ARCHIMEDEAN LOCAL HEIGHTS ON MODULAR CURVES

FRÉDÉRIC DÉGLISE

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1. Introduction: Gross-Zagier formula and archimedean local heights

1.1. Gross-Zagier formula.

1.1.1. Recall that the aim of this workgroup is to present the formula of Gross and Zagier. This formula is modeled on the Dirichlet's class number formula, in the spirit of the conjectures of Beilinson. From the broad picture, it expresses the first derivative (special value) of a Rankin-Selberg L-function in terms of arithmetic invariants and the height of an explicit divisor. The height plays the role of the Dirichlet regulator. The aim of today's talk is to give the necessary computations for the *archimedean part* of this height.

1.1.2. Before going more deeply into this project, recall the geometric setting we have seen previously. We work with Hegneer points of the modular curve $X_0(N)$, associated with the modular group $\Gamma_0(N)$. The latter latter is seen as a scheme over $\mathbf{Q}$. A Hegneer point

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is a K-rational point of $X_0(N)$ for an imaginary quadratic field K — it corresponds to a CM-elliptic curve over K. We will therefore fix such a point $x \in X_0(N)(K)$.

1.2. **Global and local heights.**

1.2.1. In the three precedent talks, we have presented the theory of heights, via three approaches: Weil height functions defined on varieties over number fields up to bounded functions, Néron-Tate canonical heights which remove that dependence for abelian varieties, and Néron local heights. The latter allows to decompose Néron-Tate heights into a sum of local heights, which over the number field K are indexed by all places of K. In fact, we will restrict to the abelian variety $J_0(N)$, jacobian of the curve $X_0(N)$.

Today, we will focus on archimedean places v of K. As K is totally imaginary, the associated archimedean field is $\mathbf{C}$. Let $M = X_0(N)^v_K(\mathbf{C})$ be the associated Riemann surface. A local height at the complex place v, associated to a 0-degree divisor a on M is a real function

$$h_{a,v} : \big(M - Supp(a)\big) \to \mathbf{R}$$

continuous and with prescribed value when $a = \mathrm{div}(f)$ is principal (see (1.2.2.a) below). Note that this function automatically extends to 0-degree divisors b on M provided its support is disjoint from that of a. In fact, let us introduce the following useful variant.

Definition 1.2.2. An *(archimedean) local height pairing* at the complex place v of $X_0(N)$, or equivalent a local height on the Riemann surface $M = X_0(N)(\mathbf{C})$, is a real function:

$$\big(\,\mathrm{Div}^0(M) \times \mathrm{Div}^0(M) - \mathscr{Z}\big) \to \mathbf{R}, (a, b) \mapsto \langle a, b\rangle$$

where $\mathrm{Div}^0(M)$ denotes the 0-degree divisors of M and

$$\mathscr{Z} = \{(a, b) \mid \mathrm{Supp}(a) \cap \mathrm{Supp}(b) = \varnothing\}$$

which is bi-additive, bi-continuous and such that for a principal divisor $b = \mathrm{div}(f)$ and $x \in M$, one has:

$$\langle x, \mathrm{div}(f)\rangle = \log\big(|f(x)|^2\big). \tag{1.2.2.a}$$

Remark 1.2.3. (1) The definition can be extended to arbitrary places (see Leonardo's talk). One usually write $\langle -, -\rangle_v$ for local height pairings associated to a place v.
(2) Normalization (1.2.2.a) is chosen so that once we take the sum over all places, one gets a global height (or global height pairing) which will depends only on the rational equivalence class of divisors.
(3) The square appearing in the above convention accounts to the fact we are looking to complex places, and one has to take care of the corresponding two conjugate complex embedding to get the right product formula.

1.3. **Archimedean heights and Green functions.**

1.3.1. We next consider the compact Riemann surface $M = X_0(N)(\mathbf{C})$.

Fix two points $x_0, y_0 \in M$. A local height pairing on M induces a function:

$$G : M \times M - \delta_M \cup \{(x_0, *), (*, y_0)\} \to \mathbf{R} \tag{1.3.1.a}$$

according to the formula:

$$G(x, y) = \langle x - x_0, y - y_0 \rangle.$$

This function is continuous, and admits logarithmic poles along the boundary.

