Stability at ∞

\+ Østvær, Dubouloz.

diff. geom. / alg. top. / alg. geom.

singularity theory

Mumford Topology normal surf. sing. — Intersection Alg Goresky MacPherson

link, "Plumbing" — perverse sheaves

Hughes-Ranicki — Ends of complexes ⟷ Boundary motives Wildeshaus

Stable $\mathbb{A}^1$-htp ?

I - A ~~tentative~~ definitive ? answer

II - Duality and fundamental class of diagonal

III - Compactifications with NCD boundary

IV - "Mumford's plumber game."

I - Voevodsky's tly motives:

$X \underset{\text{covariant}}{\longmapsto} M(X) \longleftarrow$ singular complex $C_*^{sing}(X(\mathbb{C}))$

$\cap$

$DM(k)$ $\longrightarrow$ $C_*^{suslin}(\mathbb{Z}^{tr}(X))$

motive with compact support : $X \mapsto M^c(X)$

$$\mathrm{Hom}(M^c(X), \mathbb{Z}(n)[p]) = H^{p,n}_{M,c}(X)$$

6 functors formalism for SH

$f : X \to S$ (sep.) of f. type : $M_S(X) = f_! f^!(\mathbb{1}_S)$

$M_S^c(X) = f_* f^!(\mathbb{1}_S)$

$\exists$ nat. tr. $f_! \to f_*$

$\alpha_* : M_S(X) \longrightarrow M_S^c(X)$

Def: $\Pi_S^\infty(X) = \mathrm{hofib}\left(\alpha_X : \Pi_S(X) \to \Pi_S^c(X)\right)$

(recall: X/S sm.: $\Pi_S(X) = \Sigma^\infty X_+$
X/S proper sm. " $\Pi_S^c(X)$)

1) $X = \mathbb{A}^n_S$:

$$\begin{array}{ccc} \Pi_S(\mathbb{A}^n_S) & \xrightarrow{\alpha} & \Pi_S^c(\mathbb{A}^n_S) \\ \wr\wr & & \wr\wr \text{ exercice} \\ \mathbb{1}_S & \xrightarrow[1]{} & \mathbb{1}_S(n)[2n] \end{array}$$

if S is field (or semi-local k-alg.)

this is 0.

$$\Pi_S^\infty(\mathbb{A}^n_S) = \mathbb{1}_S \oplus \mathbb{1}_S(n)(2n-1)$$

$\leadsto$ stable htp type of $\mathbb{A}^n - \{0\}$

2) $S = \mathrm{Spec}(\mathbb{C})$: $SH(\mathbb{C}) \to SH_{top} \xrightarrow{N} D(Ab)$

$\Pi_S(X) \longmapsto C_*^{sing}(X(\mathbb{C}))$

$\Pi_S^c(X) \longmapsto C_*^{lf}(X(\mathbb{C}))$

$\mathrm{Hom}_{cont}(\Delta^*_{top}, X(\mathbb{C}))$

$\Pi_\infty(X) \longmapsto C_*^\infty(X(\mathbb{C}))$

Hughes - Ranicki

3) $\overset{\gamma^*}{\Pi} : SH(S) \longrightarrow DM(S, \Lambda)$ $\quad \Lambda = \mathbb{Z}[1/p]$ - alg.

add "transfers" $\quad S/k \quad \mathrm{char}(k) = p$

Π commutes with the 6 operations

$\Pi(\Pi_S(X)) = M_S(X)$, $\Pi(\Pi_S^c(X)) = M_S^c(X)$

$\Pi(\Pi_S^\infty(X)) = \partial M(X)$ Wildeshaus boundary motive

4) $X = V/S$ vector bundle.

$$\begin{array}{ccc} \Pi_S(V) & \longrightarrow & \Pi_S^c(V) \\ \wr\!\wr & & \wr\!\wr \\ \mathbb{1}_S & \xrightarrow{\ e(V)\ } & \mathrm{Th}(V) \end{array}$$

"motivic" Euler class (Khan, D., Jin)

II. (Jin, Khan, D.)

fundamental class of $f : X \to S$ lci smoothable

L_f = cotangent complex.

$$\exists\ \eta_f : Th(L_f) \otimes f^* \longrightarrow f^!$$

("uniquely" determined by $\eta_f \in H_0(X/S, \langle L_f \rangle)$)

Def | f is h-smooth if (in SH) $\eta_{f,\mathbb{1}_S} : Th(L_f) \to f^!(\mathbb{1}_S)$ is an isomorphism.

eg 1) f an $\cong$ h-sm.

