfies several universal properties (see [CD19, Sections 15, 16]). It satisfies a lot of good structural properties: absolute purity, constructibility of the six operations, cohomological descent theorems. An important conjectural property is the existence of the weight structure due to Bondarko. Unfortunately, the remaining properties in Beilinson's conjectures are hard: the existence of the motivic t-structure, and the conservativity conjecture.

For integral coefficients, the situation is more open. A construction has been obtained when restricting to bases which are schemes over a field, using sheaves with transfers and replacing the Nisnevich topology by the cdh-topology. A more general construction has been introduced by Spitzweck in [Spi18] for general schemes by defining a good Eilenberg-MacLane ring spectrum representing motivic cohomology (higher Chow groups for smooth schemes over Dedekind rings). Still it remains to show that the definition does not depend on the choice of the Dedekind ring, and the comparison with Voevodsky's motivic cohomology is open in general.

When working with étale coefficients, the situation is much better and two definitions were proposed and proved to be equivalent: [Ayo14], [CD15]. In fact, these definitions are equivalent to the h-motives introduced in Voevodsky's 1994 Ph. D. thesis! (after appropriate restrictions)

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