Local Heights and Néron's Formula

REMOLD Working Group: Heights and the Gross-Zagier Formula

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Contents

1 Introduction

Let K be a field together with a set M_K of absolute values satisfying the product formula with weights n_v. In previous talks we have seen global heights in various forms:

- For a point of projective space:

$$x = (x_0, \dots, x_n) \in \mathbb{P}^n(\overline{K}) \quad \to \quad h(x) = \frac{1}{[L:K]} \sum_{w \in M_L} n_w \max_i \log |x_i|_w,$$

where L is any finite extension of K over which x is defined, M_L is the set of absolute values on L extending those of M_k and

$$n_w = \frac{[L_w : K_v]}{[L:K]};$$

- more generally for a morphism $f : V \to \mathbb{P}^n$, $h \circ f$ is a height. When V is projective we have a map
$$\mathrm{Pic}(V) \to \mathbb{R}^{V(K)}/O(1);$$
- for abelian varieties we have a canonical height, the Néron-Tate height, associated to each line bundle, which satisfies all the nice properties that general global heights only satisfy up to a bounded function.

In this talk we will try to answer the following question: can we split global heights into a sum of local contributions λ_v from each v? The formula for the height on $\mathbb{P}^n$ suggests that we could choose something that looks like $\max_i \log |x_i|_v$. However, this expression is not even well defined (while the sum was well defined thanks to the product formula). To remedy this, we have to choose some meromorphic section of $\mathcal{O}(1)$ (because the standard height on $\mathbb{P}^n$ is associated with $\mathcal{O}(1)$), for example x_0, and use instead
$$\max_i \log |x_i|_v - \log |x_0|_v.$$
This is well defined, but it depends on the chosen section, and it is not defined where $x_0 = 0$. This is no coincidence: local heights will not be associated to divisor classes (line bundles), like in the global case, but to (Cartier) divisors.
The structure of the talk will be as follows:

1. Construction of local heights, and of canonical local heights for abelian varieties.
2. Connection with global heights.
3. The special case of curves. On a regular curve C, a divisor can be identified with a cycle of closed points. In particular, one might imagine to be able to construct a pairing $\langle -, - \rangle$ on pairs of divisors (with some assumptions) such that for points x, y we get $\langle x, y \rangle = h_y(x)$. This will be the case, both globally and locally, and moreover we will be able to get a canonical pairing. This pairing will be computable using intersection theory.

Notation: If X is a scheme, $x : \mathrm{Spec}\, F \to X$ is an F-point for some field F, U is a neighbourhood of x in X, then we get a map $x^* : \mathcal{O}(U) \to F$, and for $\phi \in \mathcal{O}(U)$ we denote its image by $\phi(x) \in F$.

2 Local heights

In this section we follow [Ser97, Chapter 6]. Let K be a field complete with respect to an absolute value $|-|_v$. It is convenient to assume K to be algebraically closed. The reader can consult instead [Lan83, Chapter 10] for a more general (and technical) treatment.
Put
$$v(-) = -\log |-|_v.$$
Consider a quasi-projective variety V over K.
The set of K-points $V(K)$ comes equipped with a topology induced by that of K, called the

K-*topology*: it is the coarser topology such that, for any Zariski open U and for any $f \in \mathcal{O}(U)$, the function $|f|_v : U(K) \to \mathbb{R}$ is continuous ([Lan83, Chapter 10]).

Definition 2.1. A subset $B \subset V(K)$ is *bounded* if it satisfies the following condition.
If $V = \mathrm{Spec}(A)$ is affine, B is bounded if $f(B)$ is bounded in K for all $f \in A$ (this can be checked on a finite number of generators of A).
In the general case, B is bounded if there exists a finite open affine cover $V = \bigcup U_i$ and B is the union of subsets $B_i \subset U_i(K)$ which are bounded in U_i. One can easily see that the two definitions are compatible.
Warning: This is not equivalent to asking the same condition for *all* open covers U_i and decompositions B_i.

Remark 2.2.

1. When K is locally compact a set B is bounded if and only if the closure $\overline{B}$ is compact for the K-topology.
2. Projective space is bounded: it is the union of the subsets

$$B_i = \{x \in \mathbb{P}^n(K) \mid |x_i|_v \geqslant |x_j|_v \ \forall j\} \subset \{x_i \neq 0\},$$

which are all bounded by 1. More generally, any projective variety is bounded.

