The axiomatic of Rost cycle modules I

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February 12, 2021

We are going to introduce cycle modules (and before that, cycle premodules) which generalize Milnor K-theory, with $\varphi_* : \{a_1, \dots, a_n\} \mapsto \{\varphi(a_1), \dots, \varphi(a_n)\}$, φ^* (the degree $\mathbb{Z} \to \mathbb{Z}$, the norm $E^* \to F^*$, etc.), (ring) product and residue morphism ∂_v.

(Other examples are Quillen K-theory and Galois cohomology.)

We are going to study de Rham cohomology as an example.

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From now on, k is a field and B is the base scheme, it is the (semi-)localization of a separated scheme of finite type over k at a finite (e.g. empty) family of points, or more generally the limit of étale morphisms between separated schemes of finite type over k.

Our schemes X will be "localizations" of separated B-schemes of finite type over k in the same sense.

Definition

A field over B is a field F with a morphism $\mathrm{Spec}(F) \to B$ such that $\mathrm{Spec}(F) \to B \to \mathrm{Spec}(k)$ is the spectrum of a finitely generated extension. A B-field extension $\varphi : F \to E$ is a field extension whose spectrum is a morphism of B-schemes.

Definition

A valuation over B is a non-trivial discrete valuation $v : F^* \to \mathbb{Z}$ with a morphism $\mathrm{Spec}(\mathcal{O}_v) \to B$ such that $k \subset \mathcal{O}_v$ and such that $F = \mathrm{Frac}(\mathcal{O}_v)$ and $\kappa(v) = \mathcal{O}_V/\mathfrak{m}$ are finitely generated extensions of k. (Note that F and $\kappa(v)$ are fields over B.)

We will most often consider normalized (i.e. surjective) discrete valuations.

For example, you can take $\mathcal{O}_v = \mathcal{O}_{X,x}$ with x a point of codimension one in a smooth k-scheme X.

Definition

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- D4 for all valuation $v : F^* \to \mathbb{Z}$ over B, we have a residue morphism $\partial_v : M(F) \to M(\kappa(v))$ of degree -1

satisfying some rules.

Definition

If M is a cycle premodule over B and π is a prime of v (i.e. $\mathfrak{m} = (\pi)$), we have a specialization morphism of degree 0

$s_v^\pi : \begin{cases} M(F) & \to & M(\kappa(v)) \\ \rho & \mapsto & \partial_v(\{-\pi\} \bullet \rho) \end{cases}$ (this uses D3 and D4).

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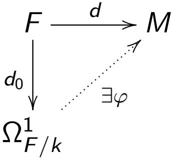
From now on, $k \subset F$ is an extension of fields of characteristic zero.

Definition

The vector space of 1-differential forms (or Kähler differentials), denoted by $\Omega^1_{F/k}$, is the quotient of $\bigoplus_{f \in F} F\mathrm{d}f$ by the sub-F-vector space generated by the $\mathrm{d}\lambda$, $\lambda \in k$, the $\mathrm{d}(f_0 + f_1) - \mathrm{d}f_0 - \mathrm{d}f_1$ and the $\mathrm{d}(f_0 f_1) - f_0.\mathrm{d}f_1 - f_1.\mathrm{d}f_0$, $f_0, f_1 \in F$. The differentiation is the k-linear map $d_0 : \begin{cases} F & \to & \Omega^1_{F/k} \\ f & \mapsto & \mathrm{d}f \end{cases}$ (it satisfies the Leibniz rule).

Universal property of the F-vector space of Kähler differentials.



with M an F-vector space, d a k-linear map satisfying the Leibniz rule, and φ the unique F-linear map given by the universal property.

Definition

Let $n \geq 2$. The vector space of n-differential forms, denoted by $\Omega^n_{F/k}$, is the exterior product of n copies of $\Omega^1_{F/k}$, i.e. it is the quotient of the tensor product over F of n copies of $\Omega^1_{F/k}$ by the sub-F-vector space generated by the $x_1 \otimes \cdots \otimes x_n$ with $i \neq j$ such that $x_i = x_j$. The differentiation is d_{n-1} :

$$\begin{array}{ccc} \Omega^{n-1}_{F/k} & \to & \Omega^n_{F/k} \\ \sum_{i \in I} f_{0,i} d_0(f_{1,i}) \wedge \cdots \wedge d_0(f_{n,i}) & \mapsto & \sum_{i \in I} d_0(f_{0,i}) \wedge d_0(f_{1,i}) \wedge \cdots \wedge d_0(f_{n,i}) \end{array}$$

(it is well-defined and verifies $d_{n-1} \circ d_{n-2} = 0$).

