31/3/26 GdT Gross-Zagier

Rankin-Selberg's method III [GZ, Section IV.6]
Explicit formula for the L-derivative in weight 2

Setup $k \geq 1$, $f \in S_{2k}^{new}(\Gamma_0(N))$

$D<0$ fundamental discriminant, $(D, 2N) = 1$.

$K = \mathbb{Q}(\sqrt{D})$, $\varepsilon(n) = \left(\frac{D}{n}\right)$, $h = |Cl(\mathcal{O}_K)|$, $w = 2u = |\mathcal{O}_K^\times|$

$\mathcal{A} \in Cl(\mathcal{O}_K)$ $r_{\mathcal{A}}(n) = \#\{I \subset \mathcal{O}_K,\ I \in \mathcal{A},\ N(I) = n\}$ $(n>0)$

$r_{\mathcal{A}}(0) = \frac{1}{w}$

$$L_{\mathcal{A}}(f,s) = L^{(N)}(\varepsilon, 2s-2k+1) \sum_{n=1}^{\infty} \frac{a_n r_{\mathcal{A}}(n)}{n^s} \qquad \left(\mathrm{Re}(s) > k + \tfrac{1}{2}\right)$$

Baptiste
$$\frac{\Gamma(s+2k-1)}{(4\pi)^{s+2k-1}} L_{\mathcal{A}}(f, s+2k-1) = \langle f, \tilde{\Phi}_s \rangle_N$$

with $\tilde{\Phi}_s \in \tilde{M}_{2k}(\Gamma_0(N))$ + formula for Fourier coeff. of $\tilde{\Phi}_s$

$\tilde{M}_{2k}$: modular of weight $2k$ but not holomorphic

$\tilde{\Phi}_s$ has analytic continuation to $s \in \mathbb{C}$, functional equation $s \leftrightarrow 2-2k-s$

$\Rightarrow$ $L_{\mathcal{A}}(f,s)$ has analytic continuation to $s \in \mathbb{C}$, functional equation $s \leftrightarrow 2k-s$

$\varepsilon(N) = 1 \Rightarrow L_{\mathcal{A}}(f,k) = 0$

$\Rightarrow L'_{\mathcal{A}}(f,k) = ?$

We have $L'_A(f,k) = * < f, \tilde{\Phi} >$

with $\tilde{\Phi} = \frac{\sqrt{|D|}}{2\pi} \frac{\partial}{\partial s} \tilde{\Phi}_s \Big|_{s=1-k} \in \tilde{M}_{2k}(\Gamma_0(N))$

Sandra. Holomorphic projection: $\tilde{\Phi} \rightsquigarrow \Phi \in S_{2k}(\Gamma_0(N))$

$$\left.\begin{array}{l} S_{2k}(\Gamma_0(N)) \longrightarrow \mathbb{C} \\ g \longmapsto < g, \tilde{\Phi} > \end{array}\right|$$ linear form, so of the form $g \mapsto < g, \Phi >$ for a unique $\Phi \in S_{2k}(\Gamma_0(N))$

We write $\Phi = \pi_{hol}(\tilde{\Phi})$.

Sandra computed the Fourier coefficients of Φ when $\boxed{k>1}$.

In this talk: $k=1$.

Putting naively $k=1$ in Sandra's formula gives a potentially divergent infinite series ...

Recall the statement of holomorphic projection.

Thm. Let $\tilde{\Phi} \in \tilde{M}_{2k}(\Gamma_0(N))$, $\tilde{\Phi}(z) = \sum_{m\in\mathbb{Z}} a_m(y)\, e^{2i\pi m z}$

Assume $\exists\, \varepsilon>0$, $\forall \alpha \in SL_2(\mathbb{Z})$, $(\tilde{\Phi}\,|_{2k}\,\alpha)(z) = O(y^{-\varepsilon})$
$y = \mathrm{Im}(z) \longrightarrow +\infty$

Let $\Phi = \pi_{hol}(\tilde{\Phi}) = \sum_{m=1}^{\infty} a_m\, e^{2i\pi m z} \in S_{2k}(\Gamma_0(N))$.

