ENS Lyon
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Non archimedean computations I

$x, \sigma(x) \in X_0(N)(H)$ Heegner points, $\sigma \in \mathrm{Gal}(H/K) \longleftrightarrow A \in Cl(\mathcal{O}_K)$.

v finite place of H, H_v completion of H at v, $\mathcal{O}_v = \mathcal{O}_{H_v}$, $\pi \in \mathcal{O}_v$, $k_v = \mathcal{O}_v/(\pi)$

$q_v = \# k_v$ $\quad p = \mathrm{char}\, k_v$.

$x, \sigma(x) \rightsquigarrow x_v, \sigma(x)_v \in X_0(N)(\mathcal{O}_v)$ sections.

$c = (x) - (\infty) \quad d = (x) - (0) \in \mathrm{Div}^0(X_0(N))(H)$

$c_v = (x_v) - (\infty) \quad \sigma(d)_v = (\sigma(x)_v) - (0) \in \mathrm{Div}^0(\underbrace{X_0(N)_v}_{X_0(N)_{\mathcal{O}_v}})$,

Fix $m \wedge N = 1$.

<u>Proposition 1</u> Assume that $r_A(m) = 0$. Then

$$\langle c, T_m \sigma(d) \rangle_v = -(x_v \cdot T_m \sigma(x)_v) \log q_v .$$

Reminder (Leonardo's talk): X regular model of $X_0(N)_{H_v}$ over $\mathcal{O}_v$.
$a, b \in \mathrm{Div}^0(X_0(N))(H_v)$ $\quad A, B \in \mathrm{Div}^0(X)_{\mathbb{Q}}$ st $A_{H_v} = a \quad B_{H_v} = b$
and $\forall\, \mathcal{C}$ irred comp of X_{k_v}, $(A \cdot \mathcal{C}) = 0$. Then

$$\langle a, b \rangle_v = -(A \cdot B) \log q_v .$$

<u>Case $p \nmid N$</u> From Michel's talk we know that $X_0(N)_v$ smooth$/\mathcal{O}_v$ with irreducible special fiber.
x_v, ∞ are sections of $X_0(N)_v/\mathcal{O}_v \Rightarrow (x_v \cdot X_0(N)_{k_v}) = (\infty \cdot X_0(N)_{k_v}) = 1$.
$\Rightarrow (c \cdot X_0(N)_{k_v}) = 0$ ✓.

<u>Case $p \mid N$</u> By assumption, p splits in K. Then $\underbrace{x_v, \sigma(x)_v}_{\text{ordinary}}, \underbrace{\infty, 0}_{\text{cusp}}$ are all in smooth$/\mathcal{O}_v$ locus of $X_0(N)_v$

$$X_0(N)^{red}_{k_v} = \bigcup_{\substack{a+b=n \\ = v_p(N)}} \mathcal{F}_{a,b}, \quad \mathcal{F}_{a,b} \text{ irreducible + smooth.}$$

$\rightsquigarrow \exists!\, (a,b)$ st $x_v \cap \mathcal{F}_{a,b} \neq \emptyset$, $\infty_{k_v} \in \mathcal{F}_{n,0}$, $0_{k_v} \in \mathcal{F}_{0,n}$

Let $\mathfrak{n} \subset \mathcal{O}_K$ be the type of $x \doteq (\phi : E \to E/\ker\phi)$ st $\ker\phi \simeq \mathcal{O}_K/\mathfrak{n}$
$N = \mathfrak{n}\bar{\mathfrak{n}}$, $\mathfrak{n} + \bar{\mathfrak{n}} = \mathcal{O}_K$.

Remarks 1) n is also the type of $\sigma(d)$

2) $T_m\sigma(d)_v$ is contained in the smooth$/_{\mathcal{O}_v}$ locus of $X_0(N)_v$

Namely T_m is given by

$$X_0(N)_v \longleftarrow X_0(Nm)_v \longrightarrow X_0(N)_v$$

with étale maps (wrt m).

Lemma 1 If $v|\bar{n}$, $x_v \cap \sigma_{a,b} \neq \emptyset \iff (a,b)=(0,m)$

If $v|n$, $x_v \cap \sigma_{a,b} \neq \emptyset \iff (a,b)=(m,0)$.

Consequence of the lemma If $v|n$, $(c_v \cdot \sigma_{a,b}) = 0 \quad \forall a,b$

If $v|\bar{n}$, $(\sigma(d)_v \cdot \sigma_{a,b}) = 0 \ \forall a,b \Rightarrow$

$(T_m\sigma(d)_v \cdot \sigma_{a,b}) = (\sigma(d)_v \cdot T_m\sigma_{a,b}) = 0 \quad \forall a,b$ as $T_m\sigma_{a,b} = \sigma_1(m)\sigma_{a,b}$.

