HEIGHTS IN THE CASE OF NON-DISJOINT SUPPORTS ; COMPUTATIONS IN THE ARCHIMEDEAN CASE

Understanding [GZ86, II.§5] with the help of [Gro86] and [Con04].

1. Heights in the case of non-disjoint supports

Fix H a number field, and X be a curve defined over H. Recall that we have a global height pairing $\langle , \rangle$ on $\mathrm{Div}^0(X)(H)$, and also for each place v of H a local height pairing $\langle , \rangle_v$ on the set of pairs of elements of $\mathrm{Div}^0(X_v)(H_v)$ with disjoint supports, satisfying a product relation :

$$\langle a, b\rangle = \sum_v \langle a, b\rangle_v$$

if a, b are in $\mathrm{Div}^0(X)(H)$ with disjoint supports.

We want to extend the local heights pairings to at least some cases with non-disjoint supports, in a way that the product formula still holds. We will see how we can do this when $|a| \cap |b|$ is a union of points of $X(H)$, but the extension will not be canonical.

1.1. **Definition of generalized local height.**

1.1.1. *Generalized evaluation.* Fix a place v, and $x \in X_v(H_v)$. If v is complex, we set $|a|_v = |a|^2$ for $a \in \mathbb{C}$.

Definition 1.1.1. *A uniformizer t_x at x is a function defined on a neighborhood of x such that $t_x \in \mathfrak{m}_{X_v,x} \setminus \mathfrak{m}^2_{X_v,x}$. That is, a rational function on X_v that has a simple zero at x.*

Definition 1.1.2. *Let f be a non-zero element of $H_v(X_v)$, and let t_x be a uniformizer at x. We define the generalized evaluation of f at x along t_x by :*

$$f_{t_x}[x] = \left(\frac{f}{t_x^{v_x(f)}}\right)(x) \in H_v^\times$$

Note that $f_{t_x}[x] = f(x)$ if $v_x(f) = 0$, and that generalized evaluation is multiplicative.

If t_x is a uniformizer at x, let ω_x be its image in the cotangent space at x, that is, $\mathfrak{m}_{X_v,x}/\mathfrak{m}^2_{X_v,x}$ (essentially this is $t'_x(x)$). We see easily that $f_{t_x}[x]$ depends only on ω_x, we denote it by $f_{\omega_x}[x]$.

1.1.2. *Generalized local height.* Let a and b be in $\mathrm{Div}^0(X_v)(H_v)$. Assume that $|a| \cap |b| = \{x_1, \ldots, x_n\}$ with each x_i in $X_v(H_v)$. Let $\underline{\omega} = \{\omega_1, \ldots, \omega_n\}$ be a set of cotangent vectors at each x_i.

Definition 1.1.3. *Let $f \in H_v(X_v)^\times$ such that $a + \mathrm{div}(f)$ and b have disjoint supports. We define :*

$$\langle a, b\rangle_{v,\underline{\omega}} = \langle a + \mathrm{div}(f), b\rangle_v - \log |f_{\underline{\omega}}[b]|_v$$

where $f_{\underline{\omega}}[b]$ is defined as follows : write $b = \sum n_i[x_i] + b'$ so that the support of b' does not contains any x_i (note that b' is in $\mathrm{Div}^0(X_v)(H_v)$, and that b' and $\mathrm{div}(f)$ have disjoint support). Then

$$f_{\underline{\omega}}[b] = \left(\prod_i f_{\omega_i}[x_i]^{n_i}\right) f(b')$$

Proposition 1.1.4. *The value of $\langle a, b\rangle_{v,\underline{\omega}}$ does not depend on the choice of f satisfying the conditions. In particular if a and b have disjoint supports it is equal to $\langle a, b\rangle$.*

Démonstration. It is enough to show this when a and b already have disjoint supports, as the generalized evaluation is multiplicative. Then $\mathrm{div}(f)$ and b also have disjoint supports. Then the result follows from general properties of local heights : $\langle \mathrm{div}(f), b\rangle = \log |f(b)|_v$. □

Proposition 1.1.5. *Assume that $\alpha_1, \ldots, \alpha_n$ are elements of H_v with $|\alpha_i|_v = 1$. Let $\omega'_i = \alpha_i\omega_i$. Then $\langle a, b\rangle_{v,\underline{\omega}} = \langle a, b\rangle_{v,\underline{\omega'}}$.*

1.2. **Alternative definition.** Let a and b be in $\mathrm{Div}^0(X_v)(H_v)$. Assume that $|a| \cap |b| = \{x_1, \dots, x_n\}$ with each x_i in $X_v(H_v)$. Let $\underline{\omega} = \{\omega_1, \dots, \omega_n\}$ be a set of cotangent vectors at each x_i, and $t_1, \dots, t_n$ uniformizers corresponding to these cotangent vectors.