3. If $V \xrightarrow{f} W$ is a morphism, the image of any bounded set is bounded.

Definition 2.3. A real valued function on $V(K)$ is *strongly continuous* if it is continuous (for the K-topology) and preserves bounded subsets.

Remark 2.4.

1. When K is locally compact this is equivalent to being continuous.
2. When $V = \mathrm{Spec}\, A$ is affine, any $f \in A$ is strongly continuous (by definition of bounded set).
3. In what follows we will work with projective varieties: in this case the second condition is equivalent to being bounded on all $V(K)$.

Definition 2.5 (Local height). Let D be a Cartier divisor on V. A *local height* associated to D is a real valued function

$$\lambda : V(K) - \mathrm{Supp}\, D \to \mathbb{R}$$

satisfying the following condition: if U is an affine open of V where $D = (\phi)$ is principal, with $\phi \in K(V)$, then the function

$$x \mapsto \lambda(x) - v(\phi(x))$$

defined on $U(K) - \mathrm{Supp}\, D$ extends to a strongly continuous function on $U(K)$. If λ exists it is unique up to a strongly continuous function.

Example 2.6. On $\mathbb{P}^n$, with D the hyperplane $x_0 = 0$ (with local equation x_0/x_i where $x_i \neq 0$) we can put

$$\lambda(x) = v(x_0) - \inf_{0 \leqslant i \leqslant n} v(x_i) = \max_i \log|x_i|_v - \log|x_0|_v$$

REMARK 2.7. 1. If $D = (\phi)$ is a principal divisor, we may take $\lambda_D(x) = v(\phi(x))$.

2. $\lambda_{D_1} + \lambda_{D_2}$ is a local height for $D_1 + D_2$.

3. If $V \xrightarrow{f} W$ is a morphism, $C = f^*D$, then $\lambda_C = \lambda_D \circ f$ is a local height for C. This follows from the definition of local height, the fact that any local equation of D pulls back to a local equation of C, and from 2.4(2).

COROLLARY 2.8. Local heights exist for quasi-projective varieties. Using the properties of the remark one reduces to the cases of the example.

3 NÉRON'S THEOREM

As before K is an algebraically closed field complete with respect an absolute value $|-|_v$.

THEOREM 3.1. To all pairs (A, D), where A is an abelian variety over K and D is a divisor on A, there is an associated local height

$$\lambda_D : A(K) - \operatorname{Supp} D \to \mathbb{R},$$

which is defined up to a constant function, satisfying the following properties (all up to constants).

1. If $D = (\phi)$ is principal, then $\lambda_D(x) = v(\phi(x))$.

2. $\lambda_{D_1} + \lambda_{D_2} = \lambda_{D_1+D_2}$.

3. If $\pi : A \to B$ is a morphism (not necessarily such that $0 \mapsto 0$), and if π^*D is defined, then $\lambda_{\pi^*D} = \lambda_D \circ \pi$.

For the proof we will use the following lemma.

LEMMA 3.2. Let S be a set, $\pi : S \to S$ a map, and E the Banach space of bounded real valued functions on S, with the sup-norm. Let $A : E \to E$ be the operator given by $f \mapsto f \circ \pi$. Then the operator

$$\mathrm{Id} - \frac{1}{\lambda}A$$

is invertible for all $\lambda > 1$, with inverse given by the formula

$$f \mapsto \sum_0^\infty \frac{1}{\lambda^n} A^n.$$

PROOF. We may assume that the divisor class of D is either symmetric or antisymmetric. Indeed, the divisor group is generated by divisors which are either principal, symmetric or antisymmetric. Property (1) already defines λ for principal divisors, and (2) implies that we may restrict to a generating subset of the group.

Under the map $x \mapsto 2x$ on A, by the formula for pullback of line bundles by multiplication maps (see for example [Mum70, I.6, Corollary 3]) we find

$$[2]^*D = rD + (\phi), \quad r = 4 \text{ or } r = 2,$$

where r is either 2 or 4 and ϕ is a rational function. Let μ be any local height associated to D, then we have

$$\mu(2x) = r\mu(x) + \nu(\phi(x)) + \varepsilon(x),$$

where ε is a strongly continuous (and thus bounded) function on $A(K)$. From Lemma 3.2 we find a bounded function η such that $\varepsilon = \eta(2x) - r\eta(x)$, and we put

$$\lambda = \mu - \eta.$$

This yields

$$(*) \qquad r\lambda(2x) = r\lambda(x) + \nu(\phi(x)).$$

This local height satisfies the required conditions: $(*)$ has to be satisfied by our required heights. Moreover, we have the following fact:

LEMMA 3.3. For any $\omega \in \mathbb{R}$, the only bounded function $f : A(K) \to \mathbb{R}$ satisfying

$$f(2x) = rf(x) + \omega$$

is the constant function $f = \omega/(1-r)$.