2) $i: Z \to X$ closed im. between schemes sm. over some base the i h-smooth

3) $X \xrightarrow{i} \mathbb{A}^2$ i is not h-sm.

$X = \{x=0\} \cup \{y=0\}$

> X/S sep. of f. type.
> $K \in D_{perf}(\mathcal{O}_X\text{-mod})$
> $\Pi_S(X, \langle K \rangle) = f_!\big(\mathrm{Th}(K) \otimes f^!(\mathbb{1}_S)\big)$

Th. $f : X/S$ k-smooth.

Then η_f induces an iso:

$$\eta_f' : \underline{\mathrm{Hom}}\big(\Pi_S(X, -\langle L_f \rangle), \mathbb{1}_S\big) \xrightarrow{\sim} \Pi_S^c(X)$$

The composite: $\Pi_S(X) \xrightarrow{\alpha_X} \Pi_S^c(X) \simeq \underline{\mathrm{Hom}}\big(\Pi_S(X, -\langle L_f \rangle), \mathbb{1}_S\big)$

corr. to $\alpha'_X : \Pi_S(X) \otimes \Pi_S(X, -\langle L_f \rangle) \longrightarrow \mathbb{1}_S$

f is smooth: α'_X fits into:

$$\begin{array}{ccc}
\Pi_S(X) \otimes \Pi_S(X, -\langle L_f \rangle) & \xrightarrow{\alpha'_X} & \mathbb{1}_S \\
\text{Künneth} \; \wr\!\wr & \circlearrowleft & \uparrow f_* \\
\Pi_S(X \times_S X, -\langle p_1^* L_f \rangle) & \xrightarrow{\delta^!} & \Pi_S(X)
\end{array}$$

α_X can be computed as the fund class of the diagonal

$\delta : X \to X \times_S X$ lci

"$[\Delta_{X/S}]$" $\in \Pi_0(X \times_S X, \langle p_1^* L_f \rangle)$

exercice in the 6 functors formalism.

<u>Def</u>. X/S f.t. scheme; $(X_i)_{i \in I}$ irr. comp. of X.

$J \subset I$ $\quad X_J = \bigcap_{j \in J} X_j$ (scheme) $\quad \cap = - \times_X -$

$X'_J = (X_J)_{red}$

X has (reg) sm. crossings over S if $\forall J \neq \emptyset$, X'_J/S is (reg) sm.

$X \subset \Omega$ smooth Ω/S sm.

say X is a normal crossing ~~dif~~ div. $/S$

if in add. to be an. cross. $/S$

$\forall J \neq \emptyset$, $\mathrm{Codim}_\Omega(X_J)$

$$= \sum_{j \in J} \mathrm{codim}_\Omega(X_j).$$

eg. $\bar{X}/S$ sm. proper, $\partial X \subset \bar{X}$ closed ~~normal~~ subschem

$X = \bar{X} - \partial X$ & $\partial X/S$ sm crossings

X/S sm. $\Pi_S(X)$ is rigid with dual $\Pi_S^c(X, -\langle\langle 1 \rangle\rangle)$

III -

Prop. X/S separated of f. type $\partial X = (\bar{X} - X)_{red}$

$$X \overset{j}{\hookrightarrow} \bar{X} \xrightarrow{p} S$$

j open im

p proper.

Then, there exists comm. diagram.

$$\begin{array}{ccccc}
0 & \longrightarrow & \Pi_S(\partial X) & = & \Pi_S(\partial X) \\
\downarrow & & \downarrow i_* & & \downarrow p_* \\
\Pi_S(X) & \xrightarrow{j_*} & \Pi_S(\bar{X}) & \xrightarrow{\pi} & \Pi_S(\bar{X}/X) = \Pi_S(\bar{X}/\bar{X}-\partial X) \\
\| & & \downarrow j^* & & \downarrow \\
\Pi_S(X) & \xrightarrow{\alpha_*} & \Pi_S^c(X) & \longrightarrow & \Pi_S^\infty(X)[1]
\end{array}$$

Rem: hint: cross the localization htp exact sequences:

