ENS Lyon
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Non archimedean computations II

Recall $x, \sigma(x) \in X_0(N)(H)$ Heegner points, $\sigma \in Gal(H/K) \leftrightarrow A \in Cl(\mathcal{O}_K)$.
$v | p$ place of H, H_v completion of H, F completion of max unr ext of H_v.
$W = \mathcal{O}_F$, $\pi \in W$ uniformizer. $k = W/\pi \simeq \overline{\mathbb{F}_p}$.

$x, \sigma(x) \rightsquigarrow \underline{x}, \underline{\sigma(x)} \in X_0(N)(W)$. $\quad m \wedge N = 1$.

Proposition Assume $r_A(m) = 0$, $(\underline{x} \cdot T_m \underline{\sigma(x)}) = \frac{1}{2} \sum_{n \geq 0} \# \mathrm{Hom}_{W_n}(\underline{\sigma(x)}, \underline{x})$.
$(W_n = W/(\pi^{n+1}))$.

Proof: Assume p inert in K. $m = r p^t$ $\quad r \wedge p = 1$.
$(p | m)$

Recall: $T_r \underline{\sigma(x)} = \sum_{\underline{z} \in Z} \underline{z}$ $\quad Z \subset X_0(N)(W)$, $\underline{z} = (\phi : E \twoheadrightarrow E/\mathrm{Ker}\,\phi) / W$
cyclic N-isogeny

$*$ Over F, $T_{p^t} \underline{z}_F = \sum_{\substack{C \subset E[p^t] \\ \# C = p^t}} \underline{z}_{C,F}$, $\quad T_{p^t} \underline{z} = \sum \underline{z}_C$
$\underline{z}_C = \left(E/C \xrightarrow{\phi_C} E/C + \mathrm{Ker}\,\phi \right)$.

We have $0 \to E[p^{\frac{t-s}{2}}] \to C \to \underbrace{C'}_{\text{cyclic of } \# p^s} \to 0$, $\quad s \equiv t\ [2]$.
$\underline{z}_C \simeq \underline{z}_{C'}$

Let $\mathcal{C}(s) = \{ C \subset E[p^s] \mid C \text{ cyclic of } \# p^s \}$.

We have $T_{p^t} \underline{z} = \sum_{\substack{0 \leq s \leq t \\ s \equiv t \bmod 2}} \underbrace{\sum_{C' \in \mathcal{C}(s)} \underline{z}_{C'}}_{\underline{y}(s)}$, divisor on $X_0(N)_W$.

Recall $E[p^\infty]_W$ is the canonical lift of $E[p^\infty]_k$.
$\Rightarrow E[p^s] \simeq \mathcal{O}_{K_p}/p^s \circlearrowleft Gal(\bar{F}/F) \simeq I_{K_p} \xrightarrow{\chi_{LT}} \hat{\mathcal{O}}_{K_p}^\times$
transitive action on $\mathcal{C}(s)$.

Rk $\underline{y}(0) = \underline{z}$.

$*$ Over k, $E[p^\infty]_k$ connected, has a unique subgroup of order p^t: $\mathrm{Ker}\,\mathrm{Frob}^t$.
$\Rightarrow \mathrm{Supp}(T_{p^t} \underline{z})_k = \begin{cases} \underline{z}_k & \text{for } t \text{ even} \\ \underline{z}_k^{(p)} & \text{— odd} \end{cases}$

<u>Lemma</u> For $s>0$, we have $(\underline{x}\cdot\underline{y}(s)) = \frac{1}{2}\#\mathrm{Isom}_k\left(\underline{x}, \begin{cases}\underline{z} & t \text{ even}\\ \underline{z}^{(p)} & t \text{ odd}\end{cases}\right)$.

Proof: similar to the previous case, use that q canonical lifts of degree $s>0$ are not $\simeq$ to canonical lift over $W/\pi^{s/2}$. $\square$

$$\Rightarrow (\underline{x}\cdot T_{p^t}\underline{z}) = \begin{cases}(\underline{x}\cdot\underline{z}) + \frac{1}{2}\,\frac{t}{2}\,\#\mathrm{Isom}_k(\underline{z},\underline{x}) & t \text{ even}\\ \frac{1}{2}\,\frac{t+1}{2}\,\#\mathrm{Isom}_k(\underline{z}^{(p)},\underline{x}) & t \text{ odd.}\end{cases}$$

<u>Lemma</u> 1) $\underline{z}, \underline{z}'$ canonical lifts of their supersingular special fiber.