There is a well-known class of real functions on open subsets of M with logarithmic poles. These are the *Green functions*, which we will review shortly in the next section.

2. Geometry and analysis on modular curves

Here we give some brief overview of the underlying geometro-analytic theory underlying Gross and Zagier construction of archimedean local heights via Green functions. Good references are [Jos17, For81]. We will also quickly recall the definition of Eisenstein series (in the complex case). For that part, the reader is advised to consult [Iwa02].

2.1. **Orbifold structure.**

2.1.1. The topological space $X_0(N)(\mathbf{C})$ is obtained as the quotient

$$M = \mathcal{H}^*/\Gamma_0(N),$$

where $\mathcal{H}$ is the upper half-plane and $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbf{Q})$. The action of $\Gamma_0(N)$ is not free, so one can naturally define a orbifold $\mathcal{H}^*//\Gamma_0(N)$, which is a geometric enrichment of the above topological quotient. However, the above quotient has a natural structure of a compact Riemann surface,

$$M = X_0(N)(\mathbf{C})$$

as we have seen in the preceding talks. This is in fact the associated coarse moduli space. This is the usual geometry that one uses on the latter space. But nevertheless, one has to take care about the orbifold points, which correspond to points of $\mathcal{H}$ with non-trivial stabilizers. Indeed, non-trivial stabilizers will be reflected in the choice of an adequate uniformizing parameter for the Riemannian surface M, that is a complex local coordinate.

Recall that there are two types of orbifold points:

1. *elliptic points* in $\mathcal{H}$, with finite stabilizer of order 2 or 3;
2. cusps corresponding to $\mathbb{P}^1(\mathbf{Q})$, whose stabilizers are infinite parabolic subgroups of $\mathrm{SL}_2(\mathbf{R})$.

In the sequel, we will also consider the open Riemannian surface

$$M^o = Y_0(N)(\mathbf{C}) = \mathcal{H}^*/\Gamma_0(N) \subset M$$

obtained by removing the cusps.

2.1.2. Convention on actions.– Note that we will view $\Gamma_0(N)$ as a subgroup of $\mathrm{PSL}_2(\mathbf{R})$, with its natural action on $\mathcal{H}$. This is slightly unusual compare to the classical notation. This justified as the natural action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathcal{H}$ always factor as an action of

$$\mathrm{PSL}_2(\mathbf{R}) = \mathrm{SL}_2(\mathbf{R})/\{\pm \mathrm{Id}\}.$$

This is also justified as the latter group is precisely the group of isometries of the Poincaré upper half-place $\mathcal{H}$ with is canonical hyperbolic metric (see Example 2.2.3 for recall).

2.2. **Analysis on Riemann surfaces.**

2.2.1. Let M be a smooth variety. A Riemannian metric g on M is a smooth family of scalar products $g_p : T_pM \times T_pM \to \mathbf{R}$. One associates to g the Laplacian differential operator Δ_g of order 2. Recall that a Riemannian manifold is a smooth variety equipped with a conformal class of Riemannian metrics.

Example 2.2.2. Consider the flat Riemann surface $\mathbf{C}$ with local coordinate (x, y). Then the Laplacian differential operator is simply defined as:

$$\Delta = \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y}.$$

Example 2.2.3. On the (Poincaré) upper half-plane $\mathcal{H}$ with local coordinate (x, y) (such that the complex coordinate is $z = x + iy$), one can consider the *hyperbolic metric*:

$$h = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} = \frac{|\,\mathrm{d}z|^2}{(\Im z)^2}.$$

Then, the associated Laplacian is:

$$\Delta_h = y^2 \left(\frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y} \right).$$

Beware that the convention for automorphic forms is to take opposite signs. Therefore we will just define the Laplacian with arithmetic conventions as

$$\Delta = -\Delta_h.$$

2.3. **Harmonic functions and Eisenstein series.**

Let us recall the following classical definition.