$i : \partial X \to X$

$$\begin{array}{ccccc} & & i_* i^! & & \\ & & \downarrow & & \\ j_! j^* & \longrightarrow & 1 & \longrightarrow & i_* i^* \\ & & \downarrow & & \\ & & j_* j^* & & \end{array}$$

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & \bar{X} \\ g \searrow & & \swarrow f \\ & S & \end{array}$$

$$\rightsquigarrow \quad \Pi_S^\infty(X) = g_* \underbrace{i^* j_*}_{\text{"nearly cycle"}} f^! (1_S)(-1)$$

$$\Pi_S^\infty(X) \to \Pi_S(\partial X) \to \Pi_S(\bar X / \bar X - \partial X)$$

$$(1) \qquad\qquad (2)$$

Th X/S sm. $X \hookrightarrow \bar X/S$ proper sm.

$\partial X = (\bar X - X)_{red}$ is a N.C.D. $/S$

$\Pi_S^\infty(X)$ is the htp fiber of:

$$\underset{n \in (\Delta^{inj})^{op}}{\operatorname{hocolim}} \overset{(1)}{''} \Big(\bigoplus_{\substack{J \subset I \\ \#J = n+1}} \Pi_S(\partial_J X) \Big) \longrightarrow \underset{n \in \Delta^{inj}}{\operatorname{holim}} \overset{(2)}{''} \Big(\bigoplus_{\substack{K \subset I \\ \#K = n+1}} \Pi_S(\partial_K X, \langle N_K \rangle) \Big)$$

$\partial X = \bigcup_{i \in I} \partial_i X$, $\quad \partial_J X = \Big(\bigcap_{j \in J} \partial_j X\Big)_{red}$

N_j normal bundle of $\partial_J X \hookrightarrow \bar X$

$$\nu_i : \partial_i X \to \bar{X}$$

$$\left(\begin{array}{c} \vdots \\ \bigoplus_{i_1<i_2} \Pi_S(\partial_{i_1 i_2} X) \\ \left(\nu^{i_1}_{i_1 i_2}\right)_* - \downarrow \quad \downarrow \left(\nu^{i_2}_{i_1 i_2}\right)_* \\ \bigoplus_{i \in I} \Pi_S(\partial_i X) \end{array}\right) \xrightarrow[\mu]{\left(\nu_j^! \nu_{i*}\right)_{i,j \in I^2}} \left(\begin{array}{c} \bigoplus_{j \in I} \Pi_S(\partial_j X, (N^!_j)) \\ \left(\nu^{i_1}_{i_1 i_2}\right)^! \downarrow \quad \downarrow \left(\nu^{i_2}_{i_1 i_2}\right)^! \\ \bigoplus_{j_1<j_2} \Pi_S(\partial_{j_1 j_2} X, (N^!_{j_1 j_2})) \\ \vdots \end{array}\right)$$

$$\nu_i : \partial_i X \to \bar{X}$$

$$\begin{array}{c}
\vdots \\
\bigoplus_{i_1<i_2} \Pi_S(\partial_{i_1 i_2} X) \\
(\nu^{i_1}_{i_1 i_2})_* \; - \downarrow \quad \downarrow (\nu^{i_2}_{i_1 i_2})_* \\
\bigoplus_{i\in I} \Pi_S(\partial_i X)
\end{array}
\xrightarrow[\mu]{(\nu_j^{!}\,\nu_{i*})_{i,j\in I^2} \; =}
\begin{array}{c}
\bigoplus_{j\in I} \Pi_S(\partial_j X, (N^{!}_{j})) \\
(\nu^{i_1}_{i_1 i_2})^{!} \quad \downarrow \; \downarrow (\nu^{i_2}_{i_1 i_2})^{!} \\
\bigoplus_{j_1<j_2} \Pi_S(\partial_{j_1 j_2} X, (N^{!}_{j_1 j_2})) \\
\vdots
\end{array}$$

Proof : (1) $\Pi_S(\partial X) = \operatorname{colim}_{n \in \Delta_{inj}} (\mathcal{O} \;\; \ldots)$

↗ refined check-descent.

(2) Atiyah-duality / refined purity

$$\Pi_S(\overline{X} / \overline{X} - \partial X) \simeq \Pi_S(\partial X, -\langle T|_{\partial X} \rangle)^{\vee}$$

Rem ↪ $i^* j_*$, $i^! j_!$

IV – Mumford's plumbing.