The de Rham complex, denoted by $\Omega^*(F/k)$, is the complex of differential forms and differentiations as above. For all $n \in \mathbb{N}$, we define $H^n_{dR}(F/k) := H^n(\Omega^*(F/k)) = \operatorname{Ker} d_n / \operatorname{Im} d_{n-1}$ (the associated n-th cohomology group (d_{-1} being the zero map by convention)).

If F is a field over Spec(k) then $M(F) = \bigoplus_{n\in\mathbb{N}} H^n_{dR}(F/k)$ (it is a $\mathbb{Z}$-grading with terms zero in negative degree).

D1) If $\varphi : F \to E$ is a k-field extension then we define $\varphi_* : M(F) \to M(E)$ by : $(\varphi_*)^n_n : H^n_{dR}(F/k) \to H^n_{dR}(E/k)$ is the morphism deduced from the morphism $\Omega^n_{F/k} \to \Omega^n_{E/k}$ which verifies

$$f_0 d_0(f_1) \wedge \cdots \wedge d_0(f_n) \quad \mapsto \quad \varphi(f_0) d_0(\varphi(f_1)) \wedge \cdots \wedge d_0(\varphi(f_n)).$$

(Such a morphism exists because of the universal properties of the vector space of Kähler differentials and of the exterior product.)

D2) Let $\varphi : F \to E$ be a finite k-field extension of (finite) Galois closure $\overline{E}$ (we have $\psi : E \to \overline{E}$ and $\psi \circ \varphi : F \to \overline{E}$ is Galois). Denote $G = \mathrm{Gal}(\overline{E}/F)$.

We have a group action of G on $H^n_{dR}(\overline{E}/k)$ given by

$$\sigma \bullet \overline{\sum_{i \in I} f_{0,i} d_0(f_{1,i}) \wedge \cdots \wedge d_0(f_{n,i})} = \overline{\sum_{i \in I} \sigma(f_{0,i}) d_0(\sigma(f_{1,i})) \wedge \cdots \wedge d_0(\sigma(f_{n,i}))}$$

Note that for all $n \in \mathbb{N}$, $H^n_{dR}(\overline{E}/k)^G \simeq H^n_{dR}(F/k)$ (canonically).

We define $\mathrm{Tr}(\omega) = \sum_{\sigma \in G} \sigma \bullet \omega$ and $\varphi^* = \mathrm{Tr} \circ \psi_*$ (via the isomorphism).

The first set of rules is the following :

R1a) Whenever defined, $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$;
R1b) Whenever defined, $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$;
R1c) If $\psi : F \to L$ is a B-field extension and $\varphi : F \to E$ is a finite B-field extension, $R = L \otimes_F E$, then

$$\psi_* \circ \varphi^* = \sum_{p \in \mathrm{Spec}(R)} \mathrm{length}(R_p) \bullet (\varphi_p)^* \circ (\psi_p)_*$$

with $\varphi_p : L \to R/p$ and $\psi_p : E \to R/p$ the canonical morphisms (φ_p is finite since φ is)

Note that R1c) implies R2e) If $\varphi : E \to F$ is a finite and totally inseparable B-field extension then $\varphi_* \circ \varphi^* = \deg(\varphi) \bullet \mathrm{Id}$.

For D3, let F be a k-field.
If $m \in \mathbb{Z}$, define an additive morphism $m\bullet$ by
$m \bullet (f_0 d_0(f_1) \wedge \cdots \wedge d_0(f_n)) = m f_0 d_0(f_1) \wedge \cdots \wedge d_0(f_n)$.

If $f_1', \ldots, f_l' \in F^*$, define an additive morphism $\{f_1', \ldots, f_l'\}\bullet$ by
$\{f_1', \ldots, f_l'\} \bullet (f_0 d_0(f_1) \wedge \cdots \wedge d_0(f_n)) =$
$f_0 f_1'^{-1} \ldots f_l'^{-1} d_0(f_1') \wedge \cdots \wedge d_0(f_l') \wedge d_0(f_1) \wedge \cdots \wedge d_0(f_n)$.
(It is well defined since for all $f \in F^* \setminus \{1\}$,
$f^{-1}(1-f)^{-1} df \wedge d(1-f) = -f^{-1}(1-f)^{-1} df \wedge df = 0$)

The second set of rules is the following :

R2a) Whenever defined, $\varphi_*(x \bullet \rho) = \varphi_*(x) \bullet \varphi_*(\rho)$;
R2b) Whenever defined, $\varphi^*(\varphi_*(x) \bullet \mu) = x \bullet \varphi^*(\mu)$;
R2c) Whenever defined, $\varphi^*(y \bullet \varphi_*(\rho)) = \varphi^*(y) \bullet \rho$

Note that in the expressions $\varphi_*(x)$ and $\varphi^*(y)$, the morphisms are the ones from Milnor K-theory (for instance, φ_* is the identity of $\mathbb{Z}$ or the morphism induced by φ).