$X \twoheadrightarrow X_0(N)_v$ resolution of singularities. Everything is in the smooth locus of $X_0(N)_v$, so lift uniquely to X with same intersection numbers.

$$\Rightarrow \langle c, T_m\sigma(d)\rangle_v = -\,(c_v \cdot T_m(\sigma(d))_v)_X \log q_v$$
$$= -\,(c_v \cdot T_m(\sigma(d))_v)_{X_0(N)_v} \log q_v.$$

Final step: $(c_v \cdot T_m\sigma(d)_v) = (x_v \cdot T_m\sigma(d)_v) - \overbrace{(x_v \cdot \underbrace{T_m 0}_{\sigma_1(m)\cdot 0})}^{=0} - (\infty \cdot T_m\sigma(d)_v)$

$+\underbrace{(\infty \cdot T_m 0)}_{=0}$ as $N>1$ $\square$

Proof of the lemma 1 We can base change to $W = \breve{\mathcal{O}}_v$. $F = W[p^{-1}]$, $k = W/(\pi) \simeq \bar{\mathbb{F}}_p$.

Recall (Michel's talk): $x_W = x_{v/W}$ comes from a Heegner diagram

$\underline{x} = (\phi : E \to E/\mathrm{Ker}\,\phi)_{/W}$, with E ell curve with CM by $\mathcal{O}_K$,

ϕ cyclic N-isogeny, $\mathrm{Ker}\,\phi$ $\mathcal{O}_K$-stable.

$\mathrm{Ker}\,\phi$ killed by n. Let $G = (\mathrm{Ker}\,\phi)_k \simeq ?$

If $v|\bar{n}$, Lie G is a k-vs with action of $\mathcal{O}_K$ given by

$\mathcal{O}_K \twoheadrightarrow \mathbb{F}_p \hookrightarrow k$ with kernel $\bar{n}$ $\Rightarrow$ Lie G killed by $n + \bar{n} = \mathcal{O}_K$

$\Rightarrow G \simeq (\mathbb{Z}/N\mathbb{Z})_k$ étale and $x_k \in \sigma_{0,m}(k)$ (smooth point)

If $v|n$, $E[p]/G$ killed by $\bar{n}$ ... is étale $\Rightarrow G \simeq \mu_{p^m} \times \mathbb{Z}/N\mathbb{Z}$, $x_k \in \sigma_{m,0}(k)$ $\square$

Now we want to compute $(x_W \cdot T_m \sigma(x_W))$.

<u>Proposition 2</u> Let $\underline{y} = (\phi : E \to E/\mathrm{Ker}\,\phi)$, $\underline{y}' = (\phi' : E' \to E'/\mathrm{Ker}\,\phi')$ be cyclic N-isogenies between ell curves $/W$. Assume that $\underline{y}_F \neq \underline{y}'_F$ and $\underline{y}, \underline{y}'$ are contained in the smooth locus of $X_0(N)_W$.

Then $(\underline{y} \cdot \underline{y}') = \frac{1}{2} \sum_{m \geq 0} \# \mathrm{Isom}_{W_m}(\underline{y}, \underline{y}')$ $\quad \left(W_m = W/(\pi^{m+1}) \right)$.

Proof: $\underline{y}_0 \neq \underline{y}'_0$, $\underline{y} \cap \underline{y}' = \emptyset$ and everything is 0 (recall $\mathrm{Hom}_{W_{m+1}}(\underline{y}, \underline{y}') \hookrightarrow \mathrm{Hom}_{W_m}(\underline{y}, \underline{y}')$ ---- $\mid$ Fix $\underline{y}_0 \simeq \underline{y}'_0$.

Let A be the universal deformation ring of the N-isogeny $\underline{y}_0$:
$\forall$ (R, m) local artinian W-algebra st $R/m \simeq k$,
$\mathrm{Hom}_W(A, R) \simeq \left\{ \text{cyclic } N\text{-isogenies } \underline{z}/R + \underline{z}_{R/k} \simeq \underline{y}_0 \right\} / \simeq$.

Serre-Tate + def theory of p-div groups $\Rightarrow$ A is local complete noeth regular of dim 2 ($A \simeq W[[T]]$.