$X/\mathbb{k}$ sm. surface $\mathbb{k}$ = field (≃ smooth semi-local ring over a field)

$X \hookrightarrow \bar{X}$ sm. proj. comp. $/\mathbb{k}$

∂X = union of $\mathbb{P}^1_n$.

$$(1)\quad \bigoplus_{i<j} \Pi(\partial_{ij} X) \rightrightarrows \bigoplus_i \Pi_s(\partial_i X) \xrightarrow{\mu} \bigoplus_j \Pi_s(\partial_j X, \langle N_j \rangle) \rightrightarrows \bigoplus_{j<n} \Pi_s(\partial_{jn} X, \langle N_{jn} \rangle) \quad (2)$$

(N_j: rk 1; N_{jn}: rk 2)

$$\Pi(\partial_j X) \simeq \underline{\mathbb{1}_*} \oplus \mathbb{1}(1)[2]$$

$$(1) = \begin{cases} \mathcal{D}_X = \text{hocof}\left(\bigoplus_{i<j} \mathbb{1}^{ij}_* \xrightarrow{\sum_{i<j} (p^i_{ij})_* - (p^j_{ij})_*} \bigoplus_{i\in I} \mathbb{1}_* \right) \\ \oplus \\ \bigoplus_{i\in I} \mathbb{1}(1)[2] \end{cases}$$

$\partial_{ij} X = \{\text{Spec } k\}$

$$\begin{pmatrix} 1 & & 0 \\ -1 & \ddots & \\ 0 & & \ddots \end{pmatrix} = M$$

$\partial_j X \simeq \mathbb{P}^1_*$

th of bundle on $\partial_j X \simeq \mathbb{P}^1_*$

$N_j \simeq \mathcal{O}(d)$

<u>Lemma</u>: $\left(\underline{\Pi_*}(\partial_j X, \langle N_j \rangle) \right) \simeq Th(N_j^\circ) \oplus Th(N_j^\infty)(1)[2]$

$\to Th(N_j^\infty)(1)[2]$ iff d is even

i.e. N_j is $\underline{SL^c}$-orientable

$Th(N_j^\circ) \to \Pi \to Th(N_j^\infty)(1)[2]$

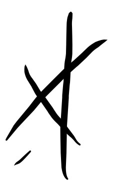
$\uparrow = 0$

$Th(N_j^\infty)(1)(1)$

$$(2) = \left(\mathcal{D}'_X = \mathrm{hofib}\left(\bigoplus_{i \in I} \mathbb{1}^i(2)[4] \longrightarrow \bigoplus_{i<j} \mathbb{1}^{ij}(2)[4] \right)\right)$$

transpose of σ

$\forall$ a, b vanish



$$\bigoplus$$

$$\bigoplus_{j \in I} \mathbb{1}_u(1)[2] -$$

Conf. part. $\qquad$ dual

$$\hookrightarrow \hat{\Pi}_X(X) = \mathrm{hofib}\left(\mathcal{D}_X \oplus \bigoplus_{i \in I} \mathbb{1}(1)[2] \longrightarrow \mathcal{D}'_X \oplus \bigoplus_{j \in I} \mathbb{1}(1)[2] \right)$$

$$\begin{pmatrix} a & b' \\ b & \mu \end{pmatrix}$$

μ = quadratic Mumford matrix

$$\mu = (\mu_{ij})_{i,j \in I^2} \qquad \mu_{ij} : \underline{\mathbb{1}}(r)[2] \longrightarrow \underline{\mathbb{1}}(r)[2]$$

$$\in GW(k)$$

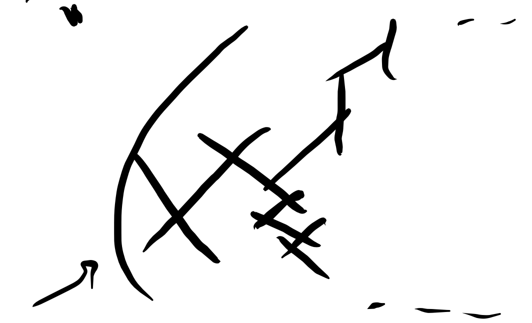
$$= \text{" } \widetilde{\deg}\left([\partial_i X] \cdot [\partial_j X] \right) \text{ "}$$

