QUADRATIC INVARIANTS FOR ALGEBRAIC VARIETIES

F. DÉGLISE

ABSTRACT. These are the (still incomplete) notes of a 4 talks course given at the Spring School "Invariants in Algebraic Geometry ", organized by Daniele Faenzi, Frédéric Déglise, Adrien Dubouloz and Ronan Terpereau.

CONTENTS

1. INTRODUCTION

1.1. **Motivic homotopy theory.** Grothendieck's theory of pure motives aims at defining an abelian invariant of projective smooth varieties universal for the so-called Weil cohomology theory. The idea is based on the fact any such cohomology theory admits cycle classes, and therefore becomes functorial with respect

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to algebraic correspondences. The theory is, till now, stopped by the *standard conjectures* on algebraic cycles.

Nevertheless, the dream of finding a universal, "motivic", invariant is still vivid. The perspective was completely renewed, after Beilinson, by Voevodsky when he uncovered motivic homotopy theory. The main idea is that Weil cohomology theories all extend to open smooth varieties in such a way that the affine line becomes contractible. Therefore, one should do homotopy theory as in classical geometry and topology by replacing the interval with the affine line.

Voevodsky's idea, realized in collaboration with Morel, has since then attracted much attention and obtained several successes, the first of all being the proof of the Milnor and then Bloch-Kato conjectures. Most of all, it has renewed and deepened the link between algebraic topology and algebraic geometry.

There are many well-written surveys on motivic homotopy. In this lecture, I will focus in an unexpected invariant which pops out of this theory, and is fundamental for applications on vector bundles and enumerative geometry: the so-called *quadratic intersection theory.*

1.2. **The Brouwer degree.** As a motivation, I will explain Morel's motivc generalization of the classical Brouwer's degree. Recall the Brouwer degree is the isomorphism from the homotopy classes $[S^n, S^n]$ of endomorphisms of S^n to $\mathbb{Z}$, which basically when $n = 1$ "counts the number of turns".

Consider a dominant algebraic map $f : \mathbb{P}^1_K \to \mathbb{P}^1_K$ defined say over some number field K, with a rational point x such that f is étale over x (this always exists after some finite extension of K).

Given a complex embedding of K, and taking the analytical topology, we get a map $f_\mathbb{C} : S^2 \simeq \mathbb{P}^1_\mathbb{C}(\mathbb{C}) \to \mathbb{P}^1_\mathbb{C}(\mathbb{C}) \simeq S^2$. The point x corresponds to a regular value x of $f_\mathbb{C}$, and the Brouwer degree of $f_\mathbb{C}$ is the integer:

$$d_\mathbb{C} = \deg(f_\mathbb{C}) = \sum_{y/x} +1$$

where the sum runs over the fiber of f at x.

Consider now a real embedding of K, and consider the euclidean topology to get a map: $f_\mathbb{R} : S^1 \simeq \mathbb{P}^1_\mathbb{R}(\mathbb{R}) \to \mathbb{P}^1_\mathbb{R}(\mathbb{R}) \simeq S^1$. We choose an orientation of S^1: this gives us a notion of positivity on its tangent bundle T_{S^1}. Again the point x is a regular value of $f_\mathbb{R}$ and the Brouwer degree is now given by the formula:

$$d_\mathbb{R} = \deg(f_\mathbb{R}) = \sum_{y/x} \epsilon_y(f) \tag{1.2.0.a}$$

where $\epsilon_y(f) = \pm 1$ if $d_y f/dt$ respects or reverse the orientation (in fact this counts the number of turns of f) !

We thus have two notions of degree $(d_{\mathbb{C}}, d_{\mathbb{R}})$ associated with f. They are in fact related. Indeed, consider the K-vector space

$$V = \sum_{y/x} \kappa(y)$$

given by the residue fields of the elements of the algebraic fiber of f over x. Then one defines a quadratic form on V by the formula:

$$\delta_f : V \to K, \sum_y v_y \mapsto \sum_{y/x} \mathrm{Tr}_{\kappa(y)/K}(df/dt(y).v_y^2).$$

When $k = \mathbb{R}$, one can check that the rank of δ_f is $d_{\mathbb{C}}$ and the signature of δ_f is $d_{\mathbb{R}}$. Thus, this invariant captures both the real and complex topological behavior of f: its isomorphism class, in the ring $\mathrm{GW}(K)$, is called the motivic degree of f.

In algebraic geometry, degrees (for example in enumerative geometry) have been incarnated by the notion of a degree of 0-cycles. The aim of this lecture is to explain how one can build a theory of algebraic cycles with coefficients in *quadratic forms*.

1.3. **Bloch's formula.** To motivate and explain our constructions, I want to first recall Bloch's beautiful formula. Let X be a smooth k-scheme, and let $\mathrm{CH}^n(X)$ be the group of algebraic cycles modulo rational equivalence. Then Bloch's formula gives an isomorphism:

$$\mathrm{CH}^n(X) \simeq H^n_{\mathrm{Zar}}(X, \mathcal{K}_n)$$

where the right hand-side is the Zariski cohomology of X with coefficients in the Zarski sheaf associated with algebraic K-theory in degree n.

The main explanation for this isomorphism comes out of a theorem of Quillen, which says that the group $\mathrm{CH}^n(X)$ can be computed by the Gersten complex associated with K-theory, which for a regular scheme X has the following form:

(1.3.0.a) $$\ldots \to \oplus_{y \in X^{(n-2)}} \underbrace{\mathrm{K}_2(\kappa_z)}_{\mathrm{K}_2^M(\kappa_z)} \to \bigoplus_{y \in X^{(n-1)}} \underbrace{\mathrm{K}_1(\kappa_y)}_{\kappa_y^\times} \xrightarrow{\mathrm{div}} \underbrace{\oplus_{x \in X^{(n)}} \underbrace{\mathrm{K}_0(\kappa_x)}_{\mathbb{Z}}}_{Z^n(X)}$$

according to the following notation:

(1) $X^{(n)}$ denotes the set of codimension n points x of X – the dimension of the local ring $\mathcal{O}_{X,x}$.
(2) κ_x the residue field of such a point, $\kappa_x = \mathcal{M}_x/\mathcal{M}_x^2$, $\mathcal{M}_x \subset \mathcal{O}_{X,x}$ maximal ideal.
(3) $Z^n(X)$ is the group of n-codimensional algebraic cycles: the free abelian group generated by the closed integral subschemes of codimension n, which amounts to the $x \in X^{(n)}$ (take the reduced closure)
(4) $K^\times$ is the group of units of a field K

(5) div associates to $Y \subset X$ integral closed with generic point $y \in X^{(n-1)}$, to a global invertible function $f \in \kappa_y^\times$, its divisor $\mathrm{div}(f) = \sum_{x \in Y^{(1)}} \mathrm{ord}_x(f).x$, seen as a codimension n algebraic cycle of X.[1]

By definition, the Chow group of X in codimension n is the cokernel of the last map div. This will be our guiding definition to formulate a quadratic intersection theory, the so-called Chow-Witt ring introduced by Barge and Morel ([BM00]), developed by Fasel ([Fas08, Fas07]) with recent important developments due to Feld ([Fel20a]).

Conventions

Schemes are always assumed to be noetherian of finite dimension. Given such an X, $X^{(p)}$ is the set of points $x \in X$ of codimension p.

Notations:

$X^{(p)}$	codimension p points
κ_x	residue field of a point x (of a given scheme)
κ_v	residue field of a valuation v (of a given field)
$\mathcal{L}_x$	pullback to a point x (of a given scheme)

2. Morel's K-theory

2.1. Grothendieck-Witt groups and symmetric bilinear forms.

2.1.1. (cf. [MH73]) Let K be a field. An *inner product space* (E, ϕ) over K is a finite K-vector space E with a bilinear form

$$\phi : E \otimes_K E \to K$$

which is symmetric and non degenerate: $E \to E^\vee, x \mapsto \phi(x, -)$ is an isomorphism. The dimension of E/K is called the *rank* of the inner product space (E, ϕ).

The category of inner product spaces admits direct sums and tensor products:

$$(E, \phi) \perp (F, \psi) \to (E \oplus F, \phi + \psi)$$
$$(E, \phi) \otimes (F, \psi) \to (E \otimes_K F, \phi.\psi).$$

Therfore the set of I_K isomorphism classes of inner product spaces over K is commutative monoid for $\oplus$, and a commuative semi-ring for $\oplus, \otimes$.

Definition 2.1.2 (Knebusch)**.** The *Grothendieck-Witt ring* $\mathrm{GW}(K)$ of K is the group completion of the monoid $(I_K, \oplus)$, and products induced by the tensor product $\otimes$.

The rank of an inner product space induces a ring map:

(2.1.2.a) $$\mathrm{GW}(K) \xrightarrow{\mathrm{rk}} \mathbb{Z}.$$

[1] See below for a definition of $\mathrm{ord}_x(f)$, the order of f at x.

Remarque 2.1.3. If the characteristic of K is different from 2, for any K-vector space V, there is a one to one correspondence between the symmetric bilinear form ϕ on V and the quadratic forms q: $q(x) = \phi(x, x)$. Then the Grothendieck-Witt ring can be defined in terms of isomorphism classes of quadratic forms.

This is no longer true in characteristic 2, but the definition based on inner product spaces is the correct one for $\mathbb{A}^1$-homotopy. Nevertheless, one abusively use terms such as *quadratic* intersection theory, in any characteristic — this sounds much better than *inner product space intersection theory* !

Example 2.1.4.
(1) Let a be a unit in K. Then $K \otimes K \xrightarrow{a.m} K, (x, y) \mapsto a.xy$ is an inner product space of rank 1. Its class in the Grothendieck-Witt ring is denoted by $\langle a\rangle$. Obviously, $\langle ab^2\rangle = \langle a\rangle$. Therefore, one has a canonical map:
$$Q(K) := K^\times/(K^\times)^2 \to \mathrm{GW}(K).$$
(2) Given units $a_i \in K^\times$, we put $\langle a_1, \dots, a_n\rangle = \langle a_1\rangle + \dots + \langle a_n\rangle$.
A bilinear form on a framed K-vector space is defined by a symmetric invertible matrix. The above element of $\mathrm{GW}(K)$ is represented by the K-vector space K^n and the diagonal matrix with coefficients a_i.

Example 2.1.5.
(1) Obviously if K is an algebraically closed field the rank map $\mathrm{rk} : \mathrm{GW}(K) \to \mathbb{Z}$ is an isomorphism. In fact, rk is an isomorphism whenver (-1) is a square in K.
(2) It is well-known that a quadratic form over a real vector space is determined by its signature. In other words, any $\sigma \in \mathrm{GW}(\mathbb{R})$ can be uniquely written as $\sigma = p.\langle 1\rangle + q.\langle -1\rangle$, $\mathrm{rk}(\sigma)p + q$ and the signature of σ is defined as the pair (p, q). The map $\mathrm{GW}(\mathbb{R}) \to \mathbb{Z} \oplus \mathbb{Z}, \sigma \mapsto (p, q)$ is an isomorphism.
(3) Let $K = \mathbb{F}_p$ be a finite field of characteristic $p > 0$. Then the following sequence of abelian groups is exact:
$$0 \to Q(\mathbb{F}_p) \to \mathrm{GW}(\mathbb{F}_p) \xrightarrow{\mathrm{rk}} \mathbb{Z} \to 0$$
$$a \mapsto \langle a\rangle - 1$$
Note that this fits with item (1) above !