For each i, denote by y_i an element of $X_v(H_v)$ which is disjoint from the x_j and the other y_j and the support of b'. Write $a_{\underline{y}}$ for the divisor that we obtain by replacing each x_i by y_i.

Proposition 1.2.1.

$$\langle a, b\rangle_{v,\underline{\omega}} = \lim_{\underline{y}\to\underline{x}} \left(\langle a_{\underline{y}}, b\rangle_v - \sum_i \mathrm{ord}_{x_i}(a)\,\mathrm{ord}_{x_i}(b) \log |t_{x_i}(y_j)|_v \right)$$

and the limit actually exists.

Démonstration. Assume for simplicity of notation that $n = 1$ (which is what we use anyway in the end), and denote by t a uniformizer at x.

Write $b = n[x] + b'$, $a = m[x] + a'$ so that $n = \mathrm{ord}_x(b)$, and $m = \mathrm{ord}_x(a)$. Let f be as above, so that $\mathrm{ord}_x(f) = -m$ and f is of the form $t^{-m}g$ with $\mathrm{div}(g)$ prime to a and b.

Then we need to show that the limit of the following is zero as $y \to x$:

$$\begin{aligned}
\mathrm{LHS} - \mathrm{RHS} &= \langle a + \mathrm{div}(f), b\rangle_v - \log|f_t[b]|_v - \langle a_y, b\rangle_v + mn\log|t(y)|_v \\
&= \langle a + \mathrm{div}(f) - a_y, b\rangle_v - \log|f(b')|_v - n\log|f_t[x]|_v + mn\log|t(y)|_v \\
&= \langle (m([x]-[y]-\mathrm{div}(t)) + \mathrm{div}(g), b\rangle_v - \log|f(b')|_v - n\log|g(x)|_v + mn\log|t(y)|_v \\
&= \langle -m\,\mathrm{div}(t)_y + \mathrm{div}(g), b\rangle_v - \log|f(b')|_v - n\log|g(x)|_v + mn\log|t(y)|_v \\
&= -m\left(\langle \mathrm{div}(t)_y, b\rangle_v - \langle \mathrm{div}(t), b_y\rangle_v\right)
\end{aligned}$$

Hence it is enough to show that for any pair of divisors of degree 0, with intersection of supports supported in $\{x\}$, $\langle a_y, b\rangle_v - \langle a, b_y\rangle_v$ goes to zero as $y \to x$. Fix some w far away, and write a and b as sums of $[u] - [w]$, then it follows from continuity of the local height and symmetry. □

1.3. **Product formula.**

Theorem 1.3.1. *Let a and b be in $\mathrm{Div}^0(X)(H)$. Assume that $|a| \cap |b| = \{x_1, \dots, x_n\}$ with each x_i in $X(H)$. Let $\underline{\omega} = \{\omega_1, \dots, \omega_n\}$ be a set of cotangent vectors at each x_i. For each place v, $\underline{\omega}$ defines a set of cotangent vectors at the images of each x_i in $X_v(H_v)$.*

Let $f \in H(X)^\times$ such that $a + \mathrm{div}(f)$ and b have disjoint supports.

Then

$$\langle a, b\rangle = \sum_v \langle a, b\rangle_{v,\underline{\omega}}$$

and the latter sum is independent of the choice of f.