Then for $\boxed{k>1}$:

Converges for $k=1$ and gives the right answer, but in our case the assumption on $\tilde{\Phi}$ is not satisfied.

$$a_m = \frac{(4\pi m)^{2k-1}}{(2k-2)!} \int_0^\infty a_m(y)\, e^{-4\pi m y}\, y^{2k-2}\, dy$$

Problems:

1. The result is still true for $k=1$, but the proof for $k>1$ uses Poincaré series P_m which do not converge for $k=1$ (weight 2)

2. Our function $\tilde{\Phi}$ does not satisfy $\tilde{\Phi}|_2 \alpha\,(\tau) = O(y^{-\varepsilon})$

We first address 1.

Recall, for $m \geq 0$,

$$P_m(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} e^{2i\pi m \tau} \Big|_{2k} \gamma \qquad \left(\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \right)$$

$$= \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma_0(N) \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \frac{e^{2i\pi m \gamma\tau}}{(c\tau+d)^{2k}}$$

$k=1$: does not converge absolutely.

Let us put ourselves in a simple situation: $N=1$, $m=0$

$$\sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(c\tau+d)^2}$$

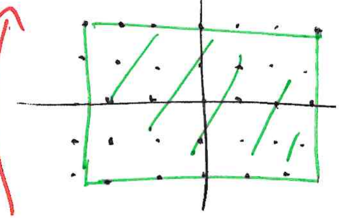
Two approaches:

* Eisenstein summation

(4)

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad \mathbb{Z}z+\mathbb{Z} \longrightarrow \frac{1}{cz+d}(\mathbb{Z}z+\mathbb{Z})$

$$\underbrace{\lim_{M\to\infty} \lim_{M'\to\infty}} \sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1 \\ |c|\le M,\ |d|\le M'}} \frac{1}{(cz+d)^2}$$



Order is important!

parallelogram gets distorted when $z \to \gamma z$, $\gamma \in SL_2(\mathbb{Z})$.

$\rightsquigarrow$ converges, holomorphic in z, but not modular.

see Serre, Cours d'arithmétique p. 155

[Sometimes called "sweeper's trick" but this is rude ...]

I don't know what gives this approach for Poincaré series of weight 2.

* <u>Hecke's trick</u>

$$E_{2,s}(z) = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)=1}} \frac{y^s}{(cz+d)^2|cz+d|^{2s}} \qquad \text{converges for } \mathrm{Re}(s)>0.$$

$$= \sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathbb{Z})} y^s \big|_2 \gamma$$

By construction, $E_{2,s}$ is modular of weight 2 for $SL_2(\mathbb{Z})$.

<u>Def</u>. $E(z) = \lim_{s\to 0} E_{2,s}(z)$

$\rightsquigarrow$ Still modular of weight 2, but we lose holomorphicity.

$E \in \widetilde{M}_2(SL_2(\mathbb{Z}))$

Here, use Hecke's trick for P_m:

Def. $$P_{m,s}(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \left(e^{2i\pi m z} \cdot y^s\right)\Big|_2 \gamma$$

$$= \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma_0(N) \\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}} \frac{e^{2i\pi m \gamma z} \, y^s}{(cz+d)^2 \, |cz+d|^{2s}}$$ converges for $\mathrm{Re}(s) > 0$.

By construction $P_{m,s} \in \tilde{M}_2(\Gamma_0(N))$

Fact. • $P_m(z) = \lim_{s \to 0} P_{m,s}(z)$ exists and is modular for $\Gamma_0(N)$.

• Moreover, for $m \geq 1$, we have $P_m \in S_2(\Gamma_0(N))$.