Consider the notations of Example 2.1.4. The element $h = \langle 1, -1\rangle$ is called (slightly abusively) the *hyperbolic form*. The following definition is the most famous one in the theory of "quadratic forms".

Definition 2.1.6. One defines the Witt ring of a field K as the quotient ring:
$$\mathrm{W}(K) = \mathrm{GW}(K)/(h).$$

The hyperbolic form bieng of rank 2, the map (2.1.2.a) induces a morphism of rings:
$$W(K) \to \mathbb{Z}/2$$
which is again called the *rank map*.

Here is an important theorem, which is in essence due to Milnor and Husemoller (but from what I know first remarked by Morel under this form):

Theorem 2.1.7. *The ring* $\mathrm{GW}(k)$ *admits a presentation (as a ring) whose generators are given by symbols* $\langle a\rangle$ *for* $a \in K^\times$ *(mapping to the elements of Example 2.1.4) and relations are given by, whenever they make sense:*
(GW1) $\langle ab^2\rangle = \langle a\rangle$
(GW2) $\langle a, -a\rangle = \langle 1, -1\rangle$
(GW3) $\langle a, b\rangle = \langle a+b, (a+b)ab\rangle$
where we have used the notation $\langle a, b\rangle := \langle a\rangle + \langle b\rangle$.

Remarque 2.1.8. From the theory of symmetric bilinear forms, it is natural to consider $\mathcal{L}$-valued inner product space for an arbitrary invertible K-vector space L: symmetric bilinear form $V \otimes_K V \to \mathcal{L}$ such that the adjoint map $V \to \mathrm{Hom}_K(V, \mathcal{L})$ is an isomorphism.

This gives well-defined rings $\mathrm{GW}(K, \mathcal{L})$ and $\mathrm{W}(K, \mathcal{L})$. These rings are *non canonically* isomorphic to their untwisted counter-parts. We will meet this variation again soon. Let us already mention that the twists will later be responsible for an orientation theory in the quadratic intersection theory.

2.2. **Main definition.**

2.2.1. *Milnor K-theory.* The first step in our "quadratic" adventure is to find a proper "quadratic" analogue of K-theory. We first elaborate on the computation of the last three terms of the Gersten complex (1.3.0.a). In fact, Milnor defined what we now called the Milnor K-theory $KM*(K)$ of a field K as the $\mathbb{Z}$-graded algebra generated by symbols $\{a\}$ in degree $+1$ for $a \in K^\times$ modulo the relations
(M1) $\{a, 1-a\} = 0$
(M2) $\{ab\} = \{a\} + \{b\}$
where we have put $\{a_1, \ldots, a_n\} = \{a_1\} + \ldots + \{a_n\}$.

Note in particular:
$$\mathrm{K}_0^M(K) = \mathbb{Z}, \mathrm{K}_1^M(K) = K^\times.$$
Then we have in fact used in (1.3.0.a) that there is a canonical *symbol map* with values in (Quillen) algebraic K-theory:
$$\mathrm{K}_n^M(K) \to \mathrm{K}_n(K)$$
which induces an isomorphism if (and only if) $n \leq 2$.

Remarque 2.2.2. The cases $n < 2$ follow from the definitions, while the case $n = 2$ is a difficult theorem (given the definition of Quillen K-theory), Matsumoto's theorem. The cokernel of the symbol map is called the *indecomposable part* of algebraic K-theory.

We have all the tools to formulate the Milnor conjecture, now a theorem due to Orlov, Vishik and Voevodsky:

Theorem 2.2.3 (Orlov, Vishik, Voevodsky)**.** *Let K be a field of characteristic not 2.*

Let $I(K)$ be the kernel of the canonical rank map $\mathrm{rk} : \mathrm{W}(K) \to \mathbb{Z}/2$ *— called the* fundamental ideal.

Given a unit $u \in K^\times$, we put $\langle\langle u \rangle\rangle := 1 - \langle u \rangle$ seen as an element in $\mathrm{W}(K)$ *— this is called the* Pfister form *associated with u.*

Then for any $n \geq 0$, the map $K^\times \to \mathrm{W}(K), u \mapsto \langle\langle u \rangle\rangle$ induces a ring morphism:

$$\mathrm{K}_*^M(K)/2\,\mathrm{K}_*^M(K) \to \oplus_{n\geq 0} I^n(F)/I^{n+1}(F)$$

which is an isomorphism.

See [OVV07] or [Mor05].

2.2.4. In our quadratic quest, we now look for an analogue which will mix generators and relations of Milnor K-theory and of the Grothendieck-Witt ring:

Definition 2.2.5 (Morel)**.** Let K be any field. We define the *Milonr-Witt ring* $\mathrm{K}_*^{MW}(K)$ of K, that we will call *Morel K-theory* as the $\mathbb{Z}$-graded ring with the following presentations:

Generators are given by symbols $[a]$ of degree $+1$ for $a \in K^\times$, and a symbol η of degree -1 called the *Hopf element*. Let us introduce the following notations to formulate the relations:

$$\begin{aligned} [a_1, \ldots, a_n] &= [a_1] \ldots [a_n] \\ h &= 2 + \eta[-1] \end{aligned}$$

Relations are given as follows, whenever they make sense:

(MW1) $[a, 1-a] = 0$
(MW2) $[ab] = [a] + [b] + \eta.[a, b]$
(MW3) $\eta[a] = [a]\eta$
(MW4) $\eta h = 0$

Obviously, Morel K-theory is a covariant functor with respect to inclusion morphisms of fields. Given such a map $\varphi : K \to L$, the natual morphism of $\mathbb{Z}$-graded ring (homogeneous of degree 0):

$$\varphi_* : \mathrm{K}_*^{MW}(K) \to \mathrm{K}_*^{MW}(L)$$

is usually called the *corestriction*.

Remarque 2.2.6. Given any (Noetherian) ring A, the preceding definition makes sense so that we can define the ring $\mathrm{K}_*^{MW}(A)$. The resulting $\mathbb{Z}$-graded ring is covariantly functorial in the ring A. This extended definition will especially be useful when A is a semi-local ring.

2.2.7. *Relation with Milnor K-theory.* Given the above presentation, compared to that of Milnor K-theory, if one kills η in the latter, one recovers the former. In other words, the composite map

$$f : \mathrm{K}^M_*(K) \hookrightarrow \mathrm{K}^{MW}_*(K) \to \mathrm{K}^{MW}_*(K)/(\eta)$$

is an isomorphism of $\mathbb{Z}$-graded rings. In other words, we get an exact sequence for $q \in \mathbb{Z}$: Moreover, because of

$$\mathrm{K}^{MW}_{q+1}(K) \xrightarrow{\eta} \mathrm{K}^{MW}_q(K) \xrightarrow{f} \mathrm{K}^M_q(K) \to 0. \tag{2.2.7.a}$$

Let us introduce some further terminology.

Definition 2.2.8. The morphism of $\mathbb{Z}$-graded ring $f : \mathrm{K}^{MW}_*(K) \to \mathrm{K}^M_*(K)$ is called the *forgetful map*.
We also consider the following morphism of $\mathbb{Z}$-graded abelian groups,

$$H : \mathrm{K}^M_*(K) \to \mathrm{K}^{MW}_*(K), \sigma \mapsto h.\sigma$$

called the *hyperbolic map*.

2.2.9. By definition, the forgetful map f is characterized by the properties:

$$f(\eta) = 0, \quad f([a]) = \{a\}.$$

One easily deduces from this characterization and the definition of h, that the composite map:

$$\mathrm{K}^M_*(K) \xrightarrow{H} \mathrm{K}^{MW}_*(K) \xrightarrow{f} \mathrm{K}^M_*(K)$$

is multiplication by 2. Finally, relation (MW4) implies that the image of the hyperbolic map H lands in the kernel of the Hopf map $\eta : \mathrm{K}^{MW}_q(K) \to \mathrm{K}^{MW}_q(K)$.

Remarque 2.2.10. According to relation (MW4), the image of H is the ideal $(h) \subset \mathrm{K}^{MW}_*(K)$ generated by the hyperbolic element. Moreover, if K is of characteristic not 2, it follows from the Milnor conjecture Theorem 2.2.3 that $\mathrm{Im}(H) = \mathrm{Ker}(\eta)$.

Therefore, if $\mathrm{char}(K) \neq 2$, one can extend (2.2.7.a) in an exact sequence

$$\mathrm{K}^M_{q+1}(K) \xrightarrow{H} \mathrm{K}^{MW}_{q+1}(K) \xrightarrow{\eta} \mathrm{K}^{MW}_q(K) \xrightarrow{f} \mathrm{K}^M_q(K) \to 0. \tag{2.2.10.a}$$

Let us come back to the study of the general groups $\mathrm{K}^{MW}_*(K)$. It is easy to get the following presentation of each individual graded parts, as abelian groups:

Lemma 2.2.11. *Consider an integer $n \in \mathbb{Z}$. Then the abelian group $\mathrm{K}^{MW}_n(K)$ is generated by symbols of the form:*

$$[\eta^r, a_1, \dots, a_{n+r}], r \geq 0, a_i \in K^\times$$

modulo the following three relations:

(MW1ab) $[\eta^r, a_1, \dots, a_{n+r}] = 0$ *if* $a_i + a_{i+1} = 1$ *for some* i

(MW2ab) $[\eta^r, a_1, \dots, a_i b_i, \dots, a_{n+r}] = [\eta^r, a_1, \dots, a_i, \dots, a_{n+r}] + [\eta^r, a_1, \dots, b_i, \dots, a_{n+r}] + [\eta^{r+1}, a_1, \dots, a_i, b_i, \dots, a_{n+r}]$

(MW4ab) $[\eta^r, a_1, \dots, -1, \dots, a_{n+r-1}] = -2[\eta^r, a_1, \dots, \cancel{-1}, \dots, a_{n+r-1}]$ *for* $r \geq 2$

2.3. **Relations with quadratic forms.** Using the presentation obtained in the lemma just above, together with the presentation of Grothendieck-Witt groups Theorem 2.1.7, Morel deduces the following computation (for full details, see [Car, Prop. 1.9, Lem. 1.3]):

Proposition 2.3.1 (Morel)**.** *The following map is well-defined*

$$\mathrm{GW}(K) \to \mathrm{K}_0^{MW}(K), \langle a \rangle \mapsto 1 + \eta.[a]$$

and induces an isomorphism of rings.