Démonstration. We compute that

$$\sum_v \langle a, b\rangle_{v,\underline{\omega}} = \sum_v \left(\langle a + \mathrm{div}(f), b\rangle_v - \log|f_{\underline{\omega}}[b']|_v\right) = \langle a + \mathrm{div}(f), b\rangle = \langle a, b\rangle$$

as

$$\sum_v \log|f_{\underline{\omega}}[b']|_v = 0$$

by the product formula. □

Proposition 1.3.2. *Assume that $\alpha_1, \dots, \alpha_n$ are elements of H that are roots of unity. Let $\omega_i' = \alpha_i\omega_i$. Then $\langle a, b\rangle_{v,\underline{\omega}} = \langle a, b\rangle_{v,\underline{\omega'}}$ for all places v.*

Démonstration. This is simply Proposition 1.1.5, as $|\alpha_i|_v = 1$ for all places v. □

2. Height computations

2.1. **What we want to compute.** Let $x \in Y_0(N)(H)$ be a Heegner point. Let $c = (x) - (\infty)$ and $d = (x) - (0)$, so that c and d are in $\mathrm{Div}^0(X_0(N))(H)$. Let $m \geq 1$ be prime to N. We want to compute $\langle c, T_m\sigma(d)\rangle$ for $\sigma \in \mathrm{Gal}(H/K)$. For this we decompose $\langle c, T_m\sigma(d)\rangle$ in a sum of local heights.

Let $\mathcal{A} \in \mathrm{Cl}(\mathcal{O}_K)$ corresponding to σ.

Proposition 2.1.1. *We have $|c| \cap |T_m\sigma(d)| \subset \{x\}$, and $\mathrm{ord}_x(c) = 1$ and $\mathrm{ord}_x(T_m\sigma(d)) = r_{\mathcal{A}}(m)$ where $r_{\mathcal{A}}(m)$ is the number of ideals of $\mathcal{O}_K$ in the class $\mathcal{A}$ that have norm m.*

Démonstration. Let $x = ([\mathfrak{a}], \mathfrak{n})$. Let $\sigma = \sigma_{\mathcal{A}}$ and let $\mathfrak{b}$ in the class of $\mathcal{A}^{-1}$, so that $\sigma(x) = ([\mathfrak{a}\mathfrak{b}], \mathfrak{n})$.

Recall that $T_m y = \sum_C y_C$ the sum is over all subgroups of E_y of order m and with trivial intersection to $\ker \phi_y$, and then y_C is $(E_y/C, E'_y/\phi_y(C))$.

An isogeny of order m $x \to \sigma(x)$ is the same as finding an inclusion $\alpha^{-1}\mathfrak{a} \subset \mathfrak{a}\mathfrak{b}$ of index m for some $\alpha \in K$, that is, $(\alpha\mathfrak{a})^{-1} \subset \mathcal{O}_K$ an integral ideal of norm m □

In particular, the divisors do not necessarily have disjoint supports, so we will use the generalization of the local heights. Let t be a uniformizer at x, then

$$\langle c, T_m\sigma(d)\rangle_{v,t} = \lim_{y\to x} \left(\langle c_y, T_m\sigma(d)\rangle_v - r_{\mathcal{A}}(m)\log|t(y)|_v\right)$$

2.2. **Choice of uniformizer.** We want to choose t the uniformizer at x.

One way to do this is as follows : Choose a lift $\hat{x}$ of x in $\mathcal{H}$. Choose a 1-form ω over $\mathcal{H}$, that is of the form $f(z)dz$ where $f(\hat{x}) \neq 0$. We will choose ω to be $2i\pi\eta^4(z)dz$, where $\eta(z) = q^{1/24}\prod_{n\geq 1}(1-q^n)$ (so that $\eta^{12} = \Delta$). It is defined only up to some 6-th root of unity but this won't change the local heights computations.

Let t be any uniformizer at x. Then it lifts locally around $\hat{x}$ to $\hat{t}$.

Assuming that x is not an elliptic point then we can normalize t (by multiplying by a non-zero constant) so that $\hat{t}'(\hat{x}) = f(\hat{x})$, we take this to be our uniformizer.

Assume now that $\mathcal{H} \to X$ ramifies at $\hat{x}$ with multiplicity $e = 2, 3$. Then we normalize so that we have for z close to $\hat{x}$: $\hat{t}(z) = f(\hat{x})(z-\hat{x})^e(1+o(1))$.