Proof (omitted): compute the Fourier expansion of $P_{m,s}(z)$ using Poisson summation with respect to d.

$$\sum_{\substack{c,d \\ c \neq 0}} = \sum_{c \neq 0} \Big(\sum_{d \in \mathbb{Z}} \cdots \Big)$$

+ check that after $s \to 0$, the constant term in the Fourier expansion is 0.

Rk. In fact $s \mapsto P_{m,s}(z)$ has analytic continuation to $\mathrm{Re}(s) > -\frac{1}{2}$

We can perform the same computation as in Sandra's talk for the Petersson scalar product, for $\mathrm{Re}(s) > 0$

$$\langle \tilde{\Phi}, P_{m,\bar{s}} \rangle = \int_{\Gamma_0(N)\backslash\mathcal{H}} \tilde{\Phi}(\tau) \overline{\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \left(e^{2i\pi m \tau} y^{\bar{s}} \right)\Big|_2 \gamma} \, dx\, dy$$

$$= \int_{\Gamma_\infty\backslash\mathcal{H}} \tilde{\Phi}(\tau)\, \overline{e^{2i\pi m \tau}}\, y^{s}\, dx\, dy$$

$$= \int_0^\infty a_m(y)\, e^{-4\pi m y}\, y^{s}\, dy$$

here, need convergence of $\int_{\Gamma_0(N)\backslash\mathcal{H}}$ when the terms in $P_{m,\bar{s}}$ are replaced by their absolute value. This is where the assumption $\tilde{\Phi}(\tau) = O(y^{-c})$ at all cusps is needed.

The same is true for Φ:

$$\langle \Phi, P_{m,\bar{s}} \rangle = \int_0^\infty a_m\, e^{-4\pi m y}\, y^{s}\, dy$$

$$= \frac{\Gamma(s+1)}{(4\pi m)^{1+s}}\, a_m$$

Now we can compute a_m for $m \geq 1$:

$$a_m = 4\pi m \lim_{s\to 0} \langle \Phi, P_{m,\bar{s}} \rangle$$

$$= 4\pi m \langle \Phi, P_m \rangle \quad \text{(the integrability is uniform in } s\text{)}$$

$$= 4\pi m \langle \tilde{\Phi}, P_m \rangle \quad \text{(def. of hol. projection)}$$

$$= 4\pi m \lim_{s\to 0} \langle \tilde{\Phi}, P_{m,s} \rangle$$

$$= 4\pi m \lim_{s\to 0} \int_0^\infty a_m(y)\, e^{-4\pi m y}\, y^{s}\, dy$$

$$= 4\pi m \int_0^\infty a_m(y)\, e^{-4\pi m y}\, dy \quad \left(a_m(y) = O_{y\to\infty}\left(e^{2\pi m y}\right) \right)$$

$\square$

Problem 2 is more serious.

In our case: $\tilde{\Phi}(z) = A \log y + B + \underset{y\to\infty}{O}(y^{-\varepsilon})$. $\varepsilon > 0$

and similarly at each cusp:

$$\tilde{\Phi}|_2 \alpha (z) = A_\alpha \log y + B_\alpha + O(y^{-\varepsilon})$$

First we deal with the cusp ∞

$\rightsquigarrow$ We "cheat" and do as if $N=1$. (only one cusp)

<u>Assume $A=0$</u>

<u>Idea</u>: subtract from $\tilde{\Phi}$ a suitable Eisenstein series having the same behavior as $y \to \infty$.

The Eisenstein series $E(z) = \lim_{s\to 0} E_{2,s}(z)$ will do the job:

- $E(z) \in \tilde{M}_2(SL_2(\mathbb{Z}))$ as we saw
- $E(z) = \underset{\substack{\uparrow \\ \text{term } (c,d)=(0,1)}}{1} + O(y^{-\varepsilon})$ when $y \to \infty$
- E is orthogonal to cusp forms.