For any $n > 0$, the multiplication map: $\mathrm{K}_0^{MW}(K) \xrightarrow{\eta^n} \mathrm{K}_{-n}^{MW}(K)$ induces an isomorphism:

$$\mathrm{W}(K) = \mathrm{GW}(K)/(h) \to \mathrm{K}_{-n}^{MW}(K).$$

Finally, for any $n \geq 0$, the abelian group $\mathrm{K}_n^{MW}(K)$ is generated by symbols of the form $[a_1, \ldots, a_n]$ for units $a_i \in K^\times$.

As a consequence, we will view the elements of $\mathrm{GW}(K)$ as elements in degree 0 of Morel K-theory. Note moreover, that $\mathrm{GW}(K)$ lands in the center of the ring $\mathrm{K}_*^{MW}(K)$.

Example 2.3.2.
(1) The notation $h \in \mathrm{K}_0^{MW}(K)$ in relation (MW4) was therefore justified, as it corresponds to the hyperbolic form. Note that relation (GW2) in Theorem 2.1.7 can be written as:

$$\forall u \in K^\times, \langle u \rangle.h = h. \tag{2.3.2.a}$$

Remark also that $h^2 = 2.h$ (direct computation).

(2) Another important element is:

$$\epsilon = -\langle -1 \rangle \in \mathrm{K}_0^{MW}(K).$$

Then relation (MW4) translates to $\epsilon.\eta = \eta$. Moreover, one gets that the $\mathbb{Z}$-graded ring $\mathrm{K}_*^{MW}(K)$ is *ϵ-commutative*:

$$\forall(\alpha, \beta) \in \mathrm{K}_n^{MW}(K) \times \mathrm{K}_m^{MW}(K), \alpha\beta = \epsilon^{nm}.\beta\alpha.$$

To prove this formula, one reduces to the case $\alpha = [a]$, $\beta = [b]$ for units a, b (see [Car, Cor. 1.5]).

Example 2.3.3. Using Example 2.1.5(1), if every unit in K admits a square root, one deduces:

$$\mathrm{K}_n^{MW}(K) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}/2 & n < 0 \\ K_n^M(K) & n > 0. \end{cases}$$

Recall also that if K is algebraically closed, $\mathrm{K}_n^{MW}(K) = \mathrm{K}_n^M(K)$ is torsion free for all n and divisible for $n > 1$. These are therefore huge groups !

2.3.4. *Quadratic multiplicities.* One associates to any integer $n \in \mathbb{Z}$ the following element of $K_0^{MW}(K)$:

$$n_\epsilon = \begin{cases} \sum_{i=0}^{n-1}(-\epsilon)^i & n \geq 0 \\ \epsilon.(-n)_\epsilon & n < 0. \end{cases}$$

An equivalent computation:

$$n_\epsilon = \begin{cases} m.h & n = 2m \\ m.h + 1 & n = 2m+1 \end{cases}$$

Beware that the induced arrow $\mathbb{Z} \to \mathrm{K}_0^{MW}(K), n \mapsto n_\epsilon$ is a monoid morphism for multiplication

$$(nm)_\epsilon = n_\epsilon m_\epsilon$$

but not for the addition (compute 3_ϵ and 4_ϵ).

Remarque 2.3.5. (1) A principle of quadratic enumerative geometry is that, under a careful choice of orientations, degrees of classical enumerative geometry should be replaced by ϵ-degrees as defined above.

(2) With the previous notation, relation (MW4) translates to:

$$2_\epsilon.\eta = 0$$

This should remind the reader of the fact the classical/topological Hopf map $\eta : S^3 \to S^2$ induces a 2-torsion element in homotopy, which account for the isomorphism:

$$\pi_3^{st}(S^2) = \mathbb{Z}/2.\eta$$

where the left hand-side group is the third stable homotopy group of S^2.

(3) Note also that one ca write the image of the hyperbolic map H as: $\mathrm{Im}(H) = 2_\epsilon.\,\mathrm{K}_*^M(K)$ as a (graded) ideal of $\mathrm{K}_*^{MW}(K)$. We will also retain from Remarque 2.2.10, for future purposes the following computation if $\mathrm{char}(K) \neq 2$:

$$\mathrm{Ker}(\eta) = 2_\epsilon.\,\mathrm{K}_*^M(K).$$

(4) In negative degree, the quadratic multiplicities n_ϵ became drastically simpler ! Indeed, modulo h or equivalently in $\mathrm{W}(K)$,

$$n_\epsilon = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd.} \end{cases}$$

2.4. **Twists.** To account to orientability theory that we have already encountered in the definition of the real degree $d_\mathbb{R}(f)$ in (1.2.0.a), we introduce along the lines of Remark 2.1.8 the following definition.

Definition 2.4.1. Let K be a field, and $\mathcal{L}$ an invertible (*i.e.* of dimension one over K) K-vector space. Consider the set: $\mathcal{L}^\times = \mathcal{L} - \{0\}$. The action of $K^\times$ on

$\mathrm{K}_*^{MW}(K)$, via the map $K^\times \to \mathrm{K}_0^{MW}(K), a \mapsto \langle a \rangle$, and on $L^\times$ by scalar multiplication, gives morphism of rings:

$$\mathbb{Z}[K^\times] \to \mathrm{K}_*^{MW}(K), \mathbb{Z}[K^\times] \to \mathbb{Z}[\mathcal{L}^\times].$$

We defined the *$\mathcal{L}$-twisted Morel K-theory* of K (or simply the Morel K-theory of $(K, \mathcal{L})$) in degree $n \in \mathbb{Z}$ as the following abelian group:

$$\mathrm{K}_n^{MW}(K, \mathcal{L}) : \mathrm{K}_n^{MW}(K) \otimes_{\mathbb{Z}[K^\times]} \mathbb{Z}[\mathcal{L}^\times].$$

Elements of $\mathrm{K}_n^{MW}(K, \mathcal{L})$ are therefore formals sums of elements of the form $\sigma \otimes l$ where $\sigma \in \mathrm{K}_n^{MW}(K)$ and $l \in \mathcal{L}^\times$. Note that l is a base of K-vector space $\mathcal{L}$, which we call the *parameter* (hinted "of $\mathcal{L}/K$").

Example 2.4.2. Let $(K, \mathcal{L})$ as above. Then for any $n \geq 0$, the isomorphism of Proposition 2.3.1 induces a *canonical* isomorphism:

$$\mathrm{GW}(K, \mathcal{L}) \xrightarrow{\simeq} \mathrm{K}_0^{MW}(K, \mathcal{L})$$
$$\mathrm{W}(K, \mathcal{L}) \xrightarrow{\simeq} \mathrm{K}_n^{MW}(K, \mathcal{L}), n < 0.$$

where the left hand-side was defined in Remarque 2.1.8. Indeed, it suffices to use the isomorphism:

$$\mathrm{GW}(K) \otimes_{\mathbb{Z}[K^\times]} \mathbb{Z}[\mathcal{L}^\times] \to \mathrm{GW}(K, \mathcal{L}), [\phi] \otimes l \mapsto [\phi.l].$$

Remarque 2.4.3. We consider again the situation of Remark 2.2.6, and assume that A is regular semi-local (noetherian). Let $\mathcal{L}$ be an invertible[2] A-module. As A is regular semi-local, $\mathcal{L}$ is trivializable: in fact, for any $l \in \mathcal{L}^\times := \mathcal{L} - \{0\}$, the map $\Theta_l : A \to \mathcal{L}, \lambda \mapsto \lambda.l$ is an isomorphism.[3]

Moreover, the definition of $\langle a \rangle = 1 + \eta.[a]$ makes sense for any unit $a \in A^\times$. Thus we can define:

$$\mathrm{K}_n^{MW}(A, \mathcal{L}) = \mathrm{K}_*^{MW}(A) \otimes_{\mathbb{Z}[A^\times]} \mathbb{Z}[\mathcal{L}^\times]$$

2.4.4. *Basic operations on twisted Morel K-theory.* We have the following structure on twisted Morel K-theory:

(1) products:

$$\mathrm{K}_n^{MW}(K, \mathcal{L}) \otimes \mathrm{K}_m^{MW}(K, \mathcal{L}') \to \mathrm{K}_{n+m}^{MW}(K, \mathcal{L} \otimes \mathcal{L}'), (\sigma \otimes l, \tau \otimes l') \mapsto (\sigma.\tau, l \otimes l').$$

(2) First functoriality: given a morphism of field $\varphi : K \to L$, one gets:

$$\varphi_* : \mathrm{K}_n^{MW}(K, \mathcal{L}) \to \mathrm{K}_n^{MW}(L, \mathcal{L} \otimes_K L), (\sigma, l) \mapsto (\varphi_*(\sigma), l \otimes_K 1_L).$$

(3) Second functoriality: given an isomorphism of K-vector spaces $\theta : \mathcal{L} \to \mathcal{L}'$ one gets:

$$\theta_* : \mathrm{K}_n^{MW}(K, \mathcal{L}) \to \mathrm{K}_n^{MW}(L, \mathcal{L}'), (\sigma, l) \mapsto (\sigma, \Theta(l)).$$

which is an isomorphism of abelian groups.

[2] *i.e.* locally free of rank 1

[3] This comes from the isomorphism $\mathrm{Pic}(A) = \mathbb{Z}$.

Remarque 2.4.5. It is possible to unite the first and second functoriality. One considers the category of *twisted fields* $\mathscr{T}\mathscr{F}$ objects are pairs $(K, \mathcal{L})$ of a field and invertible K-vector space and morphisms are given by

$$(\varphi, \Theta) : (K, \mathcal{L}) \to (L, \mathcal{L}')$$

where $\varphi : K \to L$ is a morphism of fields, and $\Theta : \mathcal{L} \otimes_K L \to \mathcal{L}'$ is an isomorphism. Composition is defined in the obvious way. Then K_*^{MW} becomes a covariant functor from the category of twisted fields to the category of graded abelian groups.

The category of twisted fields is *cofibred* over the category of fields $\mathscr{F}$. To interpret correctly the tensor product, via a *symmetric* monoidal structure, one has to consider the graded category of twisted fields. This is obtained via the Grothendieck construction applied to the graded Picard category over fields (see [Del87] for this category and [Fas] for the monoidal structure).

2.4.6. We will identify the untwisted group $\mathrm{K}_*^{MW}(K)$ with $\mathrm{K}_*^{MW}(K, K)$ via the canonical isomorphism:

$$\mathrm{K}_*^{MW}(K) \to \mathrm{K}_*^{MW}(K, K), \sigma \mapsto \sigma \otimes 1.$$

Further, given any choice of $l \in \mathcal{L}^\times$, we get an isomorphism of invertible K-vector spaces $\Theta_l : K \to \mathcal{L}, \lambda \mapsto \lambda.l$ and therefore a *trivialization*:

$$\mathrm{K}_*^{MW}(K, \mathcal{L}) \xrightarrow{\Theta_l^{-1}} \mathrm{K}_*^{MW}(K, K) = \mathrm{K}_*^{MW}(K).$$

Thus the twisted groups $\mathrm{K}_*^{MW}(K, \mathcal{L})$ are all abstractly isomorphic, but (again!) via a *non-canonical* isomorphism.