$\{x\in \mathrm{Hom}_{W_{m+1}}(\underline{z},\underline{z}') \mid p^2 \mid \deg x\} \xrightarrow{\sim} p\,\mathrm{Hom}_{W_m}(\underline{z},\underline{z}')\ (\xleftarrow{\sim} \mathrm{Hom}_{W_m}(\underline{z},\underline{z}'))$

2) $\mathrm{Hom}_{W_m}(\underline{z},\underline{z}')_{\deg p}^{-} = 0$ for $m\geq 1$.

<u>Consequence</u>: $\forall m\geq 0$, $\mathrm{Hom}_{W_{m+\frac{t}{2}}}(\underline{z},\underline{z}')_{\deg p^t} \simeq \mathrm{Isom}_{W_m}(\underline{z},\underline{z}')$.

t even, $(\underline{x}\cdot T_{p^t}\underline{z}) = \frac{1}{2}\left(\sum_{m\geq 0}\#\mathrm{Isom}_{W_m}(\underline{z},\underline{x}) + \sum_{i=0}^{\frac{t}{2}-1}\#\mathrm{Isom}_k(\underline{z},\underline{x})\right)$

$$= \frac{1}{2}\left(\sum_{m\geq 0}\#\mathrm{Hom}_{W_{m+\frac{t}{2}}}(\underline{z},\underline{x})_{\deg p^t} + \sum_{i=0}^{\frac{t}{2}-1}\#\mathrm{Hom}_{W_{\frac{t}{2}}}(\underline{z},\underline{x})_{\deg p^t}\right).$$

$\nwarrow W_i$

For $i\leq \frac{t}{2}-1$, we have

$$\mathrm{Hom}_{W_i}(\ )_{\deg p^t} \simeq \mathrm{Hom}_k(\ )_{\deg p^{t-2i}} \simeq \mathrm{Isom}_k(\underline{z},\underline{x})$$

using $[p]: \underline{x}\twoheadrightarrow\underline{x}$ of degree p^2.

t odd similar, use $\mathrm{Hom}_{W_m}(\ )_{\deg p^t} \simeq \mathrm{Hom}_{W_{m-\frac{t-1}{2}}}(\ )_{\deg p} = 0$ for $m>\frac{t-1}{2}$.

<u>Proof of the lemma</u>: Using Serre-Tate theorem we are reduced to prove that $\{x\in \mathrm{End}_{W_{m+1}}\breve{\mathcal{G}} \mid p^2\mid \deg x\} \simeq p\,\mathrm{End}_{W_m}\breve{\mathcal{G}}$.

direct computation with $\mathrm{End}_{W_m}\breve{\mathcal{G}} = \mathcal{O}_{K_p} + p^m R$ $\square$

p ramified, similar

Computation

* Split case, p split in K. Then $\lambda_A(m)=0 \Rightarrow (\underline{x}\cdot T_m \underline{\sigma(x)})_v = 0$.

Proof: $\underline{x} = (\phi : E \longrightarrow E/\ker\phi)$ $\quad E, E'/W$ canonical lifts of their special fiber.
$\underline{\sigma(x)} = (\phi' : E' \longrightarrow E'/\ker\phi')$

$\rightsquigarrow \mathrm{Hom}_W(E', E) \xrightarrow{\sim} \mathrm{Hom}_{W_m}(E', E) \xrightarrow{\sim} \mathcal{O}_K \quad \forall m$.

$\ker\phi, \ker\phi'$ are $\mathcal{O}_K$-stable $\Rightarrow \underbrace{\mathrm{Hom}_W(\underline{\sigma(x)}, \underline{x})_{\deg m}}_{=0 \text{ as } \lambda_A(m)=0} \Rightarrow \underbrace{\mathrm{Hom}_{W_m}(\underline{\sigma(x)}, \underline{x})_{\deg m}}_{=0} \quad \forall m \geq 0$. $\square$

* p inert in K. E, E' have supersingular reduction.