Definition 2.3.1. Let (M, g) be a Riemannian manifold. A smooth function $f : M \to \mathbf{R}$ is *harmonic* if it satisfies the partial differential equation:

$$\Delta_g(u) = 0.$$

This does not depend on the choice of g in its conformal class.

In other words, harmonic functions are the smooth functions lying in the kernel of the Laplacian.

Remark 2.3.2.
(1) It is a classical theorem that harmonic functions are functions that are locally the real part of a holomorphic function.
(2) The convention of signs on Laplacian does not change the notion of harmonicity!

2.3.3. When M is compact On a compact connected manifold, harmonic functions consists only of the constant functions.

On a non-compact Riemannian manifold of finite volume, such as the open modular curve $Y_0(N) = \Gamma_0(N) \backslash \mathcal{H}$, the analytic behaviour of the Laplacian is more subtle, because the action of the Laplacian on $L^2(Y_0(N))$ has a much richer *spectral structure*.

Besides its discrete spectrum (generated by square-integrable eigenfunctions), the Laplacian has a *continuous spectrum*, which reflects the presence of the cusps. Let us give an example from classical automorphic theory.

Definition 2.3.4. For $\Re(s) > 1$, the *Eisenstein series* attached to the cusp ∞ of $\Gamma_0(N)$ is defined by

$$E_N(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \Im(\gamma z)^s,$$

where $\Gamma_\infty \subset \Gamma_0(N)$ denotes the stabilizer of the cusp ∞. The series converges absolutely for $\Re(s) > 1$ and defines a smooth $\Gamma_0(N)$-invariant function on $\mathcal{H}$.

The Eisenstein series is not harmonic in general; instead, it is an eigenfunction of the (hyperbolic) Laplacian, satisfying

$$\Delta E_N(\cdot, s) = s(s-1)\, E_N(\cdot, s).$$

As s varies, these functions parametrize the continuous spectrum of the Laplacian on $Y_0(N)$ and describe the asymptotic behaviour of eigenfunctions near the cusps.

2.4. **Potential theory and Green, functions.** In order to formulate Laplace equations with singular right-hand sides, it is convenient to enlarge the space of smooth functions by allowing generalized functions in the sense of Schwartz.

Definition 2.4.1. Let M be a smooth manifold. The $\mathbf{R}$-vector space of *distributions on* M is defined as

$$\mathcal{D}(M) = C^0\big(C_c^\infty(M), \mathbf{R}\big),$$

the space of continuous linear forms on the space $C_c^\infty(M)$ of compactly supported smooth functions, equipped with its natural Fréchet topology.

In other words, the space of distributions of M is the continuous dual of the space of *test functions* $C_c^\infty(M)$. For a distribution $T \in \mathcal{D}(M)$ and a test function $f \in C_c^\infty(M)$, we write

$$\langle T, f \rangle = T(f).$$

Example 2.4.2. Given a point $p \in M$, the *Dirac distribution* at p is defined by

$$\langle \delta_p, f \rangle = f(p).$$

It represents a unit point mass concentrated at p.

2.4.3. Consider now a Riemannian manifold (M, g), with associated Laplacian Δ_g. Fix a point $p \in M$.

A *Green function* with pole at p is, informally, a function $G(\cdot, p)$ which is harmonic on $M \setminus \{p\}$ and has a logarithmic singularity at p.

Example 2.4.4. Consider the flat Riemannian manifold $\mathbf{C}$, with Laplacian as in (2.2.2).

Then, a Green function is a real function $G(z, z')$ of two complex variables, which is defined for $z \neq z'$ by the partial differential equation (in the distribution sense):

$$\Delta_z G(z, z') = \delta_{z'}$$

using the Dirac distribution as above. Solutions of this differential equation are of the form:

$$G(z,z') = \frac{1}{2\pi}\log|z-z'| + h(z) = \frac{1}{4\pi}\log|z-z'|^2 + h(z)$$

where h is a harmonic function on $\mathbf{C} - \{z'\}$ (locally bounded near z').