Example 2.4.7. Consider $(K, \mathcal{L})$ as above. Remark that the action of $K^\times$ on $\mathrm{K}_*^{MW}(K)$ via the map $u \mapsto \langle u \rangle$ is trivial: indeed, $\langle u \rangle = 1 \mod \eta$! This implies that $\mathrm{K}_*^{MW}(K, \mathcal{L})/\eta$ is canonical isomorphic to $\mathrm{K}_*^{MW}(K)/\eta$, which I recall is just the Milnor K-theory of K. This will later explain why no orientation theory is visible on usual Chow groups!

We further extends Definition 2.2.8 as follows.

Definition 2.4.8. Let $(K, \mathcal{L})$ be a twisted field. Then one defines the twisted forgetful (resp. Hopf) maps:

$$\begin{aligned} f &: \mathrm{K}_*^{MW}(K, \mathcal{L}) \to \mathrm{K}_*^{M}(K), (\sigma \otimes l) \mapsto f(\sigma) \\ H &: \mathrm{K}_*^{M}(K) \to \mathrm{K}_*^{MW}(K, \mathcal{L}), \sigma \mapsto (h\sigma) \otimes l \end{aligned}$$

where the last formula is valid for any choice of $l \in \mathcal{L}^\times$, given Equation (2.3.2.a).

Obviously, the relation $H \circ f = 2.Id$ still holds with twists.

2.5. **Residues.**

2.5.1. Residues are a famous part of the functoriality of Milnor K-theory (see [BT73]). A *discretely valued field* will be a pair (K, v) of a field K with a discrete valuation v. We let $\mathcal{O}_v$ be its ring of integers, $\mathcal{M}_v$ the maximal ideal of $\mathcal{O}_v$ and $\kappa_v = \mathcal{O}_v/\mathcal{M}_v$ its residue class field.

Given a valuation $v : K^\times \to \mathbb{Z}$, with residue field κ_v, one deduces for any $n > 0$ a canonical morphism:

$$\partial_v : \mathrm{K}_n^M(K) \to \mathrm{K}_{n-1}^M(\kappa_v)$$

uniquely characterized by the property:

$$\partial_v(\{u_1, \dots, u_n\}) = m.\{\overline{u_2}, \dots, \overline{u_n}\}$$

for units $u_i \in K^\times$ such that $v(u_1) = m$ and for $i > 1$, $v(u_i) = 0$, $\overline{u_i}$ being the residue class of u_i.

The analogous construction exists on Morel's K-theory, but the twists are now necessary.

Theorem 2.5.2. *Consider as above a discretely valued field (K, v). The κ_v-space $\mathcal{C}_v := (\mathcal{M}_v/\mathcal{M}_v^2)$ is the* conormal cone *associated with (K, v). It is an invertible κ_v-space (*i.e.* of dimension 1) and we let $\omega_v := (\mathcal{M}_v/\mathcal{M}_v^2)^*$ be its κ_v-dual, which is the* normal cone*.*

Then for any integer $n \in \mathbb{Z}$, there exists a unique morphism of abelian groups:

$$\partial_v : \mathrm{K}_n^{MW}(K) \to \mathrm{K}_{n-1}^{MW}(\kappa_v, \omega_v)$$

satisfying the two following properties:

(Res1) $\partial_v(\eta.\sigma) = \eta.\partial_v(\sigma)$, *for all $\sigma \in \mathrm{K}_{n+1}^{MW}(K)$.*

(Res2) *For any uniformizer $\pi \in K$ and any units $u_1, \dots, u_n$ such that $u_1 = v_1\pi^m$, and $v(u_i) = 0$ for $i > 1$, one has:*

$$\partial_v([u_1, u_2, \dots, u_n]) = m_\epsilon \langle \overline{v_1} \rangle [\overline{u_2}, \dots, \overline{u_n}] \otimes \bar{\pi}^*.$$

We call ∂_v the residue map associated with (K, v).

This theorem makes the residue morphisms on Morel's K-theory much more subtle than its analogue on Milnor K-theory.[4]

Proof. (See [Mor12, 3.15, 3.21]) We first choose some uniformizing parameter $\pi \in \mathcal{M}_v$ of v. Then we introduce the following quotient ring:

$$A_* = \mathrm{K}_*^{MW}(\kappa_v)[\xi]/(\xi - [-1].\xi)$$

which we view as a graded ring by putting ξ in degree 1. Then the proof reduces to show that the canonical map:

$$K^\times \to A_*, (u = a\pi^m) \mapsto [\bar{a}] + m_\epsilon \langle \bar{a} \rangle.\xi$$

[4] The reader can check however that modulo η, the formula in (Res2) agree with that defining the residue in Milnor K-theory!

extends uniquely to a morphism of $\mathbb{Z}$-graded ring

$$\Theta_\pi : \mathrm{K}_*^{MW}(K) \to A_*$$

such that $\Theta_\pi(\eta) = \eta$.

Then given $\sigma \in \mathrm{K}_n^{MW}(K)$, one can write uniquely:

$$\Theta_\pi(\sigma) = s_v^\pi(\sigma) + \partial_v^\pi(\sigma).\xi$$

so that we get two maps

$$s_v^\pi : \mathrm{K}_*^{MW}(K) \to \mathrm{K}_*^{MW}(\kappa_v)$$
$$\partial_v^\pi : \mathrm{K}_*^{MW}(K) \to \mathrm{K}_{*-1}^{MW}(\kappa_v)$$

such that s_v^π is a (homogeneous) morphism of $\mathbb{Z}$-graded ring (obvious).

Both maps depend on the choice of π in general. We then get the desired canonical map by the formula:

$$\partial_v(\sigma) = \partial_v^\pi(\sigma) \otimes \bar{\pi}^*.$$

$\square$

Definition 2.5.3. With the notation of the above proof, the morphism of $\mathbb{Z}$-graded rings $s_v^\pi : \mathrm{K}_*^{MW}(K) \to \mathrm{K}_*^{MW}(\kappa_v)$ is called the *specialization map* associated with of (K, v, π).

One can derive from the construction in the preceding proof the following relation:

$$s_v^\pi(\sigma) = \partial_v^\pi([\pi].\sigma) - [-1].\partial_v^\pi(\sigma). \tag{2.5.3.a}$$

Remarque 2.5.4. Notice that in general, while the residue map ∂_v does not depend on a choice of uniformizing parameter, in concrete computations, one typically choose such a parameter π and reduce to compute ∂_v^π.

2.5.5. Consider the assumptions of the previous theorem. One can further define, for any invertible $\mathcal{O}_v$-module $\mathcal{L}$, a twisted version:

$$\partial_v^{\mathcal{L}} : \mathrm{K}_n^{MW}(K, \mathcal{L}_K) \to \mathrm{K}_{n-1}^{MW}(K, \omega_v \otimes_{\kappa_v} \mathcal{L}_{\kappa_v})$$

where $\mathcal{L}_E = \mathcal{L} \otimes_{\mathcal{O}_v} E$ for $E = K, \kappa_v$. The procedure is a bit intricate: take an element $\sigma \otimes l$ on the left hand-side: $\sigma \in \mathrm{K}_n^{MW}(K)$ and $l \in (\mathcal{L}_K)^\times$. By definition, there exists elements $l_0 \in (\mathcal{L} - \{0\})$ (recall $\mathcal{L}$ is an $\mathcal{O}_v$-module) and $a \in K^\times$ such that $l = l_0 \otimes_K a$. Then one deduces by definition:

$$\sigma \otimes l = (\langle a \rangle \sigma) \otimes (l_0 \otimes_K 1_K). \tag{2.5.5.a}$$

One puts:

$$\partial_v^{\mathcal{L}}(\sigma \otimes l) = \partial_v(\langle a \rangle \sigma) \otimes (l_0 \otimes_K 1_{\kappa_v})$$

or simply ∂_v when $\mathcal{L}$ is clear from the context.

Remarque 2.5.6. The necessity to "renormalize" the parameter, as in (2.5.5.a), when considering residues makes the computation in quadratic intersection theory sometime quite cumbersome ! Intuitively, we will be following a given orientation from open subschemes to the complementary (reduced) closed subscheme.

Example 2.5.7. We can specialize the definition of the above residue map to negative degree. Then according to Proposition 2.3.1, we get a canonical residue map:

$$\partial_v : \mathrm{W}(K) \to \mathrm{W}(\kappa_v, \omega_v)$$

such that

$$\partial_v(\langle u\rangle) = \begin{cases} 0 & v(u) \text{ even,} \\ \langle \bar{a}\rangle \otimes \bar{\pi}^* & u = a\pi^{2r+1}, v(a) = 0, v(\pi) = 1. \end{cases}$$

(Use Remarque 2.3.5!) This residue map (whence untwisted) is well-known in Witt theory: see [MH73, IV, §1], under the notation ψ^1.

Note also that in degree 0, we get a more regular formula:

$$\partial_v : \mathrm{GW}(K) \to \mathrm{W}(\kappa_v, \omega_v), \partial_v(\langle u\rangle) = m_\epsilon\langle \bar{a}\rangle \otimes \bar{\pi}^*,$$

for $u = a\pi^m$, $v(a) = 0$, $v(\pi) = 1$.

Example 2.5.8. Comparing the formulas in 2.5.1 and Theorem 2.5.2, it is clear that the residue in Morel K-theory "modulo η" coincides with the residue in Milnor K-theory. One can be more precise using the maps of Definition 2.4.8. Given a discretely valued field (K, v), and an invertible $\mathcal{O}_v$-module $\mathcal{L}$, one gets a commutative diagram:

$$\begin{tikzcd}
\mathrm{K}^M_*(K) \arrow[r, "H"] \arrow[d, "\partial_v^M"'] & \mathrm{K}^{MW}_*(K, \mathcal{L}_K) \arrow[r, "f"] \arrow[d, "\partial_v"] & \mathrm{K}^M_*(K) \arrow[d, "\partial_v^M"] \\
\mathrm{K}^M_*(\kappa_v) \arrow[r, "H"] & \mathrm{K}^{MW}_*(K, \omega_v \otimes \mathcal{L}_v) \arrow[r, "f"] & \mathrm{K}^M_*(K)
\end{tikzcd}$$

where, for clarity, ∂_v^M is the residue on Milnor K-theory. The commutativity of the right-hand square was just explained, while the second one follows from the formula $\partial_v(h.\sigma) = h.\partial_v^M(\sigma)$ (indeed h is unramified with respect to v).

The following computation is an analogue of the Gersten exact sequence (1.3.0.a) for Morel K-theory:

Theorem 2.5.9. *Let (K, v) be a discretely valued field, and $\mathcal{L}$ be an invertible $\mathcal{O}_v$-module.*

(1) *Then the following sequence (see Remarque 2.4.3 for the first term) is exact:*

$$\mathrm{K}^{MW}_n(\mathcal{O}_v, \mathcal{L}) \xrightarrow{\nu_*} \mathrm{K}^{MW}_n(K, \mathcal{L}_K) \xrightarrow{\partial_v} \mathrm{K}^{MW}_{n-1}(\kappa_v, \omega_v \otimes_{\kappa_v} \mathcal{L}_{\kappa_v}) \to 0$$

where $\nu : \mathcal{O}_k \to K$ is the obvious inclusion and ν_* is defined as in 2.4.4(2).