computed in $\widetilde{CH}(\bar{X})$

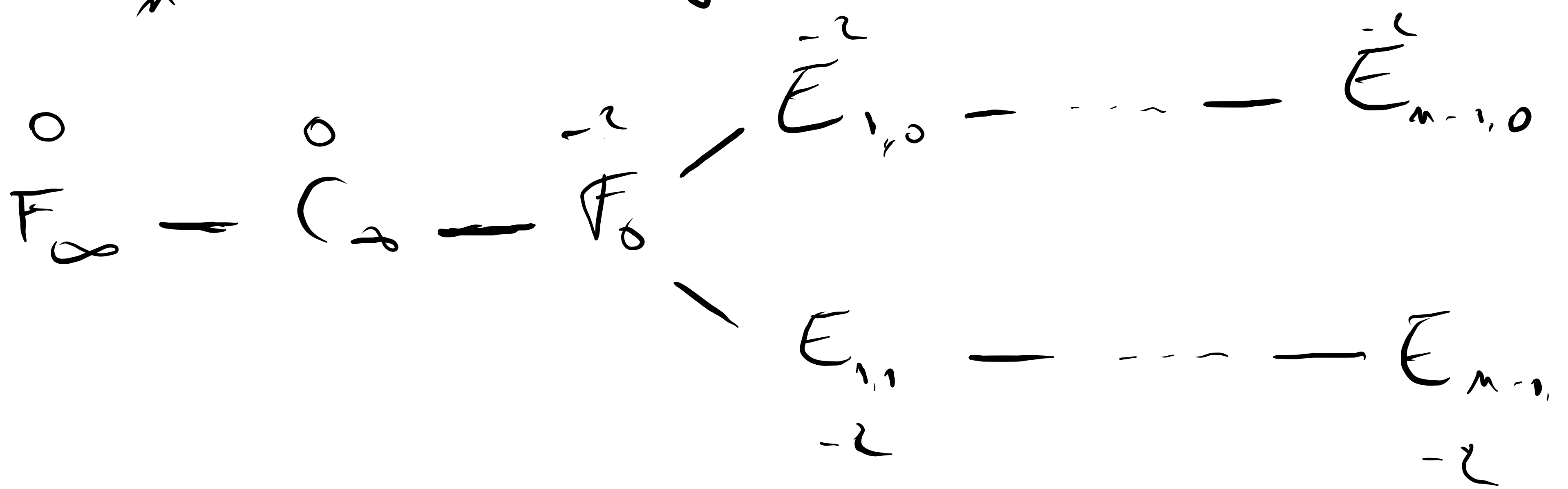
Danilewski: $D_n = \{ x^n z - y(y-1) = 0 \text{ in } \mathbb{A}^3_{\mathbb{C}} \}$

$\exists\ D_n \hookrightarrow \bar{D}_n$ compactification

$$\begin{array}{ccc} \mathbb{A}^2 & \longrightarrow & \mathbb{P}^1 \times \mathbb{P}^1 \\ \uparrow & & \uparrow \\ D_n & \longrightarrow & \bar{D}_n \end{array}$$



∂D_n is NCD by union of $\mathbb{P}^1_{\mathbb{C}}$



$n = 2$

$\mathcal{D}_x = \mathbb{1}_{\mathbb{K}}$ $\qquad$ $\mathcal{D}'_x = \mathbb{1}_{\mathbb{K}}\langle 2\rangle[4]$

$$\begin{array}{c|ccccc} & F_\infty & C_\infty & F_0 & E_{1,0} & E_{1,1} \\ \hline F_\infty & 0 & \langle 1\rangle & 0 & 0 & 0 \\ C_\infty & \langle 1\rangle & 0 & \langle 1\rangle & 0 & 0 \\ F_0 & 0 & \langle 1\rangle & -h & \langle 1\rangle & \langle 1\rangle \\ E_{1,0} & 0 & 0 & \langle 1\rangle & -h & 0 \\ E_{1,1} & 0 & 0 & \langle 1\rangle & 0 & -h \end{array} \in M_{5,5}(GW(k))$$

$\sim$ sm. form $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & 1 & \\ & & 2h \end{pmatrix}$

$e(\mathcal{O}(2)) = -h$

Th

$$\pi_n^{\infty}(D_n) = \mathbb{1}_k \oplus \mathbb{1}_k(2)[3] \oplus \mathrm{hofib}\left(\underset{\substack{\uparrow \\ \mathbb{1}(1)[2] \xrightarrow{\cdot h} \mathbb{1}(1)[2]}}{n} \cdot h\right)$$