Let us be more precise, using the notion of smooth measure on M.

Definition 2.4.5. Let again (M, g) be a Riemannian manifold. Fix a point $p \in M$. A *Green function* on M with pole at p is a function $G(-,p)$ satisfying the differential equation (in the distributional sense):

$$\Delta_g G(-,p) = \delta_p - \mu,$$

where $\mu = \varphi.\,\mathrm{dvol}_g$ is a "smooth density". Here φ is a smooth function and $\mathrm{d\,vol}_g$ is the volume form associated to g.

Remark 2.4.6.

(1) In other words, $G(-,p)$ is harmonic away from p and has logarithmic singularity at p. The latter property is encoded by the Dirac distribution.

(2) Note that the right-hand side must have total mass zero:

$$\langle \delta_p - \mu, 1\rangle = \int_M \big(\delta_p - \mu\big) = 0$$

in order for the equation to admit solutions. When M is compact and connected, the latter condition forces the equality:

$$\varphi = \frac{1}{\mathrm{vol}(M)}.$$

In particular, Green functions are unique, up to a constant function. A mean value zero condition therefore guarantees uniqueness.

(3) In general, the presence of the term μ reflects the fact that the Laplacian has non-trivial kernel. When M is non-compact, as will be the case for modular curves due to the presence of cusps, uniqueness is no longer automatic. One needs additional normalization and growth conditions at infinity in order to single out a canonical Green function.

2.4.7. Let us explain the problematic that Gross and Zagier (following previous works) faced in their paper,, in order to define the Green function that will induce the correct archimedean local height, as alluded in 1.3.1.

On the hyperbolic plane $\mathcal{H}$ endowed with its hyperbolic metric, Green functions depend only on the hyperbolic distance between two points. This leads to an explicit differential equation in one real variable, whose solutions can be expressed in terms of classical special functions. Passing to the quotient by a congruence subgroup $\Gamma_0(N)$, one obtains Green functions on the modular curve $Y_0(N)$, provided suitable invariance and growth conditions at the cusps are imposed.

In the non-compact case, such as modular curves, the Laplace equation above has no solution without further modification. This difficulty is a manifestation of the continuous spectrum of the Laplacian and necessitates additional normalization terms. In the construction of Gross-Zagier, these corrections are achieved using Eisenstein series, whose role is precisely to control the behavior of functions at the cusps while preserving harmonicity away from the diagonal.

We will see these principles in action in the next section.

3. Gross-Zagier archimedean local height

3.1. **Axiomatic description.** Let us go back to the definition of an appropriate local height pairing on the Riemann surface $X_0(N)(\mathbf{C})$. We continue the discussion started in 1.3.1.

Lemma 3.1.1. *Let M be a Riemann surface, and consider a function*

$$G : M \times M - \big(\Delta_M \cup \{(x_0,*),(*,y_0)\}\big) \to \mathbf{R}$$

as in (1.3.1.a)*. If $G(x,y)$ is bi-harmonic (see Definition 2.3.1) and has logarithmic singularities of residues -2 (resp. 2) in y at $\{x,x_0\}$ (resp. in x and $\{y,y_0\}$), then the real function of 0-degree divisors $a=\sum_i n_i.x_i$ and $b=\sum_j m_j.y_j$ defined as:*

$$\langle a,b\rangle = \sum_{i,j} n_i m_j.G(x_i,y_j) \qquad \text{(3.1.1.a)}$$

defines a local height pairing on M.