(2) *If moreover the ring $\mathcal{O}_v$ contains an infinite field of characteristic not 2, then the map ν_* is injective.*

Idea of proof for (1): the surjectivity of ∂_v is obvious: given any (abelian) generator $\sigma = [\eta^r, v_1, \dots, v_{n-1+r}] \otimes \bar{\pi}^*$ of the right hand-side group, $\pi \in \emptyset 0 \to mega_v^\times$, $v_i \in \kappa_v^\times$ (see Lemma 2.2.11), there exists lifts $u_i \in \mathcal{O}_v^\times$ of v_i, along the epimorphism $\mathcal{O}_v \to \kappa_v$. Then formulas (Res1) and (Res2) implies that $[\eta^n, \pi, v_1, \dots, v_n]$ lifts σ.

Also, (Res2) implies that $\partial_v \nu_* = 0$. Therefore, one only needs to prove that the induced map $\mathrm{Im}(\nu_*) \to \mathrm{Ker}(\partial_v)$ is an isomorphism. This is the serious part! We refer the reader to the proof of [Mor12, Th. 3.22].

Point (2) is the *Gersten conjecture for Milnor-Witt K-theory* and for the local ring $\mathcal{O}_v$. This is due to Gille, Zheng and Scully: cf. [?].

3. Rost-Schmid complex of curves

3.1. **Definition.**

3.1.1. We let X be a connected 1-dimensional scheme which is assumed to be normal (or equivalently regular!). Let $\mathcal{L}$ be an invertible sheaf over X. The main examples are smooth algebraic curves over a field and the spectrum of a Dedekind ring.

Let $\kappa(X)$ be the function field of X and $\mathcal{L}_\eta$ be the pullback to $\mathrm{Spec}(\kappa(X))$ seen as an invertible $\kappa(X)$-vector space.[5] We let $X^{(1)}$ be the set of points $x \in X$ which are closed (*i.e.* of codimension 1). This amounts to ask that the local ring $\mathcal{O}_{X,x}$ is 1-dimensional, and therefore a discrete valuation ring. In particular, x uniquely corresponds to a valuation v_x on K and we can consider the associated residue map (Theorem 2.5.2)

$$\partial_x : \mathrm{K}_*^{MW}\big(\kappa(X), \mathcal{L}_\eta\big) \to \mathrm{K}_{n-1}^{MW}(\kappa_x, \omega_x \otimes \mathcal{L}_x)$$

where ω_x is the normal cone of $(\kappa(X), v_x)$ (equivalently, the normal sheaf over $\mathrm{Spec}(()\kappa_x)$ of the closed immersion $\{x\} \to X$), and $\mathcal{L}_x$ be the restriction of $\mathcal{L}$ to $\mathrm{Spec}(\kappa_x)$, seen as a κ_x-vector space.

Given an element $f \in \mathrm{K}_*^{MW}\big(\kappa(X)\big)$, we will interpret $\partial_x(f)$ as the K^{MW}-order of f at x.

Lemma 3.1.2. *With the above notations, for any $f \in \mathrm{K}_n^{MW}\big(\kappa(X)\big)$, the set:*

$$\{x \in X \mid \partial_x(f) \neq 0\}$$

is finite.

Given the definition of the residue map, and Lemma 2.2.11, this directly follows from the (more classical) fact:

Lemma 3.1.3. *Let $u \in \kappa(X)^\times$ be a unit. Then the set $\{x \in X \mid v_x(u) \neq 0\}$ is finite.*

[5] We will also use the notation $\mathcal{L}_{\kappa(X)} = \mathcal{L}_\eta$ later.

Remarque 3.1.4.
(1) The alert reader will have recognized the support of the divisor associated with the rational function u of X appearing in the previous lemma! Even in our generality, the finiteness is very classical.
(2) The fact the scheme X is noetherian is essential here. However, in case one withdraw this assumption, everything would still be fine as we will obtain a locally finite subset of X. The theory of cycles, and quadratic cycles, would be fine as we will consider locally finite sums. This fits particularly well with the fact Chow groups (as well as Chow-Witt groups) are a kind of *Borel-Moore homology* in topology, and the latter is represented by the complex of locally finite singular chains (for suitable topological spaces).
(3) In fact, one can prove the above lemma assuming only that X in noetherian, normal and finite dimensional. This will be useful later!

The following definition is a slight generalization of the known definition of the classical definition Chow-Witt groups. We refer the reader to [DFJ] for further developments.

Definition 3.1.5. Consider the previous notation. We define the *quadratic divisor class* map as the following sum:

$$\widetilde{\mathrm{div}}_X = \sum_x \partial_x : \mathrm{K}_*^{MW}\big(\kappa(X), \mathcal{L}_\eta\big) \to \bigoplus_{x \in X^{(1)}} \mathrm{K}_{*-1}^{MW}(\kappa_x, \omega_x \otimes \mathcal{L}_x)$$

which is well-defined according to Lemma 3.1.2. This is a homogeneous morphism of $\mathbb{Z}$-graded abelian groups of degree -1.

We then define the group $\widetilde{\mathrm{C}}^p(X, \mathcal{L})_q$ for $i = 0$ (resp. $p = 1$) as respectively the source (resp. target) of $\widetilde{\mathrm{div}}_X$ with $* = q$ (resp. $* - 1 = q$), and as 0 otherwise.

Therefore we have obtained a complex $\widetilde{\mathrm{C}}^*(X, \mathcal{L})_q$ and we can the define the *Chow-Witt group* $\widetilde{\mathrm{CH}}^p(X, \mathcal{L})_q$ of codimension p and $\mathbb{G}_m$-grading q as the cohomology in degree p of this complex. When $q = 0$, we call it simply the Chow-Witt group, written $\widetilde{\mathrm{CH}}^p(X, \mathcal{L})$.

Let us be more explicit. The abelian group $\widetilde{\mathrm{CH}}^1(X, \mathcal{L})$ (or equivalently the Chow-Witt groups of 0-dimensional classes of quadratic cycles) is the cokernel of the map in degree 1:

$$\mathrm{K}_1^{MW}\big(\kappa(X), \mathcal{L}_\eta\big) \to \bigoplus_{x \in X^{(1)}} \mathrm{GW}(\kappa_x, \omega_x \otimes \mathcal{L}_x).$$

The abelian group at the target will be called the group of quadratic 0-cycles of $(X, \mathcal{L})$. These are formal sums of the form

$$\sum_{i \in I} (\sigma_i \otimes \tau_i^* \otimes l_i).x_i \tag{3.1.5.a}$$

where:

- $x_i \in X$ is a closed point,

- $\tau_i = t_i^*$ where t_i is a uniformizing parameter of the valuation ring $\mathcal{O}_{X,x_i}$, (equivalently a local parameter of the closed subscheme $\{x_i\} \subset X$),
- a non-zero element $l_i \in \mathcal{L}$.

In practice, one can also view the coefficient $(\sigma_i \otimes \tau_i^* \otimes l_i)$ as a virtual inner $(\omega_{x_i} \otimes \mathcal{L}_{x_i})$-space over $\kappa(x_i)$.

In codimension 0, $\widetilde{\mathrm{CH}}^0(X,\mathcal{L})$ is the kernel of the map in degree 0:

$$\mathrm{GW}\left(\kappa(X),\mathcal{L}_\eta\right) \to \bigoplus_{x\in X^{(1)}} \mathrm{W}(\kappa_x, \omega_x \otimes \mathcal{L}_x).$$

An element in the kernel of this map, a virtual inner $\mathcal{L}_{\kappa(X)}$-space over the function field $\kappa(X)$, is said to be *unramified* (with respect to the curve X).

Remarque 3.1.6. (1) Even, if we are mainly interested in the Chow-Witt groups, the higher grading $q \neq 0$ will be crucial for computations. See Section 3.3.
(2) The groups $\widetilde{\mathrm{CH}}^p(X,\mathcal{L})_q$ are analogues of the higher Chow groups. However, they do not deserve the name higher Chow-Witt groups as they only contribute to some part of the latter (that one can interpret as the Milnor-Witt motivic Borel-Moore homology; see [BY20, BCD$^+$22]). In fact, while the latter are represented by a full ring spectrum $\mathrm{H}_{\mathrm{MW}}\mathbb{Z}$, the former are only a truncation of $\mathrm{H}_{\mathrm{MW}}\mathbb{Z}$. On the other hand, the groups just defined satisfy the same formalism than higher Chow groups.
(3) If one replaces Morel K-theory by the usual K-theory, one obtains Rost's ($\mathbb{G}_m$-)graded Chow groups $\mathrm{CH}^p(X)_q$ defined in [Ros96]. This was in fact the model for the previous ddefinition. We refer the reader to 3.1.9 for more discussion.

Example 3.1.7. In the case X is in addition local, thus the spectrum of a discrete valuation ring $\mathcal{O}_v$, Theorem 2.5.9 implies in particular:

$$\widetilde{\mathrm{CH}}^p(\mathcal{O}_v) = \begin{cases} \mathrm{GW}(\mathcal{O}_v) & p = 0, \mathcal{O}_v \supset k_0 \\ 0 & p = 1. \end{cases}$$

where k_0 is an infinite field of characteristic not 2. The vanishing of $\widetilde{\mathrm{CH}}^1(\mathcal{O}_v)$ can be interpreted by saying that every quadratic 0-cycle of X is *principal.*

3.1.8. Let us consider the previous definition modulo η. Then we get in degree $0, 1$ a map, independent of $\mathcal{L}$:

$$\kappa(X)^\times = \mathrm{K}_1^M\left(\kappa(X),\mathcal{L}_\eta\right) \xrightarrow{\widetilde{\mathrm{div}}_X \mod \eta} \bigoplus_{x\in X^{(1)}} \mathrm{K}_0^M(\kappa_x, \omega_x \otimes \mathcal{L}_x) = Z^1(X)$$

where the right hand-side is the groupe of (ordinary!) 0-cycles of X. This is precisely the divisor class map and we get in particular:

$$\widetilde{\mathrm{CH}}^p(X,\mathcal{L})/(\eta) = \begin{cases} \mathbb{Z} & p = 0 \\ \mathrm{Pic}(X) & p = 1. \end{cases}$$

Moreover, one can describe explicitly the image of the map:

$$\widetilde{\mathrm{CH}}^p(X,\mathcal{L}) \to \widetilde{\mathrm{CH}}^p(X,\mathcal{L})/(\eta) \simeq \mathrm{CH}^p(X)$$

It is just induced by the rank map: in degree 0, it sends an unramified inner $\mathcal{L}$-space σ over $\kappa(X)$ to its rank $\mathrm{rk}(\sigma)$. In degree 1, it sends a quadratic 0-cycle

$$\sigma : \sum_{i\in I} \sigma_i \otimes \tau_i^* \otimes l_i.x_i$$

to the 0-cycle:

$$\mathrm{rk}(\sigma) = \sum_{i\in I} \mathrm{rk}(\sigma_i).x_i.$$

3.1.9. We can be more precise about the relation between Chow and Chow-Witt groups, using the definitions of Definition 2.4.8. Indeed, Example 2.5.8 implies that the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{K}_*^M\big(\kappa(X)\big) \arrow[r, "H_\eta"] \arrow[d, "\mathrm{div}_X"'] & \mathrm{K}_*^{MW}\big(\kappa(X),\mathcal{L}_\eta\big) \arrow[r, "f_\eta"] \arrow[d, "\widetilde{\mathrm{div}}_X"] & \mathrm{K}_*^M\big(\kappa(X)\big) \arrow[d, "\mathrm{div}_X"'] \\
Z^1(X) \arrow[r, "\sum_x H_x"] & \widetilde{\mathrm{C}}^1(X,\mathcal{L}) \arrow[r, "\sum_x f_x"] & Z^1(X)
\end{tikzcd}$$

Taking kernel and cokernels, one gets well defined maps:

$$\mathrm{CH}^p(X)_q \xrightarrow{H} \widetilde{\mathrm{CH}}^p(X,\mathcal{L})_q \xrightarrow{f} \mathrm{CH}\,p(X)_q.$$

whose composite is multiplication by 2. We still call them respectively the hyperbolic and forgetful maps.