Note that R2c) implies R2d) If $\varphi : E \to F$ is a finite B-field extension then $\varphi^* \circ \varphi_* = \deg(\varphi) \bullet \mathrm{Id}$.

For D4, let us describe $(\partial_v)^1_0 : H^1_{dR}(F/k) \to H^0_{dR}(\kappa(v)/k) \simeq H^0_{dR}(k/k) \simeq k$, with v a Spec(k)-valuation of ring O_v (hence $F = \mathrm{Frac}(O_v)$) and residual field $\kappa(v) \simeq k$ (by hypothesis).

We will construct a morphism $\partial : \bigoplus_{f\in F} F\mathrm{d}f \to \kappa(v)$ which will induce a morphism $\partial_v : H^1_{dR}(F/k) = \mathrm{Ker}(d_1)/\mathrm{Im}(d_0) \to \mathrm{Ker}(d_0) = H^0_{dR}(\kappa(v)/k)$.

Note that $\widehat{O_v}$ is a complete discrete valuation ring such that its residual field $\kappa(v)$ and its fraction field are of characteristic zero, hence $\widehat{O_v} \simeq \kappa(v)[[X]]$ and $F_v := \mathrm{Frac}(\widehat{O_v}) \simeq \kappa(v)((X))$.
Let's denote by π a prime of v and by $\psi : F \to F_v$ the canonical morphism. For each $f \in F$, there is a unique $m \in \mathbb{Z} \cup \{+\infty\}$ and a unique decomposition $\psi(f) = \sum_{n \geq m} a_n \pi^n$ such that $a_m \neq 0$.

$$\partial : \begin{cases} \bigoplus_{f \in F} F \mathrm{d}f & \to & \kappa(v) \\ \sum_{i \in I} f_i dg_i & \mapsto & \sum_{i \in I, k \in \mathbb{Z}} a_{i,-k}\, k b_{i,k} \end{cases}$$

(with $\psi(f_i) = \sum_{n \geq n_i} a_{i,n} \pi^n$ and $\psi(g_i) = \sum_{n \geq m_i} b_{i,n} \pi^n$, i.e. the sum of the residues of $f_i g_i'$).

To define $\partial_v : \bigoplus_{n\in\mathbb{N}} H^n_{dR}(F/k) \to \bigoplus_{n\in\mathbb{N}} H^n_{dR}(\kappa(v)/k)$, we use hypercohomology.

First, for a k-scheme X we define $\Omega^1_{X/k}$ and $\underline{\Omega^1_{X/k}}$. Let $\Delta : X \to X \times_k X$ be the diagonal : it is an immersion, so X is isomorphic to a subscheme $(Y, \mathcal{O}_Y)$ of $X \times_k X$; let U be the biggest open of $X \times_k X$ in which Y is closed, and $\mathcal{I}$ be the sheaf of ideals defining the closed subscheme $(Y, \mathcal{O}_Y)$ of $(U, \mathcal{O}_U)$; $\mathcal{I}/\mathcal{I}^2$ is an $\mathcal{O}_{X\times_k X}/\mathcal{I}$-module, i.e. an $\mathcal{O}_Y$-module, i.e. an $\mathcal{O}_X$-module. We define $\Omega^1_{X/k}$ to be this $\mathcal{O}_X$-module.

We define $\underline{\Omega^n_{X/k}}$ to be the Zariski sheaf associated to the presheaf $U \mapsto \Lambda^n(\Omega^1_{U/k})$, with $\Lambda^0(\Omega^1_{U/k}) = \mathcal{O}_U$ and $\Lambda^1(\Omega^1_{U/k}) = \Omega^1_{U/k}$.

Now let us take $\mathcal{O}_v = \mathcal{O}_{X,x}$ with x a point of codimension one in a smooth k-scheme X and $Z := \overline{\{x\}}$ smooth over k (by replacing X by a suitable open neighbourhood of x and Z by $Z \cap U$ if need be).