$\underline{y}, \underline{y}' \in \mathrm{Hom}_{W}(A, W)$. We can choose T st $(T) = \mathrm{Ker}\,\underline{y}$. Let T' st $(T') = \mathrm{Ker}\,\underline{y}'$. $\Gamma = \mathrm{Aut}(\underline{y}_0) \curvearrowright A$ $\quad \overline{\Gamma} = \Gamma/\{\pm 1\}$.

Then $\hat{\mathcal{O}}_{X_0(N), \underline{y}_0} \simeq A^{\overline{\Gamma}} \hookrightarrow A$ finite flat of degree $d = \# \overline{\Gamma}$.

We have $N_\Gamma : A \to A^{\overline{\Gamma}}$, $\quad t = N_\Gamma(T) \quad t' = N_\Gamma(T')$.
$\qquad\qquad f \mapsto \prod_{\gamma \in \overline{\Gamma}} \gamma(f)$

Then $\underline{y} : \hat{\mathcal{O}}_{X_0(N), \underline{y}_0} \hookrightarrow A \to W$ has kernel (t)
$\quad \underline{y}'$ " ———————————— (t').

$\Rightarrow (\underline{y} \cdot \underline{y}') = \lg \left(A^{\overline{\Gamma}}/(t, t') \right) = \frac{1}{d} \lg \left(A/(t, t') \right)$

$$= \frac{1}{d} \sum_{\gamma, \gamma'} \lg A/(\gamma(T), \gamma'(T')) = \sum_{\gamma \in \overline{\Gamma}} \lg A/(T, \gamma(T')).$$

$\gamma(T') = \mathrm{Ker}\,\gamma(\underline{y}')$ and $A/(T, \gamma(T')) \simeq W/\pi^{k_\gamma}$.

Therefore $\exists$ $\underline{y} \bmod \pi^k \simeq \gamma(\underline{y}') \bmod \pi^k \iff k \leq k_\gamma$
$\iff \exists$ $\underline{y}_{k-1} \simeq \underline{y}'_{k-1}$ over $\gamma \in \Gamma$. (we fixed $\underline{y}_0 \simeq \underline{y}'_0$).

Finally $\sum_{m\geq 0} \# \mathrm{Isom}(\underline{y}_m, \underline{y}'_m) = \sum_{\gamma\in\Gamma} \sum_{0\leq m\leq k_\gamma - 1} 1 = \sum_{\gamma\in\Gamma} k_\gamma$

$= 2(\underline{y}\cdot\underline{y}')$ $\square$

Our goal is to prove the formula.

<u>Theorem</u> Assume $\lambda_A(m) = 0$. Then

$$(\underline{x} \cdot_W T_m \underline{\sigma(x)}_W) = \frac{1}{2} \sum_{n\geq 0} \underbrace{\# \mathrm{Hom}_{W_n}(\underline{\sigma(x)}, \underline{x})}_{}{}_{\deg m}.$$

Hom of N-isogenies, not necessarily compatible to O_K-action!

<u>Easy case</u> $p\nmid m$. $\underline{\sigma(x)} = (\phi : E \to E')/W$. $E[m]$ étale constant $/W$.

$C \subset E[m]$ subgroup of card m, $\underline{\sigma(x)}_C = (\phi : E/_C \to E'/_C)$.

$$\begin{tikzcd} \underline{\sigma(x)} \arrow[r, "\deg m"] \arrow[d, two heads] & \underline{x} \\ \underline{\sigma(x)}_C \arrow[ur, "\sim"'] & \end{tikzcd}$$
$\exists ! C$

$T_m \underline{\sigma(x)} = \sum_{C\subset E[m]} \underline{\sigma(x)}_C$ and

$$\mathrm{Hom}_{W_n}(\underline{\sigma(x)}, \underline{x}) \simeq \coprod_C \mathrm{Isom}(\underline{\sigma(x)}_C, \underline{x})$$

$$\Rightarrow (\underline{x}\cdot T_m \underline{\sigma(x)}) = \sum_C (\underline{x}\cdot\underline{\sigma(x)}_C) \underset{\lambda_A(m)=0}{=} \frac{1}{2} \sum_C \sum_{n\geq 0} \# \mathrm{Isom}_{W_n}(\underline{\sigma(x)}_C, \underline{x})$$

$$= \frac{1}{2} \sum_{n\geq 0} \# \mathrm{Hom}_{W_n}(\underline{\sigma(x)}, \underline{x}) \checkmark .$$

<u>Case $p\mid m$</u> $m = r p^t$. $p\nmid r$. $t\geq 1$ $T_m = T_{p^t} T_r$.