Proof. (See [GZ86], beginning of II.§2.) According to the above formula, the pairing is well-defined and bicontinuous on the appropriate domain as the condition of residue $+2$ shows that the logarithmic poles cancel as we consider 0-degree divisors. Then harmonicity and the condition on residues -2 shows that for any meromorphic function f, the function of x

$$\langle x, \mathrm{div}(f)\rangle - \log\big(|f(x)|^2\big)$$

is harmonic on M. As M is a compact Riemann surface, this implies that this function is a constant c_f. As we take sum of 0-degree divisors, one gets the required property of $\langle\cdot,\cdot\rangle$. □

3.1.2. We now specialize the discussion to the case of the modular curve $X_0(N)(\mathbf{C}) = \Gamma_0(N)\backslash X_0(N)$. As usual in this context, we will define the function $G(z,z')$ up-stairs, that is on $\mathcal{H}\times\mathcal{H}$ after taking uniformizing $z,z'\in\mathcal{H}$ corresponding to $x,x'\in M$. The following definition summarize the needed properties of such a function. These properties are extracted from [GZ86, (2.3)] except that we have added the normalization condition (d5) that Gross and Zagier impose later in the text.[1]

Definition 3.1.3. A function $G : \mathcal{H}\times\mathcal{H}\setminus\Delta \to \mathbf{R}$ satisfies the *Gross-Zagier conditions* if:

(a) ($\Gamma_0(N)$*-invariance*)

$$\forall z,z'\in\mathcal{H}, \forall\gamma,\gamma'\in\Gamma_0(N), G(\gamma z,\gamma' z') = G(z,z');$$

[1] This is done by considering the constant C which is determined just before the statement of Proposition (2.22).

(b) (*harmonicity*) for fixed z', the function $z \mapsto G(z,z')$ is harmonic for $z \notin \Gamma_0(N)z'$, and similarly in the other variable;

(c) (*logarithmic pole*) as $z' \to z$ one has

$$G(z,z') = e_z \log|z-z'|^2 + O(1),$$

where e_z is the order of the stabilizer of z in $\Gamma_0(N)$;

(d) (*growth at cusps*) for fixed $z = x + \mathrm{i}y$, one has

$$G(z,z') = 4\pi y' + O(1) \qquad (z' = x' + \mathrm{i}y' \to \infty), \tag{d1}$$

$$G(z,z') = O(1) \qquad (z' \to \text{ any cusp} \neq 0,\infty), \tag{d2}$$

$$G(z,z') = 0 \qquad (z' \to 0). \tag{d3}$$

Similarly, for fixed z',

$$G(z,z') = 4\pi \frac{y}{N|z|^2} + O(1) \qquad (z = x + \mathrm{i}y \to 0), \tag{d4}$$

$$G(z,z') = O(1) \qquad (z \to \text{ any cusp} \neq 0,\infty), \tag{d5}$$

$$G(z,z') = 0 \qquad (z \to \infty). \tag{d6}$$

Conditions (a), (b), and (c) implies that G descends to a *Green function* on the open Riemann surface $Y_0(N)(\mathbf{C})$: when z' is not elliptic nor a cusp, these properties translate to the differential equation on $X_0(N)$:

$$\Delta_z G(z,z') = 4\pi.\delta_{z'} + h(z,z') \tag{3.1.3.a}$$

where h is an undetermined harmonic function h.

Then condition (d) implies that $G(z,z')$, seen as a function on $z, z' \in M^o = Y_0(N)(\mathbf{C})$, can be extended harmonically at cusps that are not 0 or ∞. Moreover, it states that is has controlled growth at 0 and ∞, and this property implies that $h(z,z')$ must vanish. In other words, (d) removes the intrinsic ambiguity of the above differential equation on the non-compact Riemann surface M^o.

Remark 3.1.4. Note that condition **(c)** should be understood intrinsically in terms of the orbifold structure of $Y_0(N)$. At an elliptic point of order e_z, the natural local uniformizing parameter for the Riemann surface $X_0(N)$ is given by $\rho(z') = (z'-z)^{e_z}$; the logarithmic singularity of G is required to be logarithmic with respect to this parameter.

Similarly, at the cusps ∞ and 0, the uniformizing parameters are $e^{2\pi\mathrm{i}z}$ and $e^{-2\pi\mathrm{i}/(Nz)}$, respectively. This explains the precise form of **(d)**.

The main result we want to prove, extracted from [GZ86, §2], is the following theorem.

Theorem 3.1.5. *There exists an explicit function $G(z,z')$ that satisfies all the required properties of the preceding definition.*

In the remaining of this section, we will prove this theorem. An explicit formula for $G(z,z')$ will be given in (3.4.3.a). In particular, **we will freely use the notation from the previous proposition for the required properties**.