If two local heights satisfy $(*)$, then they are equal by the case $\omega = 0$ of the lemma. The indeterminacy up to a constant is due to the non-canonical choice of ϕ: two functions $\phi, \psi \in K(V)$ define the same principal divisor if and only if they differ by a constant factor $\alpha \in K$. However, two heights satisfying $(*)$ for different choices of ϕ will differ by a constant $\nu(\alpha)$ by the lemma. ■

REMARK 3.4. The observation that the "up to a constant" part of the theorem is actually up to a constant $\nu(\alpha)$ is important when considering all the absolute values of K together: by the product formula we have $\nu(\alpha) = 0$ for almost all ν, and when taking sums over all ν these contributions cancel out allowing for canonical global heights.

4 RELATION WITH GLOBAL HEIGHTS

In this section we sketch some of the ideas in [Lan83, Chapter 10 and 11] for connecting the local heights developed above with global heights.
Let K be a field together with a set of absolute values M_K satisfying the product formula with weights n_ν. Let V be a projective variety over K and D be a Cartier divisor on V; we know that there exists a notion of global height h_D associated to D, and we would like to have a formula

$$h_D(x) = \frac{1}{[L:K]} \sum_{w \in M_L} n_w \lambda_{D,w}(x)$$

where the $\lambda_{D,w}$ are local heights associated to D, and L is a finite extension of K over which x is defined. There are two obstructions to such a formula.

1. If we just choose any local height for each place independently, there is no guarantee that the sum will be well defined, i.e., given a point x, it might happen that $\lambda_{D,w}(x) \neq 0$ for infinitely many v. This can be circumvented by constructing the local heights all together, as functions (called *Weil functions*) of the form

$$\lambda : \big(V(\overline{K}) - \mathrm{Supp}(D)\big) \times M_{\overline{K}} \to \mathbb{R}$$

where $M_{\overline{K}}$ is the set of absolute values on $\overline{K}$ extending those of M_K. The constructions in the sections above generalize to this functions, with the core difference that the notion of a real constant is replaced by that of an M_K-constant, i.e., a function

$$M_K \to \mathbb{R}$$

which is zero for all but finitely many $v \in M_K$. A Weil function has an associated global height

$$h_\lambda(x) = \frac{1}{[L:K]} \sum_{w \in M_L} n_w \lambda_w(x).$$

2. The global height is defined over all $V(\overline{K})$, but the local heights only outside $\mathrm{Supp}(D)$. The product formula allows to extend the sum on $\mathrm{Supp}(D)$. The idea is that if $\phi \in K(V)$ then the Weil function associated to (ϕ) given by

$$(x, v) \mapsto v(\phi(x))$$

has global height zero by the product formula. Therefore we can *move* a divisor D with a ϕ without changing the height.

ABELIAN VARIETIES. Let A be an abelian variety, and D a Cartier divisor. [Lan83, Theorem 11.1.1] generalizes Theorem 3.1, giving a canonical Weil function defined up to an M_K-constant and an associated global height h_λ, which we can normalize by requiring $h_\lambda(0) = 0$. Let $\tilde{h}_c$ denote the Néron-Tate height associated to the class $c = \mathrm{Cl}(D)$. Then we have $\tilde{h}_c = h_\lambda$, because the Néron-Tate height is uniquely determined by its properties, which are satisfied by h_λ.

5 The case of curves

We let K be as in the previous section. Let C be a complete, regular, geometrically irreducible curve over K, and $J = \mathrm{Pic}^0(C)$ its Jacobian.
For more details we refer to [BG06, Chapter 9.4] and [Gro86]. As Jacobians are self-dual, there is a class $\mathbf{p} \in \mathrm{Pic}(J \times J)$, called the Poincaré class, such that for all $a \in \mathrm{Pic}^0(J) \cong J$ we have $\mathbf{p}|_{J\times\{a\}} \cong a$ and $p|_{\{0\}\times J} \cong 0$. Furthermore, $\mathbf{p}$ is invariant under the automorphism of $J \times J$ that swaps the two factors[1].