3.2. **Homotopy invariance over a field and transfers.** Our next result was first proved for Milnor K-theory by Milnor: see [Mil70, Th. 2.3] (and also [BT73, 5.2]). It was generalized by Morel in [Mor12, Th. 3.24].

Theorem 3.2.1 (Morel)**.** *Let k be an arbitrary field, $X = \mathbb{A}^1_k$ with function field $k(t) = \kappa(X)$. Let $\varphi : k \to k(t)$ be the obvious inclusion.*

Then the quadratic divisor class map fits of X into the following sequence

$$0 \to \mathrm{K}_q^{MW}(k) \xrightarrow{\varphi_*} \mathrm{K}_q^{MW}\big(k(t)\big) \xrightarrow{d_X^1} \bigoplus_{x\in X^{(1)}} \mathrm{K}_{q-1}^{MW}(\kappa_x,\omega_x) \to 0$$

which is split exact.

In other words,

$$\widetilde{\mathrm{CH}}^p(\mathbb{A}^1_k)_q = \begin{cases} \mathrm{K}_q^{MW}(k) & p=0 \\ 0 & p=1. \end{cases}$$

Note that a splitting is easy to get: considering the valuation $v = \deg$ on $k(t)$, the specialization map s_v^t (Definition 2.5.3) gives a splitting. More generally, any valuation v on $k(t)$ trivial on k with uniformizing parameter π will give a splitting s_v^π.

The proof of this proposition uses the same trick as in Milnor's proof, and argue inductively on the degree in t. The principle is to filter $\mathrm{K}_*^{MW}(k(t))$ by the subring L_d generated by η and symbols of the form $[P(t)]$ where $P(t)$ is a polynomial of degree less than t. We can then argue inductively on the $\mathbb{Z}$-graded $\mathrm{K}_*^{MW}(k)$-rings L_d using an explicit presentation of the $\mathbb{Z}$-graded $\mathrm{K}_*^{MW}(k)$-module L_d/L_{d-1}.

As an example, the reader is encouraged to work out for himself the case of L_1. The hint is to use the (obvious!) exact sequence:

$$0 \to k^\times \xrightarrow{\varphi_*} k(t)^\times \xrightarrow{\sum_x v_x} Z^1(\mathbb{A}^1_k) \to 0$$

Remarque 3.2.2. The twisted version of the previous lemma is formal as a line bundle $\mathcal{L}$ over the affine line $\mathbb{A}^1_k$ is trivializable. However, when dealing wiht orientations problems, one need to be very cautious. Interpreting $\mathcal{L}$ as an invertible $k[t]$-module (which is therefore globally free), we let $\mathcal{L}_0 = \mathcal{L} \otimes_{k[t]} k[t]/(t)$. This is the restriction of $\mathcal{L}$ to 0. Then there is a canonical isomorphism $\mathcal{L} \to \mathcal{L}_0 \otimes_k k[t]$, which maps an element l to $l \otimes 1 \otimes 1$. Using this isomorphism, one builds a canonical map ν_* which fits in the exact sequence:

$$0 \to \mathrm{K}_q^{MW}(k, \mathcal{L}_0) \xrightarrow{\varphi_*} \mathrm{K}_q^{MW}\left(k(t), \mathcal{L}_{k(t)}\right) \xrightarrow{d_X^1} \bigoplus_{x \in X^{(1)}} \mathrm{K}_{q-1}^{MW}(\kappa_x, \omega_x \otimes \mathcal{L}_x) \to 0$$

which is easily seen to be split exact from the previous theorem.

3.3. **Localization exact sequences.** In this section, we will illustrate why one needs to consider a higher analogue of the Chow-Witt groups. The aim is to compute the Chow-Witt groups of the projective line.

3.3.1. Let again X be a normal connected 1-dimensional scheme, $\mathcal{L}$ an invertible sheaf on X. To avoid some confusion, we denote from now on $\omega_{x/X}$ the normal bundle of a point x in X.

Consider in addition a finite subset $Z \subset X$ of closed points of X, seen as reduced closed subscheme, $i : Z \to X$. Let $\omega_{Z/X} = (\mathcal{I}(Z)/\mathcal{I}(Z)^2)^\vee$ be the normal bundle of i, where $\mathcal{I}(Z) \subset \mathcal{O}_X$ is the ideal sheaf. Let $U = X - Z$, and $j : U \to X$ the open immersion.

There is an obvious split epimorphism:

$$j^* : \widetilde{\mathrm{C}}^1(X, \mathcal{L})_q \to \widetilde{\mathrm{C}}^1(U, \mathcal{L})_q$$

whose kernel is the finite sum:

$$\widetilde{\mathrm{CH}}^0(Z, \omega_{Z/X} \otimes \mathcal{L}_Z)_q := \oplus_{z \in Z} \mathrm{K}_q^{MW}\left(\kappa_z, \omega_{z/X} \otimes \mathcal{L}_z\right).$$

Remark that this notation fits in the previous considerations as for any point $z \in Z$, we have a *canonical* isomorphism of invertible κ_z-vector spaces (or what amount to the same, invertible sheaf over $z = \mathrm{Spec}(\kappa_z)$):

$$\omega_{z/X} \otimes \mathcal{L}_z \simeq (\omega_{z/Z} \otimes \omega_{Z/X}|_z) \otimes \mathcal{L}_z \simeq \omega_{z/Z} \otimes (\omega_{Z/X} \otimes \mathcal{L}_Z)_x.$$

Assembling all this, we get a commutative diagram whose lines are exact:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \widetilde{\mathrm{C}}^0(X,\mathcal{L})_{q+1} \arrow[r, equal] \arrow[d, "\widetilde{\mathrm{div}}_X"'] & \widetilde{\mathrm{C}}^0(U,\mathcal{L})_{q+1} \arrow[r] \arrow[d, "\widetilde{\mathrm{div}}_U"] & 0 \\
0 \arrow[r] & \widetilde{\mathrm{CH}}^0(Z,\omega_{Z/X}\otimes\mathcal{L}_Z)_q \arrow[r, "i_*"] & \widetilde{\mathrm{C}}^1(X,\mathcal{L})_q \arrow[r, "j^*"] & \widetilde{\mathrm{C}}^1(U,\mathcal{L})_q \arrow[r] & 0
\end{tikzcd}$$

Definition 3.3.2. Consider the previous notation. The exact sequence obtained by applying the snake lemma to the preceding commutative diagram:

$$0 \to \widetilde{\mathrm{CH}}^0(X,\mathcal{L})_{q+1} \xrightarrow{j^*} \widetilde{\mathrm{CH}}^0(U,\mathcal{L}_U)_{q+1} \xrightarrow{\partial_{Z/X}} \widetilde{\mathrm{CH}}^0(Z,\omega_{Z/X}\otimes\mathcal{L}_Z)_q$$
$$\xrightarrow{i_*} \widetilde{\mathrm{CH}}^1(X,\mathcal{L})_q \xrightarrow{j^*} \widetilde{\mathrm{CH}}^1(U,\mathcal{L}_U)_q \to 0$$

is called the *localization exact sequence* associated with i.

The connecting map $\partial_{Z/X}$ is called the *residue map* associated with i. It is induced by the following restriction/corestriction of the quadratic divisor class map d_X:

$$\sum_{z\in Z} \partial_z : \mathrm{K}^{MW}_{q+1}\big(\kappa(X)\big) \longrightarrow \oplus_{z\in Z}\,\mathrm{K}^{MW}_q\big(\kappa_z, \omega_z \otimes \mathcal{L}_z\big).$$

3.3.3. We now illustrate the usage of this sequence. Let k be an arbitrary field.

Let $\mathbb{P}^1_k = \mathrm{Proj}(k[x,y])$ be the projective line, $\infty = [1:0]$ be point at infinity with complementary open subscheme $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$. We let $i^\infty : \{\infty\} \to \mathbb{P}^1_k$ be the natural closed immersion, and $j : \mathbb{A}^1_k \to \mathbb{P}^1_k$ the complementary open immersion. We fix a line bundle $\mathcal{L}$ over $\mathbb{P}^1_k$, which is therefore determined up to isomorphism by its degree, $\deg(\mathcal{L})$ (for where $\deg : \mathrm{Pic}(\mathbb{P}^1_k) \simeq \mathbb{Z}$). We let $\mathcal{L}'$ be the restriction of $\mathcal{L}$ to $\mathbb{A}^1_k$.

Then the localization exact sequence of i^∞ together with Morel's homotopy invariance theorem (Theorem 3.2.1, or rather the twisted version Remarque 3.2.2) gives us the following exact sequence:

$$0 \to \widetilde{\mathrm{CH}}^0(\mathbb{P}^1_k,\mathcal{L})_{q+1} \xrightarrow{j^*} \mathrm{K}^{MW}_{q+1}(k,\mathcal{L}_0) \xrightarrow{\partial_{Z/X}} \mathrm{K}^{MW}_q(k,\omega_\infty\otimes\mathcal{L}_\infty)$$
$$\xrightarrow{i^\infty_*} \widetilde{\mathrm{CH}}^1(\mathbb{P}^1_k,\mathcal{L})_q \to 0$$

where we have denoted by $\mathcal{L}_0$ the restriction of $\mathcal{L}'$ to the point 0 in $\mathbb{A}^1_k$.

The main problem is to determine the residue map $\partial_{Z/X}$.