3.2. **Resolvent equation for Laplacian.**

3.2.1. The first step to prove Theorem 3.1.5 is the following classical trick to ensure the periodicity property (a). We consider a diagonally $\mathrm{PSL}_2(\mathbf{R})$-invariant function $g(z, z')$ on $(\mathcal{H} \times \mathcal{H} - \Delta_{\mathcal{H}})$, that is satisfying:

$$\forall \gamma \in \mathrm{PSL}_2(\mathbf{R}), \forall z, z', g(\gamma z, \gamma z') = g(z, z') \tag{3.2.1.a}$$

and use a simple averaging procedure to define our Green function:

$$G(z, z') := \sum_{\gamma \in \Gamma_0(N)} g(z, \gamma z') \tag{3.2.1.b}$$

provided that this series converge. Then, the resulting function $G(z, z')$ will automatically satisfy property (a) — this is an easy check.

We next find appropriate properties on g that guarantee properties (b), (c), (d) of G. Recall that properties (b) and (c) are equivalent to the "distributional Laplace" differential equation (3.1.3.a). If we naively try to resolve the corresponding differential equation for $g(z, z')$ the solution will never give convergent series (3.2.1.a). Instead, we proceed by perturbing the equation by a parameter $\epsilon = s(s-1)$, and resolve the following (ordinary) differential equation (for $z \neq z'$):

$$\Delta_z g_s(z, z') = s(s-1).g_s(z, z'). \tag{3.2.1.c}$$

We will then obtain the correct function by taking limit at $s \to 1$, provided the limit exists.

3.2.2. The next step is to remark that the periodicity property (3.2.1.a) is equivalent to say that a function $g(z, z')$ satisfying this property depends only on the hyperbolic distance $w = d_h(z, z')$. In other words, it can be written as:

$$g(z, z') = F(w).$$

Then, (3.2.1.c) can be rewritten, after the change of variable

$$t = \cosh w = \cosh d_h(z, z') = 1 + \frac{|z - z'|}{2yy'}$$

as the differential equation in the variable $Q(t)$:

$$(1 - t^2).Q''(t) - 2t.Q'(t) + s(s-1).Q(t) = 0.$$

This is a Legendre differential equation of second type (up to the change of variables $l = s + 1$). It has two linearly independent solutions, and the one which is useful for us is the non-polynomial one (also called *Legendre functions of second kind*):

$$Q_{s-1}(t) := \int_0^\infty \left(t + \sqrt{t^2 - 1} \cosh u\right)^{-s} \mathrm{d}u.$$

Therefore a solution to the (3.2.1.c) is

$$g_s(z, z') := Q_{s-1}\left(1 + \frac{|z - z'|}{2yy'}\right).$$

Our choice of Legendre function ensure that $Q_{s-1}(t) = O(t^{-s})$ so that the sum deduced from the averaging procedure (3.2.1.b):

$$G_{N,s}(z,z') := \sum_{\gamma\in\Gamma_0(N)} g(z,\gamma z') \tag{3.2.2.a}$$

converges absolutely for $s > 1$. According to the construction, the resulting function is defined over $M^o \times M^o - \Delta_{M^o}$, satisfies property (a), and the (ordinary) differential equation:

$$\Delta_z G_{N,s}(z,z') = s(s-1).G_{N,s}(z,z').$$

Remark 3.2.3. In the classical terminology, $G_{N,s}(z,z')$ is called the *resolvent kernel* of the hyperbolic Laplacian on M^o.

3.3. **Renormalisations via Eisenstein series.** The second step to prove Theorem 3.1.5 is to renormalize the Green function $G_{N,s}(z,z')$ so that the limit as $s \to 1$ exists and satisfies the appropriate harmonicity property, property (b). Property (a) and (c) will automatically follow from the construction.