[1] This follows since $\mathbf{p} = m^*\theta - p_1^*\theta - p_2^*\theta$, where θ is the theta divisor on J and $m, p_1, p_2 : J \times J \to J$ are respectively the multiplication and the two projections.

COROLLARY 5.1. The Néron-Tate height of $\mathbf{p}$ defines a symmetric bilinear form

$$\tilde{h}_{\mathbf{p}} : J(\overline{K}) \times J(\overline{K}) \to \mathbb{R}$$

such that $\tilde{h}_{\mathbf{p}}(a, b) = \tilde{h}_b(a)$.

Néron in [Nér65] showed that this pairing can be decomposed as a sum of canonical local pairing. However, as with local heights, these local pairing will not be defined on divisor classes (e.g. on J) but only on divisors (of degree zero).

We will now decribe this pairings, following [Gro86]. Let K be a field with an absolute value $|-|_v$, which makes K locally compact.
Let $\phi \in K(C)$ be a function and $a = \sum m_x(x)$ a divisor of degree zero on C, such that (ϕ) and a have disjoint support. We put

$$\phi(a) := \prod \phi(x)^{m_x}.$$

As a has degree zero, this only depends on the divisor (ϕ).

PROPOSITION 5.2. For each v there exists a unique bilinear pairing $\langle -, - \rangle_v$, defined on pairs (a, b) degree-zero divisors on C which are propetly intersecting (i.e., with disjoint support, as C is a curve), and such that a is supported on the K-points of C, satisfying the following properties:

1. $\langle a, (\phi) \rangle = -\log |\phi(a)|_v$.
2. Fix $x_0 \in C(K) - \mathrm{Supp}(a)$, then the map

$$C(K) - \mathrm{Supp}(a) \to \mathbb{R} \quad x \mapsto \langle a, (x) - (x_0) \rangle_v$$

   is strongly continuous.
3. $\langle a, b \rangle = \langle b, a \rangle$ when also b has support in $C(K)$.

Moreover, we can extend the pairing to any pair (a, b) of properly intersecting divisors of degree zero in the following way: if H is a finite extension of K over which a is pointwise rational, then we put

$$\langle a, b \rangle = \frac{1}{[H : K]} \langle a, b \rangle_H,$$

where the second term is the unique pairing for the curve C_H obtained by base change to H.

PROOF. For uniqueness (see [Gro86, §3]), notice that the difference of two such pairings descends to $\mathrm{Pic}^0(C) = J$ by property (1), and thus we would get a pairing

$$J(K) \times J(K) \to \mathbb{R}$$

which is continuous and bounded in each variable when the other is fixed. This must be trivial as $\mathbb{R}$ contains no non-trivial compact subgroups.
One can show existence in the following way: take an embedding $j : C \hookrightarrow J$ (by choosing a

point $x_0 \in C$). Given a degree-zero divisor b on C, take a degree-zero divisor D on J such that $b = j^*D + (\phi)$. Then the assignment

$$P \mapsto \lambda_{D,v}(j(P)) + v \circ \phi(x)$$

gives a Weil function which is well defined up to a constant. The key observation is that this constant gets cancelled out if we use degree-zero divisors instead of a point P, thanks to remark 3.4. For details on this approach we refer to [Lan83, Chapter 11.3]. We will follow instead another approach, following [Gro86], which gives explicit formulas in terms of intersection numbers. This construction will be carried out (at least in the non-Archimedean case) in the next section. ■

6 Computation through intersection multiplicities

In this section we will sketch the construction of the pairing of Proposition 5.2 for non-Archimedean places, following [Gro86]. This proves existence and provides a way to compute the pairing, which will be used in the following talks.
Let K be a locally compact field with respect to an absolute value $|-|_v$, $\mathcal{O}_v$ is its valuation ring, $k(v)$ is the residue field.
Let C be a complete, regular, geometrically irreducible curve over K, with a regular and proper model $\mathcal{C}$ over $\mathcal{O}_v$. We will define an intersection product $(- \cdot -)$ for properly intersecting divisors on $\mathcal{C}$. Given V, W integral close subschemes of $\mathcal{C}$ of codimension 1 that intersect properly, their fibre product $V \cap W = V \times_{\mathcal{C}} W$ is a zero-cycle $\sum m_p(p)$, with

$$m_p = \operatorname{len}\left(\mathcal{O}_{\mathcal{C},p}/(\phi, \psi)\right)$$