To see this last point, note that $E_{2,s}(z) = P_{0,s}(z)$

and thus $E(z) = P_0(z)$. But we have proved:

$\forall g \in S_2(\Gamma_0(N))$, $\langle g, P_{m,\bar{s}} \rangle = \int_0^\infty b_m e^{-4\pi m y} y^s \, dy \quad \forall m \geq 0$

$g = \sum_{n\geq 1}^{\infty} b_n q^n \qquad \Rightarrow \langle g, P_{0,\bar{s}} \rangle = 0 \Rightarrow \langle g, E \rangle = 0.$ $\square$

Therefore: $\pi_{hol}(\tilde{\Phi}) = \pi_{hol}(\underbrace{\tilde{\Phi} - B \cdot E}_{= O(y^{-\varepsilon})})$

so we can use the result for the hol. projection.

Write $E(z) = \sum_{m \in \mathbb{Z}} e(m, y) e^{2i\pi m z}$

in fact $e(m,y) = -24\, \sigma_1(m)$ for $m \geq 1$ (classical Eis. series of weight 2)

Recalling $\Phi = \pi_{hol}(\tilde{\Phi}) = \sum_{m \geq 1}^{\infty} a_m e^{2i\pi m z}$, we obtain

$$a_m = 4\pi m \int_0^\infty \left(a_m(y) - B \cdot e(m,y)\right) e^{-4\pi m y}\, dy$$

$$= 4\pi m \int_0^\infty a_m(y) e^{-4\pi m y}\, dy - 4\pi m B \times \frac{-6}{\pi m} \sigma_1(m)$$

$$= 4\pi m \int_0^\infty a_m(y) e^{-4\pi m y}\, dy + 24 B\, \sigma_1(m)$$

<u>Now for arbitrary A, B</u>

We use a second function $F(z)$

$$F(z) = \frac{\partial}{\partial s} E_{2,s}(z) \Big|_{s=0}$$

Similarly, we see:

- $F \in \tilde{M}_2(SL_2(\mathbb{Z}))$, clear since we take the derivative with respect to the indep. variable s.
- $F(z) = \log y + O_{y \to \infty}\left(\frac{\log y}{y}\right)$

(Follows from $E_{2,s}(z) = y^s + O_{y\to\infty}(y^{-1-s})$ and taking $\frac{\partial}{\partial s}\Big|_{s=0}$)

- F is $\perp$ to cusp forms (because $E_{2,s}(z)$ is $\perp$ cusp forms.)

Therefore:

$$\pi_{hol}(\hat{\tilde{\Phi}}) = \pi_{hol}(\underbrace{\hat{\tilde{\Phi}} - AF - BE}_{= O_{y\to\infty}(y^{-\varepsilon})})$$

Let $F(z) = \sum_{m\in\mathbb{Z}} f(m,y)\, e^{2i\pi m z}$

Then

$$a_m = 4\pi m \lim_{s\to 0} \int_0^\infty \big(a_m(y) - A f(m,y) - B e(m,y)\big) e^{-4\pi m y} y^s\, dy$$

Tedious computations using K-Bessel functions give (see page 15)

$$\int_0^\infty f(m,y)\, e^{-4\pi m y} y^s\, dy = -\frac{6\,\sigma_1(m)}{\pi m}\frac{1}{s}$$

$$-\frac{12}{\pi m}\sigma_1'(m) + \frac{12}{\pi m}\sigma_1(m)\left(\log(2m) + \frac{1}{2} + \frac{\zeta'(2)}{\zeta(2)}\right) + o_{s\to 0}(1)$$

with $\sigma_1(m) = \sum_{d|m} d$, $\sigma_1'(m) = \sum_{d|m} d \log d$.

$\rightsquigarrow$ formula for a_m that I don't write down.

<u>Case N is arbitrary</u>.

$$Y_0(N) = \Gamma_0(N)\backslash\mathcal{H}, \qquad X_0(N) = Y_0(N) \cup \overbrace{(\Gamma_0(N)\backslash\mathbb{P}^1(\mathbb{Q}))}^{\text{cusps}}$$

<u>Def</u>. $\xi\in\mathbb{P}^1(\mathbb{Q})$, $\xi = \alpha\infty$ with $\alpha = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$

The <u>denominator</u> of ξ is (c,N).