An important point for us is :

Proposition 2.2.1. *If x is a Heegner point then $u = e$.*

That is, the ramification is a property of the field giving complex multiplication.

A consequence of this choice is :

Proposition 2.2.2. *Let $\hat{x}$ in $\mathcal{H}$ lifting x a Heegner point. Then*

$$\log|t(y)|_v - u\log|(\hat{y}-\hat{x})\cdot 2i\pi\eta^4(\hat{x})|_v \to 0$$

as $\hat{y} \to \hat{x}$ (and then $y \to x$).

Remark 2.2.3. Actually we need the uniformizer to be defined over H, in order to be able to use it for all places. It is not clear that the construction accomplishes this. Indeed, $\eta^{24} = \Delta$ is defined over $X_0(1)(\mathbb{Q})$, but $\omega = \eta^4 dq/q$ is a priori not defined on X but only on a cover of order 6 of it.

2.3. **Archimedean computations.** Recall the following formula for the computation of the local archimedean height :

Proposition 2.3.1 (Proposition 2.23)**.** *Let $(m, N) = 1$, $x, x' \in Y$ with $x \notin T_m x'$. Then :*

$$\langle (x)-(\infty), T_m((x')-(0))\rangle = A + \lim_{s\to 1}\left(B/(s-1) + G^m_{N,s}(\hat{x},\hat{x}') + CE_N(w_N\hat{x}, s) + Dm^s\sigma_{1-2s}(m)E_N(\hat{x}', s)\right)$$

where A, B, C, D are some constants, and

$$G^m_{N,s}(z,z') = \sum_{\gamma\in R_n/\{\pm 1\}, \det\gamma = m} g_s(z, \gamma z')$$

where g_s is a function of $s > 0$ defined out of the diagonal.

We want to extend the definition of $G^m_{N,s}$ to all z, z'. Let

$$Z_s(z,z') = \lim_{w\to z}\left(\sum_{\gamma\in R_n/\{\pm 1\}, \det\gamma = m, \gamma z' = z} g_s(z,w) - \log|2i\pi(z-w)\eta^4(z)|^2\right)$$

and $G^0_s(z,z') = \sum_{\gamma\in R_n/\{\pm 1\}, \det\gamma = m, \gamma z' \neq z} g_s(z,\gamma z')$ and set

$$\tilde{G}_s(z,z') = G^0_z(z,z') + Z_s(z,z')$$

which coincides with the definition of G_s when $x \notin T_m x'$: then $G_s = G^0_s$ and $Z_s = 0$. Moreover G^0_s is clearly defined everywhere. Let us see why Z_s is well-defined. Note that

$$Z_s(z,z') = \#\{\gamma, \gamma z' = z\}\lim_{w\to z}\left(g_s(z,w) - \log|2i\pi(z-w)\eta^4(z)|^2\right)$$

As $w \to z$, $g_s(z,w) = \log(|z-w|^2/(2\operatorname{im}(z)\operatorname{im}(w)) + O(1)$ so $g_s(z,w) - \log|z-w|^2 - \log(|2i\pi\eta^4(z)|^2)$ has a finite limit as $w \to z$.

Write $g_s(z)$ for $\lim_{w\to z}\left(g_s(z,w) - \log|2i\pi(z-w)\eta^4(z)|^2\right)$, so that $Z_s(z,z') = \#\{\gamma, \gamma z' = z\}g_s(z)$.

Let $z' = \hat{x}'$ be fixed, where $x \in Y(H)$ and $x' = \sigma(x)$, and set :

$$f(w,s) = A + B/(s-1) + G^0_s(w,z') + CE_N(w_N w, s) + Dm^s\sigma_{1-2s}(m)E_N(z', s)$$

Then $(z = \hat{x})$:

$$\langle c, T_m\sigma(d)\rangle_{v,t} = \lim_{w\to z}\lim_{s\to 1}\left(f(w,s) - r_{\mathcal{A}}(m)\log|t(w)|^2\right)$$

Let us admit that we can exchange the limits. Then the local height is :

$$A + \lim_{s\to 1}\Big(B/(s-1) + CE_N(w_N z, s) + Dm^s\sigma_{1-2s}(m)E_N(z', s) + \lim_{w\to z}\left(G_s(w, z') - r_{\mathcal{A}}(m)\log|t(w)|^2\right)\Big)$$

In the limit we can replace $\log|t(w)|^2$ by $\log|2i\pi(z-w)\eta^4(z)|^2$, by Proposition 2.2.2.