3.3.1. For fixed z, z', the function $G_{N,s}(z,z')$ of $s > 1$ has a simple pole at $s = 1$ with residue:

$$\kappa_N := \frac{-4\pi}{\mathrm{vol}_h(M^o)} = -\frac{12}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma_0(N)]}. \tag{3.3.1.a}$$

This is just a computation. However, the result is not surprising, given the general form of Green functions as explained in Remark 2.4.6(2). In fact, the sign can be explained as Gross-Zagier follow the tradition from arithmetician and use the "negative" (hyperbolic) Laplacian. The scalar 4π is explained by the choice of convention of Gross-Zagier, which is chosen so that their final formula has a nice expression.

Therefore substracting the function $\frac{\kappa_N}{s-1}$ to $G_{N,s}(z,z')$ will ensure the required convergence as $s \to 1$. However, the resulting limit will not be harmonic, as a simple computation of its Laplacian shows. In fact, one needs to substract a function with the same pole at $s = 1$ and the same eigenvalue. According to Definition 2.3.4, such functions are given by Eisenstein series. If one wants to guarantee at the same time guarantee respectively the growth properties (d3), for the cusp $z \to 0$, and (d1), for the cusp $z' \to \infty$, one is lead to use the following Eisenstein series:

$$-4\pi.E_N(w_N z, s), -4\pi.E_N(z', s)$$

where w_N is the Atkin-Lehner operator of order N.

Summarizing, using definition (3.2.2.a) and the above notation, the following function is both well-defined, satisfies properties (a), (b), (c) as well as the growth conditions in (d) for $z \to 0$ and $z \to \infty$:

$$\tilde{G}(z,z') := \lim_{s\to 1^+} \left[G_{N,s}(z,z') + \frac{\kappa_N}{s-1} + 4\pi.E_N(w_N z, s) + 4\pi.E_N(z', s) \right]. \tag{3.3.1.b}$$

Remark 3.3.2. Let us observe that, according to property (3.2.1.a) of $g_s(z,z')$, and the above definition, one gets the relation:

$$\tilde{G}(z,z') = \tilde{G}(w_N z', w_N z). \tag{3.3.2.a}$$

3.4. **Growth conditions at cusps.**

3.4.1. We now need to check all the growth conditions of $\tilde{G}(z,z')$ stated in property (d). Note that (d1) and (d3) are already known according to our use of Eisenstein series.

For the other properties, no miracle here: the writer of these notes is not aware of any other method than explicit computation of the Taylor expansions of the three functions $G_{N,s}(z,z')$, $4\pi.E_N(w_N z, s)$, $4\pi.E_N(z', s)$ when z' goes to cusps different from ∞ and z goes to cusps different from 0.

The most painful property is the normalizing properties (d3) and (d6). In fact, according to (3.3.2.a), it is sufficient to treat one of them, say (d6). In fact, the function $\tilde{G}(z,z')$ does not satisfy (d6), but the following lemma shows that we can correct this by adding a constant. It is proved as explained above, by computing Taylor expansions.

Lemma 3.4.2. *For any z', $G(z,z') \to \lambda_N - 2\kappa_N$ where:*

$$\lambda_N = \kappa_N \left(\log N + 2\log 2 - 2\gamma + 2\frac{\zeta'(2)}{\zeta(2)} - 2\sum_{p|N} \frac{p\log p}{p^2-1} \right) \tag{3.4.2.a}$$

where

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} - \log n$$

is the Euler constant *and $\zeta(s)$ is* Riemann's zeta function.

3.4.3. The final definition for Gross-Zagier's Green function, defined on $(\mathcal{H}\times\mathcal{H} - \Delta_{\mathcal{H}})$ and defining a meromorphic function on $M^o \times M^o - \Delta_{M^o}$ is:

$$G(z,z') := \tilde{G}(z,z') - \lambda_N + 2\kappa_N \tag{3.4.3.a}$$

using the notation of (3.3.1.a), (3.3.1.b) and (3.4.2.a).