Rk. • $\xi' = \gamma\xi$ with $\gamma \in \Gamma_0(N) \Rightarrow \mathrm{denom}(\xi') = \mathrm{denom}(\xi)$
(exercise)

• $\xi = \infty = \frac{1}{0} \Rightarrow \mathrm{denom}(\infty) = N$

and this is the only cusp of $X_0(N)$ of denominator N.

So: $\{\text{cusps of } X_0(N)\} = \bigsqcup_{N_1 | N} \{\text{cusps of denom. } N_1\}$

For $\alpha \in SL_2(\mathbb{Z})$, write $(\tilde{\Phi}\,|_2\,\alpha)(z) = A_\alpha \log y + B_\alpha + O(y^{-\varepsilon})$

$$= A(N_1)\log y + B(N_1) + O(y^{-\varepsilon})$$

in our case

We put $\tilde{\Phi}^*(z) = \tilde{\Phi}(z) - \sum_{M|N} \left(\alpha(M)\, F(Mz) + \beta(M)\, E(Mz)\right)$

for suitable constants $\alpha(M), \beta(M) \in \mathbb{C}$.

Lemma. $E(Mz)\,|_2\,\alpha = \frac{(M,N_1)^2}{M^2} + O\left(\frac{1}{y}\right)$

$$F(Mz)\,|_2\,\alpha = \frac{(M,N_1)^2}{M^2}\log y + \frac{(M,N_1)^2}{M^2}\log\left(\frac{(M,N_1)^2}{M}\right) + O\left(\frac{\log y}{y}\right)$$

$\rightsquigarrow$ We get the $\alpha(M), \beta(M)$ in terms of the $A(N_1), B(N_1)$ by solving a linear system.

Gross and Zagier compute only the Fourier coefficients a_m for $(m,N)=1$, hence only $\alpha(1)$ and $\beta(1)$ are needed.

(because $f \in S_2^{new}(\Gamma_0(N))$ is in the new subspace).

Conclusion. (Holomorphic projection of $\tilde{\Phi}$ for $k=1$)

Let $\Phi = \pi_{hol}(\tilde{\Phi}) = \sum_{m=1}^{\infty} a_m e^{2i\pi m z}$.

Then for $(m, N) = 1$, we have

$$a_m = \lim_{s\to 0}\left[4\pi m \int_0^\infty a_m(y)\, e^{-4\pi m y} y^s\, dy + \frac{24\,\alpha(1)\,\sigma_1(m)}{s}\right]$$

$$+ 24\,\beta(1)\,\sigma_1(m) + 48\,\alpha(1)\left(\sigma_1'(m) - \sigma_1(m)\left(\log(2m) + \frac{1}{2} + \frac{\zeta'(2)}{\zeta(2)}\right)\right).$$

NB. If we want a_m for all m, we need the constants $\alpha(M)$, $\beta(M)$ for every $M \mid N$ ...

Rk. $\alpha(1)$, $\beta(1)$ have an explicit expression as a linear combination of $A(N_1)$, $B(N_1)$.

We now have to compute the $A(N_1)$, $B(N_1)$ for our function $\tilde{\Phi}$.

Prop. We have for every $N_1 \mid N$:

$$A(N_1) = \frac{h}{2u^2}\,\frac{\varepsilon(N_1)\,N_1}{N}$$

$$B(N_1) = A(N_1)\left(\log\left(\frac{N_1^2\sqrt{|D|}}{N\pi}\right) - \gamma + 2\frac{L'}{L}(\varepsilon, 1)\right)$$

Proof: uses Baptiste's formula for $\Theta_A|\alpha$ and $E_s^{(1)}|\alpha$, $\alpha \in SL_2(\mathbb{Z})$

(very long computation, see pages 13 - 14).