For F a Zariski sheaf of complexes of $\mathcal{O}_X$-modules (for instance $F = \underline{\Omega^*_{X/k}}$), we define $\Gamma(X, F) = F(X)$ and $\Gamma_Z(X, F) = \{\rho \in F(X),\ \forall x \in Z,\ \rho_x = 0\}$, and $H^n_{Zar}(X, F)$ (resp. $H^n_{Z,Zar}(X, F)$) to be the n-th right derived functor of $\Gamma(X, F)$ (resp. $\Gamma_Z(X, F)$). We define $H^n_{dR}(X) = H^n_{Zar}(X, \underline{\Omega^*_{X/k}})$ and $H^n_{dR}(X, Z) = H^n_{Z,Zar}(X, \underline{\Omega^*_{X/k}})$.

We have the de Rham localization sequence :

$$0 \longrightarrow H^0_{dR}(X,Z) \longrightarrow H^0_{dR}(X) \longrightarrow H^0_{Zar}(X \setminus Z, \underline{\Omega^*_{X/k}}) \xrightarrow{d_0} H^1_{dR}(X,Z)$$

$$\cdots \to H^n_{dR}(X,Z) \longrightarrow H^n_{dR}(X) \longrightarrow H^n_{Zar}(X \setminus Z, \underline{\Omega^*_{X/k}}) \xrightarrow{d_n} H^{n+1}_{dR}(X,Z)$$

We define $(\partial_v)^n_{n-1}$ to be d_n via the isomorphisms $H^n_{Zar}(X \setminus Z, \underline{\Omega^*_{X/k}}) \simeq H^n_{dR}(F)$ and $H^{n+1}_{dR}(X,Z) \simeq H^{n-1}_{dR}(Z) \simeq H^{n-1}_{dR}(\kappa(v))$ (thanks to a purity result and the facts that $\mathcal{O}_{X,Z} = \mathcal{O}_v$ and $\kappa(Z) = \kappa(v)$ (since $Z = \overline{\{x\}}$, $\mathcal{O}_{X,x} = \mathcal{O}_v$ and $F = \mathrm{Frac}(\mathcal{O}_v)$)).

The third set of rules is the following :

R3a) If $v : F^* \to \mathbb{Z}$ is a valuation over B and $\varphi : E \to F$ is a B-field extension such that $w = v \circ \varphi$ is a valuation over B then, denoting $\overline{\varphi} : \kappa(w) \to \kappa(v)$ the morphism induced by φ, we have $\partial_v \circ \varphi_* = |v(F)/w(E)| \bullet \overline{\varphi}_* \circ \partial_w$;
R3b) If $v : F^* \to \mathbb{Z}$ is a valuation over B and $\varphi : F \to E$ is a finite B-field extension then we have $\partial_v \circ \varphi^* = \sum_w \varphi_w^* \circ \partial_w$;
R3c) If $v : F^* \to \mathbb{Z}$ is a valuation over B and $\varphi : E \to F$ is a B-field extension such that $v \circ \varphi = 0$ then $\partial_v \circ \varphi_* = 0$;
R3d) If $v : F^* \to \mathbb{Z}$ is a valuation over B and $\varphi : E \to F$ is a B-field extension such that $v \circ \varphi = 0$, and if π is a prime of v, then, denoting $\varphi : E \to \kappa(v)$ the morphism induced by φ, we have $s_v^\pi \circ \varphi_* = \overline{\varphi}_*$;
R3e) If $v : F^* \to \mathbb{Z}$ is a valuation over B, $u \in \mathcal{O}_v$ is a unit, of class $\overline{u} \in \kappa(v)$, and $\rho \in M(F)$, then $\partial_v(\{u\} \bullet \rho) = -\{\overline{u}\} \bullet \partial_v(\rho)$.

Note that R3e) implies R3f) If $v : F^* \to \mathbb{Z}$ is a valuation over B, π is a prime of v, $x \in K_n F$, and $\rho \in M(F)$, then $\partial_v(x \bullet \rho) = \partial_v(x) \bullet s_v^\pi(\rho) + (-1)^n s_v^\pi(x) \bullet \partial_v(\rho) + \{-1\}\partial_v(x) \bullet \partial_v(\rho)$ and $s_v^\pi(x \bullet \rho) = s_v^\pi(x) \bullet s_v^\pi(\rho)$.