3.5. **Archimedean local height pairing.**

Definition 3.5.1. We define the Gross-Zagier local archimedean pairing $\langle -,-\rangle_{\mathbf{C}}$ on $M^o \times M^o - \Delta_{M^o}$ according to formulas (3.1.1.a) using the function defined by (3.4.3.a), after choosing representatives in $\mathcal{H}$ of points of M^o.

Example 3.5.2. As an example, one gets for any distinct non-cuspidal points x, x' of $M = X_0(N)(\mathbf{C})$,

$$\langle x - \infty, x' - 0\rangle_{\mathbf{C}} = G(z,z')$$

for z, z' respective representatives of x, x'.

3.5.3. We need more notation for the following formula. Let $m \geq 1$ be prime to N. Choose non-cuspidal points x, x' of M such that x does not belong to the support of $T_m(x')$, where T_m is the m-th Hecke operator. Write:

$$\sigma_i(m) = \sum_{d|m} d^i.$$

and

$$\Gamma_0^m(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = m, c = 0 \mod N \right\}.$$

One can define the following *m-twisted functions*:

$$G^m_{N,s}(z, z') := \frac{1}{2}. \sum_{\gamma \in \Gamma_0^m(N)} g_s(z, \gamma z') \tag{3.5.3.a}$$

$$\tilde{G}^m(z, z') := \lim_{s \to 1^+} \Big[G^m_{N,s}(z, z') + \frac{\sigma_1(m)\kappa_N}{s-1} \tag{3.5.3.b}$$

$$+ 4\pi\sigma_1(m).E_N(w_N z, s) + 4\pi m^s \sigma_{1-2s}(m).E_N(z', s) \Big]. \tag{3.5.3.c}$$

$$G^m(z, z') := \tilde{G}^m(z, z') - \sigma_1(m)\big(\lambda_N - 2\kappa_N\big). \tag{3.5.3.d}$$

Then one deduces from the definition of T_m:

$$\langle x - \infty, T_m(x' - 0) \rangle_{\mathbf{C}} = G^m(z, z').$$

for respective representatives z and z' of x and x'. Note that $T_m(0) = \sigma_1(m).0$ so the left expression is well-defined.

4. Arakelov geometry

4.0.1. From a broader perspective, Green functions are the archimedean analytic ingredient in *Arakelov geometry* [Ara75]. The guiding idea is to enrich divisors on arithmetic varieties with analytic data at the complex places, encoded by Green functions, so that one can define intersection products incorporating both algebraic and analytic contributions.

In this framework, Arakelov constructed *global* height pairings on arithmetic surfaces as (Arakelov) degrees of intersection products of (Arakelov) divisors. Arakelov theory was then used and substantially extended by Faltings to prove the Mordell conjecture (finiteness of $C(K)$ for curves of genus > 1), see [Fal83, Fal84].

The theory was subsequently extended to general arithmetic schemes by Gillet-Soulé, who introduced Arakelov Chow groups endowed with a well-behaved intersection pairing and a corresponding notion of (Arakelov) degree. A cornerstone of this theory is the *arithmetic Riemann-Roch theorem* proved by Gillet-Soulé. See [GZ86] for an overview of these developments.

This framework was also used by Zhang, [Zha97], to extend the results of Gross and Zagier to Shimura curves associated with quaternion algebras. More precisely, Zhang developed an Arakelov-theoretic interpretation of archimedean height pairings for special cycles on Shimura curves, leading to Gross-Zagier type formulas in settings where modular curves are replaced by their quaternionic analogues.

More broadly, Gillet-Soulé's formulation of Arakelov geometry plays a central role in the Kudla program, [Kud04]. This program predicts deep relations between arithmetic intersection numbers of special cycles on Shimura varieties and Fourier coefficients, or derivatives, of suitable automorphic forms, typically Eisenstein series. In this perspective, Green functions and archimedean local height pairings provide the analytic input that bridges arithmetic intersection theory and automorphic representation theory.

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ENS de Lyon, UMPA, UMR 5669, 46 allée d'Italie, 69364 Lyon Cedex 07, FRANCE
Email address: frederic.deglise@ens-lyon.fr
URL: https://deglise.perso.math.cnrs.fr/en/index.html