$T_r \underline{\sigma(x)} = \sum_{\underline{z}\in Z} \underline{z}$. $\underline{z} \in X_0(N)(W)$, $\underline{z} = (\phi : E \twoheadrightarrow E/_{\mathrm{Ker}\,\phi})$.

cyclic N-isog (in smooth locus)

<u>Lemma 2</u> Any $\underline{z}\in Z$ is the canonical lift of $\underline{z}_0$.

Proof: $\underline{x} = (\phi : E \to E')$ has CM by O_K.

$\rightsquigarrow \mathbb{Z}_p \otimes_{\mathbb{Z}} O_K \hookrightarrow \mathrm{End}\, E[p^\infty] \simeq \mathrm{End}(E'[p^\infty])$

$\Rightarrow E$ and E' are canonical lifts of E_0 and E'_0.

Idem for $\underline{\sigma(x)}$ and $\underline{z}\in Z$ as they differ from $\underline{\sigma(x)}$ by prime to p isogenies. $\square$

• p split in K $\quad \underline{x} = (\phi : E \to E/\mathrm{Ker}\,\phi)$, $E[p^\infty] \simeq \mathbb{Q}_p/\mathbb{Z}_p \oplus \mu_{p^\infty}$.
idem for $\sigma(x)$.

$$\mathrm{End}_W(E[p^\infty]) \xrightarrow{\sim} \mathrm{End}_k(E[p^\infty])$$

Serre-Tate $\Rightarrow$ $\underbrace{\mathrm{Hom}_W(\sigma(x), x)_{\deg m}}_{=0 \text{ as } r_A(m)=0} \simeq \underbrace{\mathrm{Hom}_k(\sigma(x), x)_{\deg m}}_{=0.}$

$\Rightarrow$ RHS $= 0$.

For LHS, we use

lemma 3 — Any canonical lift of a point of $(T_{p^t}\underline{z})_k$ is in $(T_{p^t}\underline{z})_W$. — let $\underline{z}$ be a canonical lift of $\underline{z}_k$ ordinary.
$\underline{z} \in Z$

Proof: $T_{p^{t+1}} = T_p T_{p^t} - p T_{p^{t-1}}$ $\Rightarrow$ we can do it for $t=1$ by induction.

let $\underline{z} = (\phi : E \twoheadrightarrow E')$ $\quad E[p^\infty]_k \simeq \mathbb{Q}_p/\mathbb{Z}_p \oplus \mu_{p^\infty}$.

$(T_p \underline{z})_k$ has 2 points corresponding to quotients by $\mu_{p,k}$ and $\mathbb{Z}/p\mathbb{Z}, k$.
Their canonical lifts are the quotient by $\mu_{p,W}$ and $\mathbb{Z}/p\mathbb{Z}, W$ which are in $(T_p \underline{z})_W$. □

Now $\underline{x}$ is a canonical lift of $\underline{x}_k$ and $\underline{x}_W \notin T_{p^t}\underline{z}$ for $\underline{z} \in Z$.
$\Rightarrow$ $\underline{x}_k \notin (T_{p^t}\underline{z})_k$ $\Rightarrow$ $\mathfrak{l}\underline{x} \cap T_{p^t}\underline{z} = \phi$, LHS $= 0$.

• p inert in K (the case where p is ramified is similar).
$\underline{x}$, $\sigma(x)$, $\underline{z} \in Z$ have supersingular reduction.
$\underline{z} = (\phi : E \to E/\mathrm{Ker}\,\phi)$. Over k, $E[p^\infty]$ has a unique subgroup of order p^t, this is $\mathrm{Ker}\,\mathrm{Frob}^t$. Moreover $E[p] = \mathrm{Ker}\,\mathrm{Frob}_p^2$.

$$\Rightarrow \quad \mathrm{Supp}(T_{p^t}\underline{z})_k = \begin{cases} \underline{z}_0 & \text{for } t \text{ even} \\ \underline{z}_0^{(p)} & \text{for } t \text{ odd} \end{cases}$$

Over F, $(T_{p^t}\underline{z})_F = \sum_{\substack{C \subset E[p^t] \\ \#C = p^t}} \underline{z}_C$, $\quad 0 \to E[p^{\frac{t-s}{2}}] \to C \to \underline{C'} \to 0$
cyclic of order p^s
$s \equiv t \bmod 2$.

Let $\mathcal{C}(s) = \{ C \subset E[p^s] \mid C \text{ cyclic of order } p^s \}$. $\quad \hookrightarrow \underline{z}_C \simeq \underline{z}_{C'}$.