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Definition

A morphism $\omega : M \to M'$ of cycle premodules over B of even type (respectively of odd type) is given by morphisms $\omega_F : M(F) \to M'(F)$ of degree 0 which are even : $\omega_F(-x) = \omega_F(x)$ (respectively odd : $\omega_F(-x) = -\omega_F(x)$) and satisfy :

$\varphi_* \circ \omega_F = \omega_E \circ \varphi_*$;

$\varphi^* \circ \omega_E = \omega_F \circ \varphi^*$;

$\{a\} \bullet \omega_F(\rho) = \omega_F(\{a\} \bullet \rho)$ which implies $\{a_1, \ldots, a_n\} \bullet \omega_F(\rho) = \omega_F(\{a_1, \ldots, a_n\} \bullet \ \rho)$ (respectively $\{a\} \bullet \omega_F(\rho) = -\omega_F(\{a\} \bullet \rho)$ which implies $\{a_1, \ldots, a_n\} \bullet \omega_F(\rho) = (-1)^n \omega_F(\{a_1, \ldots, a_n\} \bullet \ \rho)$);

$\partial_v \circ \omega_F = \omega_{\kappa(v)} \circ \partial_v$ (respectively $\partial_v \circ \omega_F = -\omega_{\kappa(v)} \circ \partial_v$).

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From now on, if X is an irreducible scheme, we denote by ξ_X its generic point.

If X is a normal and irreducible scheme then for each $x \in X^{(1)}$ $O_v := \mathcal{O}_{X,x}$ is a discrete valuation ring, and we denote by ∂_x the residue morphism $\partial_v : M(\kappa(\xi_X)) \to M(\kappa(x))$.

If X is a scheme and $x, y \in X$, we define $\partial_y^x : M(\kappa(x)) \to M(\kappa(y))$ by : if $y \notin \overline{\{x\}}^{(1)}$ then $\partial_y^x = 0$, else $\partial_y^x = \sum_z c_{\kappa(z)/\kappa(y)} \circ \partial_z$ with z running through the points (in finite number) of the normalization of $\overline{\{x\}}$ lying over y.

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- FD (finite support of divisors) For all normal and irreducible schemes X, $\rho \in M(\kappa(\xi_X))$ and all but finitely many $x \in X^{(1)}$, $\partial_x(\rho) = 0$;
- C (closedness) For all integral and local schemes X of dimension 2, denoting x_0 the closed point of X, $\sum_{x \in X^{(1)}} \partial_{x_0}^{x} \circ \partial_{x}^{\xi_X} = 0$.

Morphisms of cycle modules are morphisms of cycle premodules between cycle modules.

Note that in (FD), $\partial_x = \partial_x^{\xi_X}$, that if $x \notin X^{(1)}$ then $\partial_x^{\xi_X} = 0$, and that more generally (FD) implies that if $y \in X$, $\rho \in M(\kappa(y))$, then for all but finitely many $z \in X$, $\partial_z^y(\rho) = 0$.

If X is an integral scheme which verifies (FD), we define $d : M(\kappa(\xi_X)) \to \bigoplus_{x \in X^{(1)}} M(\kappa(x))$ by $d = (\partial_x^{\xi_X})_{x \in X^{(1)}}$ and $A^0(X; M) := \bigcap_{x \in X^{(1)}} \ker(\partial_x^{\xi_X})$.

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If M is a cycle module and F is a field over B then we have :

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- H (homotopy property for $\mathbb{A}^1$) We have the short exact sequence $0 \to M(F) \to M(F(X)) \to \bigoplus_{x \in (\mathbb{A}^1_F)_{(0)}} M(\kappa(x)) \to 0$, the second map being $r_{F(x)/F}$ and the third map being d (with $(\mathbb{A}^1_F)_{(0)}$ the points of $\mathbb{A}^1_F$ whose closure is of dimension 0);

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- H (homotopy property for $\mathbb{A}^1$) We have the short exact sequence $0 \to M(F) \to M(F(X)) \to \bigoplus_{x \in (\mathbb{A}^1_F)_{(0)}} M(\kappa(x)) \to 0$, the second map being $r_{F(x)/F}$ and the third map being d (with $(\mathbb{A}^1_F)_{(0)}$ the points of $\mathbb{A}^1_F$ whose closure is of dimension 0);
- RC (reciprocity for curves) For each proper curve X over F we have

$$c \circ d = 0, \text{ with } c : \begin{cases} \bigoplus_{x \in X_{(0)}} M(\kappa(x)) & \to & M(F) \\ (\rho_i \in M(\kappa(x_i))) & \mapsto & \sum_i c_{\kappa(x_i)/F}(\rho_i) \end{cases}.$$

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- Co (continuity) $A^0(X; M) \subset A^0(Y; M)$ i.e. $\partial_v(A^0(X; M)) = 0$;
- E (evaluation) There exists a unique morphism $\mathrm{ev} : A^0(X; M) \to M(\kappa(x_0))$ such that for all prime π of v, $r_{\kappa(v)/\kappa(x_0)} \circ \mathrm{ev} = s^{\pi}_{v|A^0(X;M)}$.

Thanks for your attention !