Lemma 3.3.4. *Consider the above assumptions and notations.*

Then if $\deg(\mathcal{L})$ *is even,* $\partial_{Z/X} = 0$*. If* $\deg(\mathcal{L})$ *is odd, there exists a choice of isomorphisms* $\mathcal{L}_0 \simeq k$*,* $\mathcal{L}_\infty \simeq k$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{K}_{q+1}^{MW}(k, \mathcal{L}_0) \arrow[r, "\partial_{Z/X}"] \arrow[d, "\sim"'] & \mathrm{K}_q^{MW}(k, \mathcal{L}_\infty) \arrow[d, "\sim"] \\
\mathrm{K}_{q+1}^{MW}(k) \arrow[r, "\eta"] & \mathrm{K}_q^{MW}(k)
\end{tikzcd}$$

Proof. One reduces to the case $\mathcal{L} = \mathcal{O}(d)$. We consider the $U = U_\infty$ and U_0 the open complement of ∞ (resp. 0), so that $U_\infty = \mathrm{Spec}(k[x])$ and $U_0 = \mathrm{Spec}(k[y])$. The glueing map $U_0 \cap U_\infty \to U_\infty \cap U_0$ is given by mapping x to y^{-1}. Then the line bundle $\mathcal{L} = \mathcal{O}(d)$ is given on U_∞ (resp. U_0) by a free module $\mathcal{L}'_\infty = k[x].u$ (resp. $\mathcal{L}'_0 = k[y].v$) with a glueing map $u \mapsto y^{-d}.v$.

Note therefore that one has prefered chosen isomorphisms: $\mathcal{L}_0 \simeq_u k$ and $\mathcal{L}_\infty \simeq_v k$. Therefore we deduce a canonical map

$$\mathrm{K}_{q+1}^{MW}(k) \simeq_{u_*^{-1}} \mathrm{K}_{q+1}^{MW}(k, \mathcal{L}_0) \xrightarrow{\partial_{Z/X}} \mathrm{K}_q^{MW}(k, \omega_\infty \otimes \mathcal{L}_\infty) \simeq_{y_* \otimes v_*} \mathrm{K}_q^{MW}(k)$$

denoted by $\partial'_{Z/X}$.

We compute the image of $\sigma \in \mathrm{K}_q^{MW}(k)$ under $\partial'_{Z/X}$. First, $u_*^{-1}(\sigma) = \sigma \otimes u$. Then we need to use the map φ_* of Remarque 3.2.2, which sends the latter to

$$\sigma \otimes (u \otimes 1) \in \mathrm{K}_q^{MW}\, k(t), \mathcal{L}_0 \otimes_k k(t).$$

In order to compute its residue at ∞, one needs to write it as an element of $\mathrm{K}_q^{MW}\, k(t), \mathcal{L}_\infty \otimes_k k(t)$. Therefore, one uses the above change of variables:

$$\sigma \otimes (y^{-d} v \otimes 1) = (\langle y^{-d} \rangle \sigma) \otimes (v \otimes 1).$$

Now if d is even, $\langle y^{-d} \rangle = 1$ and we get: $\partial_\infty^y(\langle y^{-d} \rangle \sigma) = 0$ as σ comes from $\mathrm{K}_*^{MW}(k)$. Thus $\partial_{X/Z}(\sigma) = 0$.

If on the contrary, d is odd, $\langle y^{-d} \rangle = \langle y \rangle$. Therefore

$$\partial_\infty^y(\langle y^{-d} \rangle \sigma) = \partial_\infty^y(\langle y \rangle \sigma) = \eta.\sigma$$

and one deduces that $\partial'_{Z/X}(\sigma) = \eta.\sigma$ as expected. □

As a corollary we get the following fundamental result, first proved by Jean Fasel for a perfect base field of characteristic not 2 (see [Fas13]):

Theorem 3.3.5. *Consider the above assumption: k is an arbitrary field, $\mathcal{L}$ an invertible sheaf over $\mathbb{P}_k^1$.*

(1) *If* $\deg(\mathcal{L})$ *is even, the following maps are isomorphism:*

$$\widetilde{\mathrm{CH}}^0(\mathbb{P}_k^1, \mathcal{L})_q \xrightarrow{j^*} \widetilde{\mathrm{CH}}^1(\mathbb{A}_k^1, \mathcal{L}')_q \simeq \mathrm{K}_q^{MW}(k, \mathcal{L}_0),$$

$$\mathrm{K}_q^{MW}(k, \omega_\infty \otimes \mathcal{L}_\infty) = \widetilde{\mathrm{CH}}^0(\{\infty\}, \omega_\infty \otimes \mathcal{L}_\infty) \xrightarrow{i_*^\infty} \widetilde{\mathrm{CH}}^1(\mathbb{P}_k^1, \mathcal{L})_q.$$

(2) *If* $\deg(\mathcal{L})$ *is odd, denoting by* $\eta : \mathrm{K}_q^{MW}(k) \to \mathrm{K}_{q-1}^{MW}(k)$ *the map induced by multiplication by* η*, one gets isomorphisms:*

$$\widetilde{\mathrm{CH}}^0(\mathbb{P}^1_k, \mathcal{L})_q \simeq \begin{cases} \mathrm{Ker}(\eta) & i = 0 \\ \mathrm{K}_q^M(k) & i = 1. \end{cases}$$

In particular, they are both 0 *(resp.* $\mathbb{Z}$*) if* $q < 0$ *(resp.* $q = 0$*). More generally, if* $\mathrm{char}(k) \neq 2$*, recall from Remarque 2.3.5 that* $\mathrm{Ker}(\eta) = 2_\epsilon . \mathrm{K}_q^M(K)$*.*

Let us draw the picture for Chow-Witt groups:

$$\widetilde{\mathrm{CH}}^p(\mathbb{P}^1_k, \mathcal{L}) = \begin{cases} \mathrm{GW}(k, \mathcal{L}_0) & p = 0, \deg(\mathcal{L}) = 2r, \\ \mathrm{GW}(k) & p = 1, \deg(\mathcal{L}) = 2r, \\ \mathbb{Z} & p = 0, 1, \deg(\mathcal{L}) = 2r + 1. \end{cases}$$

In particular, these groups depend on the twist $\mathcal{L}$ when $\mathrm{GW}(k)$ is non-trivial !

3.4. **Transfers.**

3.4.1. Before diving into the construction of transfers, we make a detour that will be necessary for all our future definition. Let $f : X \to S$ be a map of schemes. If f is a local complete intersection (=lci),[6], then one can define a cotangent complex $\mathcal{L}_{X/S}$ which is a perfect complex of coherent $\mathcal{O}_X$-modules. When one has a global factorization (say f is gci) $X \xrightarrow{i} P \xrightarrow{p} S$, i regular closed, p smooth, it is quasi-isomorphic to the complex:

$$\mathcal{C}_{Y/P} \to \Omega_{P/X}|_Y$$

where $\mathcal{C}_{Y/P} = \mathcal{I}_i/\mathcal{I}_i^2$ is the conormal sheaf associated with i, placed in homological degree $+1$, and $\Omega_{P/X}$ is the cotangent sheaf of P/X (the conormal sheaf of the diagonal of P/X) placed in degree 0. Observe for later purposes that the definition of $\mathcal{L}_{X/S}$ extends to *essentially lci* morphisms: that is projective limits of lci morphisms along affine and étale transition maps.

The usefulness of the theory comes from the fact that given a commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & \Delta & Y \arrow[dl] \\ & S & \end{tikzcd}$$

made of lci morphisms, one has a homotopy exact sequence of perfect complexes over X:

$$\mathcal{L}_{Y/S}|_X \to \mathcal{L}_{X/S} \to \mathcal{L}_{X/Y}.$$

We will always reduce to invertible sheaves using the determinant construction (Mumford, Deligne), which associates to a perfect complex $\mathcal{K}$ on X a graded line

[6]*i.e.* it admits, locally for the étale topology, a factorization into a regular closed immersion followed by a smooth morphism, see [BGI71, **?**]

bundle $(\det(\mathcal{K}), \mathrm{rk}(\mathcal{K}))$, that is a pair of a line bundle (the *determinant*) and a Zariski locally constant function $X \to \mathbb{Z}$ (the *rank*).

We put $\omega_{X/S} = \det(\mathcal{L}_{X/S})$, called the canonical sheaf of X/S and $\dim(X/S) = \mathrm{rk}(\omega_{X/S})$ called the relative dimension of X/S. Then the above homotopy exact sequence translates into a canonical isomorphism:

$$\psi_\Delta : \omega_{X/S} \simeq \omega_{X/Y} \otimes \omega_{Y/S}|_X \tag{3.4.1.a}$$

We now come back to the main application of Theorem 3.3.5:

Definition 3.4.2. Let k be an arbitrary field, $\omega = \omega_{\mathbb{P}^1_k/k}$ the canonical sheaf on $\mathbb{P}^1_k$. Applying (3.4.1.a) to the commutative diagram:

$$\begin{tikzcd} \{\infty\} \arrow[rr, "i_\infty"] \arrow[dr] & & \mathbb{P}^1_k \arrow[dl] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

we get a canonical isomorphism: $\psi : k = \omega_{\{\infty\}/k} \simeq \omega|_{\{\infty\}} \otimes \omega_\infty$ (sheaves over $\mathrm{Spec}(k)$ are identified with k-vector spaces).

One defines the *degree map*

$$\widetilde{\deg} : \widetilde{\mathrm{CH}}^1(\mathbb{P}^1_k, \omega) \to \mathrm{GW}(k)$$

as the inverse of the isomorphism:

$$\mathrm{GW}(k) \simeq_{\psi_*} \mathrm{GW}(k, \omega_\infty \otimes \omega|_{\{\infty\}}) = \widetilde{\mathrm{CH}}^0(\{\infty\}, \omega_\infty \otimes \omega|_{\{\infty\}}) \xrightarrow{i^\infty_*} \widetilde{\mathrm{CH}}^1(\mathbb{P}^1_k, \omega).$$

Using the maps of 3.1.9, we get from the previous definition the following commutative diagram:

$$\begin{tikzcd} \mathrm{CH}^1(X) \arrow[r, "H"] \arrow[d, "\deg"'] & \widetilde{\mathrm{CH}}^1(X, \mathcal{L}) \arrow[r, "f"] \arrow[d, "\widetilde{\deg}"] & \mathrm{CH}^1(X) \arrow[d, "\deg"] \\ \mathbb{Z} \arrow[r, "H"] & \mathrm{GW}(X, \mathcal{L}) \arrow[r, "\mathrm{rk}"] & \mathbb{Z} \end{tikzcd}$$

Remarque 3.4.3.
(1) Notice in particular that every cycle which comes from the hyperbolic map will have a degree of the form $n.h$ for $n \in \mathbb{Z}$ and h the hyperbolic form.
(2) By construction, the degree map gives a morphism on quadratic cycles, still denoted by $\widetilde{\deg}$, such that the following composite map is zero:

$$\mathrm{K}^{MW}_1\left(k(t), \omega_\eta\right) \xrightarrow{\widetilde{\mathrm{div}}} \widetilde{\mathrm{C}}^1(\mathbb{P}^1_k, \omega) \xrightarrow{\widetilde{\deg}} \mathrm{GW}(k).$$

This property is the analogue of what one usually calls (after Weil) the *reciprocity property*.