From all this, we get the final form of the formula for $L_A'(f, 1)$.

Thm [GZ, Thm 6.3]

$$L_A'(f,1) = \frac{8\pi^2}{\sqrt{|D|}} \langle f, \Phi_A \rangle$$

with $\Phi_A = \sum_{m=1}^{\infty} a_{m,A} \, e^{2i\pi m z} \in S_2(\Gamma_0(N))$ given by

$\forall (m, N) = 1,$

$a_{m,A} = (\dots)$

Compared to the case $k > 1$, the new terms are:

- $\frac{h\kappa}{u^2} \frac{\sigma_1(m)}{s}$ in the limit $s \to 0$, $\quad \kappa = \dfrac{-12}{N . \prod_{p|N} \left(1 + \frac{\varepsilon(p)}{p}\right)}$

- $$\frac{h\kappa}{u^2} \left[\sigma_1(m) \left(\log \frac{N}{|D|} + 2 \sum_{p|N} \frac{\log p}{p^2-1} + 2 + 2 \frac{\zeta'(2)}{\zeta(2)} - 2 \frac{L'}{L}(\varepsilon, 1) \right) + \sum_{d|m} \log\left(\frac{m}{d^2}\right) \right]$$

$\boxed{\text{Computation of } A(N_1) \text{ and } B(N_1) \text{ for our function } \tilde{\Phi}}$

We must compute the leading term of the Fourier expansion of $\tilde{\Phi}|_2\alpha$ for a cusp $\zeta = \alpha\infty$ of denominator N_1.

We do this for $\tilde{\Phi}_s = \mathrm{Tr}_N^{N|D|}\left(\theta_A E_s^{(1)}(N\tau)\right)$

and then use $\tilde{\Phi} = \frac{\sqrt{|D|}}{2\pi}\frac{\partial}{\partial s}\tilde{\Phi}_s\Big|_{s=0}$.

$$\tilde{\Phi}_s\Big|_2\alpha = \sum_{\gamma\in\Gamma_0(ND)\backslash\Gamma_0(N)\alpha}\left(\theta_A(\tau)\, E_s^{(1)}(N\tau)\right)\Big|_2\gamma$$

Write: $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$, $\quad (c,N) = N_1$

$$\left(\theta_A(\tau)\, E_s^{(1)}(N\tau)\right)\Big|_2\gamma = \left(\theta_A(\tau)\Big|_1\gamma\right)\cdot\left(E_s^{(1)}(N\tau)\Big|_1\gamma\right)$$

- $\theta_A(\tau)\big|_1\gamma = \varepsilon_{D_1}\left(\frac{c}{\delta_2}\right)\varepsilon_{D_2}(d)\,\kappa(D_1)^{-1}\delta_1^{-\frac{1}{2}}\,\chi_{D_1D_2}(A)\,\theta_{A\mathfrak{D}_1}(\tau)$

with $D = D_1D_2$, $(c,D) = |D_2|$, $\delta_i = |D_i|$

$$\kappa(D_1) = \begin{cases} 1 & \text{if } D_1>0\\ i & \text{if } D_1<0\end{cases}$$

$\mathfrak{D}_1 = [\mathfrak{d}_1]$ with $\mathfrak{d}_1^2 = (D_1)$

(14)