Note also that $\#\{\gamma, \gamma z' = z\}$ appearing in Z_s is $ur_{\mathcal{A}}(m)$ in the Heegner point case. Then the height is

$$A + \lim_{s\to 1}\Big(B/(s-1) + CE_N(w_N z, s) + Dm^s\sigma_{1-2s}(m)E_N(z', s) + \tilde{G}_s(z, z')\Big)$$

In particular, this is identical to the formula of Proposition 2.23 if we replace G_s by $\tilde{G}_s$. We redecompose this as :

$$A + \lim_{s\to 1}\Big(\Big(B/(s-1) + CE_N(w_N z, s) + Dm^s\sigma_{1-2s}(m)E_N(z', s) + \tilde{G}^0_s(z, z')\Big) + ur_{\mathcal{A}}(m)g_s(z)\Big).$$

2.4. **Evaluation at Heegner points.** We now want to find an expression for the thing appearing in the limit as $s \to 1$, when we sum over all places. We observe that : the first part (everything except the part with $g_s(z)$) can be treated exactly as in the case where $r_{\mathcal{A}}(m) = 0$, and we recover for this part the formula given by Olivier (proposition (4.2) of [GZ86]).

Now we want to understand the contribution of the $g_s(z)$ for the various places. Recall that $g_s(z, w) = -2Q_{s-1}(1 + |z-w|^2/(2\operatorname{im}(z)\operatorname{im}(w))$ and $g_s(z) = \lim_{w\to z}(g_s(z, w) - 2\log|w-z|) - 2\log|2i\pi\eta^4(z)|$.

Using the asymptotic expansion of $Q_{s-1}(t)$ as $t \to 1$ we get

$$g_s(z) = -\log|4\pi\operatorname{im}(z)\eta^4(z)|^2 + 2\left((\Gamma'/\Gamma)(s) - (\Gamma'/\Gamma)(1)\right).$$

On the other hand, Kronecker's first limit formula is :

$$2^s\zeta(2s)E(z, s) = \pi/(s-1) + 2\pi(\gamma - \log(2) + \log(|\sqrt{\operatorname{im}(z)}\eta^2(z)| + O(s-1)$$

and $\Gamma'(1) = -\gamma$ and $\Gamma(1) = 1$.

So we get

$$g_s(z) = -2\log(2\pi) + 2\left((\Gamma'/\Gamma)(s) + (\Gamma'/\Gamma)(1)\right) + (2/\pi)\lim_{\sigma\to 1}(2^\sigma\zeta(2\sigma)E(z, \sigma) - \pi/(\sigma - 1))$$

In order to compute the contribution at all places we want $\sum_{\mathcal{B}\in \mathrm{Cl}(\mathcal{O}_K)} g_s(\tau_{\mathcal{B}})$. We have the identity :

$$2^\sigma\zeta(2\sigma)E(\tau_{\mathcal{B}}, \sigma) = u|D|^{1/2}\zeta_K(\mathcal{B}, \sigma)$$

so in the sum the contribution coming from the $\sum_{\mathcal{B}} 2^\sigma\zeta(2\sigma)E(\tau_{\mathcal{B}}, \sigma)$ is $u|D|^{1/2}\zeta_K(\sigma)$.

In order to understand the part with $\zeta_K(\sigma)$ we use $\zeta_K(\sigma) = \zeta(\sigma)L(\epsilon, \sigma)$ and $\zeta(\sigma) = 1/(\sigma-1) + \gamma + O(\sigma-1)$ and $L(\epsilon, \sigma) = L(\epsilon, 1) + (\sigma - 1)L'(\epsilon, 1) + O(\sigma - 1)^2$ and $L(\epsilon, 1) = \pi h/u|D|^{1/2}$.

Finally we get that the additional contribution is :

$$2hr_{\mathcal{A}}(m)\left((\Gamma'/\Gamma)(s) - \log(2\pi) + (L'/L)(\epsilon, 1) + \log|D|^{1/2}\right)$$

Références

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