3.4.4. In fact, Theorem 3.3.5 gives a higher quadratic degree map, for any $q \in \mathbb{Z}$:

$$\widetilde{\deg} : \widetilde{\mathrm{CH}}^1(\mathbb{P}^1_k, \omega \otimes \mathcal{L})_q \to \mathrm{K}^{MW}_q(k, \mathcal{L})$$

Let us make this map more explicit. Let us fix a point $x \in \mathbb{P}^1_k$, and write $L = \kappa_x$ is residue field. Therefore, L/k is a finite monogeneous extension of k (x defines a unique irreducible polynomial P_x in $k[t]$ such that $L = k[t]/(P_x)$). Note that L/k is not necessarily separable so that this time, the canonical sheaf $\omega_{L/k}$ is non-trivial. Nevertheless, the commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(L) \arrow[rr, "x"] \arrow[dr] & & \mathbb{P}^1_k \arrow[dl] \\ & \mathrm{Spec}(k) & \end{tikzcd}$$

gives a canonical ((3.4.1.a)) isomorphism $\psi^x : \omega_{L/k} \simeq \omega_x \otimes \omega|_L$.

Specializing the quadratic degree $\widetilde{\mathrm{div}}$ at cycles supported on x, we get a map:

$$\widetilde{\mathrm{div}}_x : \mathrm{K}^{MW}_q(L, \omega_x \otimes \omega|_L) \to \mathrm{GW}(k).$$

Following Bass-Tate, Morel, and [Fel20c], we adopt the following intermediate definition:

Definition 3.4.5. Let L/k be a monogeneous extension with a chosen generator x (that we identify to $x \in \mathbb{P}^1_k$). One defines the trace/Gysin/transfer map associated with (L, x) as the composite:

$$N^x_{L/k} : \mathrm{K}^{MW}_q(L, \omega_{L/k}) \simeq_{\psi^x_*} \mathrm{K}^{MW}_q(L, \omega_x \otimes \omega|_L) \xrightarrow{\widetilde{\mathrm{div}}_x} \mathrm{GW}(k).$$

We also define a twisted version, for an invertible k-vector space $\mathcal{L}$, as follows:

$$N^{u,\mathcal{L}}_{L/k} : \mathrm{K}^{MW}_q(L, \omega_{L/k} \otimes \mathcal{L}_L) \to \mathrm{GW}(k, \mathcal{L}), (\sigma \otimes t \otimes l) \mapsto N^x_{L/k}(\langle u \rangle.\sigma \otimes t) \otimes l'$$

where $\sigma \in \mathrm{K}^{MW}_q(L)$, $t \in \omega^\times_{L/k}$, $l \in \mathcal{L}^\times_L$ and we have written: $l = l' \otimes u$ for $l' \in \mathcal{L}^\times$, $u \in L^\times$ (according to our notation $\mathcal{L}_L = \mathcal{L} \otimes_k L$).

3.4.6. Let us now give an arbitrary finite extension L/k. Let $(x, 1, \ldots, x_n)$ be an arbitrary generating family of L/k, to which we associate a tower of finite monogeneous extensions

$$k \subset \kappa_1 \subset \ldots \subset \kappa_n = L.$$

Then we can define the following composite map:

$$(*) : \mathrm{K}^{MW}_q(L, \omega_{L/k}) = \mathrm{K}^{MW}_q(L, \omega_{L/\kappa_{n-1}} \otimes \omega_{\kappa_{n-1}/k}|_L) \xrightarrow{N^{xn}_{L/\kappa_{n-1}}} \mathrm{K}^{MW}_q(\kappa_{n-1}, \omega_{\kappa_{n-1}/k})$$
$$\ldots \xrightarrow{N^{x_1}_{L/\kappa_1}} \mathrm{K}^{MW}_q(k).$$

We have the following fundamental theorem first proved by Morel, but for which the proof of [Fel20b] is better adapted to our setting (in particular k non perfect).

Theorem 3.4.7. *For any finite extension L/k, the composite map $(*)$ is independent of the chosen generating family.*

Definition 3.4.8. Given the assumption of the above theorem, the composite map $(*)$ is denoted by

$$N_{L/k} : \mathrm{K}_q^{MW}(L, \omega_{L/k}) \to \mathrm{K}_q^{MW}(k)$$

and called the (quadratic) transfer of L/k.

As in Definition 3.4.5, one finally defines for $\mathcal{L}$ an invertible k-vector space, twisted transfers:

$$N_{L/k} : \mathrm{K}_q^{MW}(L, \omega_{L/k} \otimes \mathcal{L}_L) \to \mathrm{K}_q^{MW}(k, \mathcal{L})$$

4. Rost-Schmid complex and Chow-Witt groups

4.1. **Definition.**

4.1.1. We now have all the tools to define what Morel calls the *Rost-Schmid complex.* Let us consider an arbitrary regular scheme X. We first recall that any integral closed subscheme Y of X is excellent: in particular, it is generically regular and its normalisation $\tilde{Y}$ is birational and finite over Y.

We will use the notations of 3.4.1. Given $x \in X$, we let $\omega_{x/X}$ is canonical sheaf of the essentially lci morphism $\{x\} \to X$. More concretely, ω_x is the normal sheaf of the closed immersion of x in scheme $X_{(x)}$ obtained by localization of X at x.

Definition 4.1.2. Let X be a regualr scheme and $\mathcal{L}$ an invertible sheaf on X. We define the bi-graded abelian group of quadratic chains of $(X, \mathcal{L})$, in bidegree (p, q), as:

$$\widetilde{\mathrm{C}}^p(X, \mathcal{L})_q = \bigoplus_{x \in X^{(p)}} \mathrm{K}_q^{MW}(\kappa_x, \omega_{x/X} \otimes \mathcal{L}_x).$$

Recall that p is the degree (or the codimension) and q is the $\mathbb{G}_m$-twist. Obviously, this definition extends Definition 3.1.5. The main objective of this section is to define a differential $d = \widetilde{\mathrm{div}}_X$ (a generalization of the quadratic divisor class map of the previous section) on this bigraded abelian group, which will be bihomogeneous of degree $(-1, -1)$ and does not change the $\mathcal{L}$-twist.

4.1.3. Let us consider the assumptions on the previous definition and fix points $x, y \in X$. We need to define the components d_x^y of the map d. Recall on say that x is an *immediate specialization* of y if, letting Y be the reduced closure of y in X, $x \in Y^{(1)}$.

If x is not an immediate specialization, we put $d_x^y = 0$.

Let us assume the contrary. To simplify the notation, we will now replace X by its localization at x. Therefore, one assumes that x is the unique closed point of X, and therefore of Y.

Note that the assumptions imply that Y is an integral local 1-dimensional scheme, eventually singular. Let $f : \tilde{Y} \to Y$ be the finite morphism from its normalization. Then $\tilde{Y}$ is a semi-local regular 1-dimensional scheme. The composite map $p : \tilde{Y} \to Y \to X$ is affine, of finite type, between regular schemes: thus it is lci. Finally, using (3.4.1.a) to the situation of $y/\tilde{Y}/X$, we get a canonical isomorphism of invertible κ_y-vector spaces:

$$(1) : \omega_{y/X} \simeq \omega_{\tilde{Y}/X}|_y.$$

Let us denote by $Z = Y_{(0)}$ the fintie set of closed points of $\tilde{Y}$. Any point $z \in Z$ corresponds to a valuation v_z on the function field κ_y of $\tilde{Y}$, with residue class field κ_z such that κ_z/κ_x is finite. Applying again (3.4.1.a) to the commutative diagram (of essentially lci morphisms):

$$\begin{tikzcd} \{z\} \arrow[r, hook] \arrow[d] & \tilde{Y} \arrow[d] \\ \{x\} \arrow[r, hook] & X \end{tikzcd}$$

We get a canonical isomorphism of invertible κ_z-vector spaces:

$$(2) : \omega_{z/\tilde{Y}} \otimes \omega_{\tilde{Y}/X}|_z \simeq \omega_{z/x} \otimes \omega_{x/X}|_z$$

One can now define d_x^y as the following composite map:

$$\begin{aligned}
&\mathrm{K}_q^{MW}(\kappa_y, \omega_{y/X} \otimes \mathcal{L}_y) \overset{(1)}{\simeq} \mathrm{K}_q^{MW}(\kappa_y, \omega_{\tilde{Y}/X}|_y \otimes \mathcal{L}_y) \\
&\xrightarrow{\sum_{z \in Z} \partial_{v_z}} \bigoplus_{z \in Z} \mathrm{K}_q^{MW}(\kappa_z, \omega_{z/\tilde{Y}} \otimes \omega_{\tilde{Y}/X}|_z \otimes \mathcal{L}_z) \overset{(2)}{\simeq} \bigoplus_{z \in Z} \mathrm{K}_q^{MW}(\kappa_z, \omega_{z/x} \otimes \omega_{x/X}|_z \otimes \mathcal{L}_z) \\
&\xrightarrow{\sum_{z \in Z} N_{\kappa_z/\kappa_x}} \mathrm{K}_q^{MW}(\kappa_x, \omega_{x/X} \otimes \mathcal{L}_x)
\end{aligned}$$

using residues Theorem 2.5.2 and transfers Definition 3.4.8.

We now proceed as in Lemma 3.1.2.

Lemma 4.1.4. . *Consider the above notation. Then for any point $y \in X$ and any $\sigma \in \mathrm{K}_q^{MW}(\kappa_y, \omega_{y/Y} \otimes \mathcal{L}_y)$, the set:*

$$\{x \in X \mid d_x^y(\sigma) \neq 0\}$$

is finite.

The proof works as in *loc. cit.* except that one considers the normalization $\tilde{Y}$ of the reduced closure of y in Y. Then one uses Remarque 3.1.4 in the case of $\tilde{Y}$ (see Remarque 3.1.4(3)) after having written $\sigma = [\eta^{r-q}, u_1, \ldots, u_r]$ (Lemma 2.2.11).

Definition 4.1.5. Let $(X, \mathcal{L})$ be a regular scheme with an invertible sheaf $\mathcal{L}/X$. Using the previous lemma, one define thes higher quadratic divisor class map for $(X, \mathcal{L})$ as the map:

$$d_X : \sum_{(x,y) \in X^{(p)} \times X^{(p-1)}} d_x^y : \widetilde{\mathrm{C}}^p(X, \mathcal{L})_q \to \widetilde{\mathrm{C}}^{p-1}(X, \mathcal{L})_{q-1}.$$

It is sometime convenient to see the above map as a bigraded endo-morphism d_X of $\widetilde{\mathrm{C}}^*(X, \mathcal{L})$. The last step in the construction is in fact a difficult result which, until recently, was only known for essentially smooth k-schemes.

Proposition 4.1.6. *Let $(X, \mathcal{L})$ be as in the above definition. Then d_X is a differential: $d_X \circ d_X = 0$.*

The method uses the proof of the analogous result by Kato (see [Kat86]). We refer the reader to [BHP22, DFJ].

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