and ϕ, ψ are local equations of V, W around p.
We define $(V \cdot W)$ as the degree $\sum m_p[k(p) : k(v)]$ (notice that a closed point of $\mathcal{C}$ must be in the special fibre, which is a scheme over $k(v)$, the residue field of $\mathcal{O}_v$). This pairing can be extended by linearity to properly intersecting divisors, and is clearly symmetric.
Now let (a, b) be a pair as in the statement of 5.2. Let A be an extension of a to $\mathcal{C}$ such that $A \cdot F = 0$ for every component F of the special fibre of $\mathcal{C}$, and let B be any extension of b. We put

$$\langle a, b \rangle = -(A \cdot B) \log \# k(v).$$

The link between this construction and the absolute value $|-|_v$ is due to the following observation: let x be a K-point in the support of a, and put $V = \overline{\{x\}}$ its closure in $\mathcal{C}$ (with the unique reduced closed subscheme structure). A closed point p in V must be contained in the special fibre, and $\mathcal{O}_{\mathcal{C},p}$ is a regular local ring of dimension 2, with maximal ideal (ϕ, π) for some π and a local equation ϕ of V around x, and $\mathcal{O}_{\mathcal{C},x}$ is then a DVR with maximal ideal (ϕ). We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_p & \xhookrightarrow{\text{loc. at } (\phi)} & \mathcal{O}_x \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_p/(\phi) & \xhookrightarrow[\text{loc. at } (0)]{} & K. \end{array}$$

As $\mathcal{O}_p/(\phi)$ is a DVR contained in K and which contains $\mathcal{O}_v$ (as the scheme is over $\mathcal{O}_v$), it must be $\mathcal{O}_p/(\phi) = \mathcal{O}_v$ and $v(\pi) = 1$. With this setup, we finally find that, if B is positive around p, with local equation ψ, then the multiplicity of p in $V \cap B$ is given by

$$m_p = \mathrm{len}\left(\mathcal{O}_p/(\phi, \psi)\right) = v(\psi(x)) = -\frac{\log|\psi(x)|_v}{\log \#k(v)}. \qquad (1)$$

In particular, when $b = (\psi)$ is principal and we take again (ψ) as its extension, this formula gives property (1) of 5.2.

References

[BG06] Enrico Bombieri and Walter Gubler. *Heights in Diophantine Geometry*. New Mathematical Monographs. Cambridge University Press, 2006.

[Gro86] Benedict H. Gross. 'Local heights on curves'. In: *Arithmetic Geometry*. Ed. by Gary Cornell and Joseph H. Silverman. New York, NY: Springer, 1986, pp. 327–339. ISBN: 978-0-387-96311-2.

[GZ86] Benedict H. Gross and Don B. Zagier. 'Heegner points and derivatives of L-series'. English. In: *Invent. Math.* 84 (1986), pp. 225–320. ISSN: 0020-9910. DOI: 10.1007/BF01388809. URL: https://eudml.org/doc/143341.

[Lan83] Serge Lang. *Fundamentals of Diophantine Geometry*. New York, NY: Springer, 1983. ISBN: 978-1-4419-2818-4 978-1-4757-1810-2. DOI: 10.1007/978-1-4757-1810-2. URL: http://link.springer.com/10.1007/978-1-4757-1810-2 (visited on 21/11/2025).

[Mum70] David Mumford. *Abelian Varieties Tata Institute of Fundamental Research*. 1970.

[Nér65] A. Néron. 'Quasi-fonctions et Hauteurs sur les Variétés Abéliennes'. In: *Annals of Mathematics* 82.2 (1965). Publisher: [Annals of Mathematics, Trustees of Princeton University on Behalf of the Annals of Mathematics, Mathematics Department, Princeton University], pp. 249–331. ISSN: 0003-486X. DOI: 10.2307/1970644. URL: https://www.jstor.org/stable/1970644 (visited on 20/11/2025).

[Ser97] Jean-Pierre Serre. *Lectures on the Mordell-Weil Theorem*. Ed. by Martin Brown. Red. by Klas Diederich. Vol. 15. Aspects of Mathematics. Wiesbaden: Vieweg+Teubner Verlag, 1997. ISBN: 978-3-663-10634-0 978-3-663-10632-6. DOI: 10.1007/978-3-663-10632-6. URL: http://link.springer.com/10.1007/978-3-663-10632-6 (visited on 20/11/2025).