• $N.\gamma z = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \gamma z = \gamma' z'$

with $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ $\quad a' = N_2 a$, $\quad c' = \dfrac{c}{N_1}$ $\quad (N = N_1 N_2)$

$$\Rightarrow E_s^{(1)}(Nz) \big|_1 \gamma = \frac{1}{N_2} \left(E_s^{(1)} \big|_1 \gamma'\right)(z')$$

$$\left\{\begin{array}{l} c^* c \equiv 1 \ (D/\delta_2) \\ c^* \equiv 0 \ (\delta_2) \end{array}\right.$$

$$= \frac{1}{N_2} \varepsilon_{D_1}(c') \, \varepsilon_{D_2}(d'\delta_1) \, \delta_1^{-s-1} E_s^{(D_1)}\left(\frac{z' + c^* d}{\delta_1}\right)$$

$$\Theta_{A\mathfrak{D}_1}(z) = \frac{1}{2u} + \underbrace{\cdots}_{\text{exponentially small}} \qquad (2u = w = |\mathcal{O}_K^\times|)$$

$$E_s^{(D_1)}(z) = \sum_{n \in \mathbb{Z}} e_s^{(D_1)}(n, y) \, e^{2\pi i n x}$$

$$e_s^{(D_1)}(0, y) = \begin{cases} L(\varepsilon, 2s+1) \, y^s & \text{si } D_1 = 1,\ D_2 = D \\ V_s(0) \, L(\varepsilon, 2s) \, y^{-s} & \text{si } D_1 = D,\ D_2 = 1 \\ 0 & \text{sinon} \end{cases}$$

$$\Rightarrow E_s^{(D_1)}(z) = e_s^{(D_1)}(0, y) + \cdots$$

In $\Gamma_0(ND) \backslash \Gamma_0(N)\alpha$ there is $\left\{\begin{array}{ll} \text{one coset with } D \mid c & \leftarrow \underline{D_2 = D} \\ |D| \text{ cosets with } (c, D) = 1 & \leftarrow D_2 = 1 \end{array}\right.$

$$\Rightarrow \left(\tilde{\Phi}_s \big|_2 \alpha\right)(z) = \frac{1}{2u} \frac{\varepsilon(N_1)}{N_2} \left(L(\varepsilon, 2s+1) \left(\frac{N_1 y}{N_2}\right)^s - \frac{i\, V_s(0)}{|D|^{1/2}} L(\varepsilon, 2s) \left(\frac{N_1 y}{N_2}\right)^{-s} \right) + \cdots$$

and $V_s(0) = -\dfrac{\pi^{1/2}\,\Gamma(s + \frac{1}{2})}{\Gamma(s+1)}\, i$

$\rightsquigarrow$ Take $\dfrac{\partial}{\partial s}\Big|_{s=0}$ $\qquad \square$.

$$\boxed{\text{Computation of } \int_0^\infty f(m,y)\, e^{-4\pi m y}\, y^s\, dy}$$

$$F(z) = \frac{\partial}{\partial s} E_{2,s}(z)\Big|_{s=0} = \sum_{m\in\mathbb{Z}} f(m,y)\, e^{2i\pi m z}$$

We have $\frac{\partial}{\partial z}(y^\alpha) = \frac{\alpha}{2i}\, y^{\alpha-1}$

$$\Rightarrow \frac{\partial}{\partial z}\left(\operatorname{Im}(\gamma z)^\alpha\right) = \frac{\alpha}{2i} \operatorname{Im}(\gamma z)^{\alpha-1} \underbrace{\frac{\partial}{\partial z}(\gamma z)}_{\frac{1}{(cz+d)^2}}$$

$$\Longrightarrow E_{2,s}(z) = \frac{2i}{s+1} \frac{\partial}{\partial z} \underbrace{E(z,s+1)}$$

know the Fourier exp.
(uses K-Bessel function)

$$\rightsquigarrow E_{2,s}(z) = y^s - \frac{\pi^{1/2} s\, \Gamma(s+\frac{1}{2})\, \zeta(2s+1)}{\Gamma(s+2)\, \zeta(2s+2)}\, y^{-1-s} + \sum_{m\neq 0} e_{2,s}(m,y)\, e^{2i\pi m z}$$

$\rightsquigarrow$ Compute $\int_0^\infty e_{2,t}(m,y)\, e^{-4\pi m y}\, y^s\, dy$ (integral is tabulated)

and then take $\frac{\partial}{\partial s}\Big|_{s=0}$.