Recall $R' = \mathrm{End}_{\underline{k}}(\underline{E})$ is a maximal order in B quaternion algebra$/\mathbb{Q}$, split exactly outside of p and ∞.

$\mathcal{O}_K \hookrightarrow R'$, $\rightsquigarrow K \hookrightarrow B$. $\quad B = K \oplus Kj$, $\quad j^2 \in K$, $\forall a \in K, jaj^{-1} = \bar{a}$.

$b \in B$, $b = \underset{\in K}{b^+} + \underset{\in Kj}{b^-}$.

Let $R = \mathrm{End}_{\underline{k}}(\underline{x}) \subset R'$ order (non max) in B $\left(R_\ell \simeq \begin{pmatrix} \mathbb{Z}_\ell & \mathbb{Z}_\ell \\ \ell^? \mathbb{Z}_\ell & \mathbb{Z}_\ell \end{pmatrix} \right)$ if $\ell | N$

Proposition If $\lambda_A(m)=0$, $a \in Cl$ $\quad (\underline{x}\cdot T_m \underline{\sigma(x)}) = \dfrac{1}{2} \displaystyle\sum_{\substack{b \in Ra \\ Nb = mNa}} \dfrac{1}{2}(1 + v_p(N(b^-)))$.

Proof: Describe $\underline{\sigma(x)}$ in terms of CM action.

Serre's construction: $\mathcal{O}_K^{m_1} \longrightarrow \mathcal{O}_K^{m_0} \longrightarrow a' \longrightarrow 0$ presentation

$a' \otimes_{\mathcal{O}_K} E := \mathrm{Coker}(E^{m_1} \to E^{m_0})$ ell curve with CM $/W$.

We have $E' \simeq a' \otimes_{\mathcal{O}_K} E$ (check it over $\mathbb{C}$).

$\rightsquigarrow \mathrm{Hom}_{W_m}(\underline{\sigma(x)}, \underline{x}) \simeq \mathrm{Hom}_{W_m}(a' \otimes_{\mathcal{O}_K} \underline{x}, \underline{x}) \simeq a \otimes_{\mathcal{O}_K} \mathrm{End}_{W_m}(\underline{x})$.

$\hookrightarrow K \otimes_{\mathcal{O}_K} R \simeq B$.

check that $\phi \longleftrightarrow b \in B$ st $N(a)\deg\phi = N(b)$

(indeed, $E \to a' \otimes_{\mathcal{O}_K} E$ isogeny of degree $N(a)$).

~~Recall from Michal's talk:~~

~~$\mathrm{End}_{W_m} E[p^\infty] \simeq$~~

As $p \nmid N$, $\mathrm{End}_{W_m}(\underline{x}) = \mathrm{End}_{\underline{k}}(\underline{x}) \cap \mathrm{End}_{W_m}(E)$

$= \mathrm{End}_{\underline{k}}(\underline{x}) \cap \mathrm{End}_{W_m}(E[p^\infty])$ by Serre-Tate.

Recall from Michel's talk that

$$\mathrm{End}_{W_m}(E[p^\infty]) = \mathcal{O}_{K_p} + p^m R_p = \{ x_p \in R_p \mid v_p(N(b^{-})) \geq 1+2m \}.$$

Rk $v_p(j^2)$ odd $\Rightarrow$ $v_p(N(b^{-}))$ has to be odd.

$$\Rightarrow (\underline{x} \cdot T_m \sigma(x)) = \frac{1}{2} \sum_{\substack{b \in R\mathfrak{a} \\ N(b) = mN(\mathfrak{a})}} \# \{ m \geq 0 \mid v_p(N(b^{-})) \geq 1+2m \} \ \square.$$

$\exists$ similar computation in the ramified case.

* Final reduction We can (and do) choose $\mathfrak{a} \in \mathcal{A}$ prime to D and N.

We need an explicit description of R. $\left(+ (D, -2q)_2 = -1 \text{ if } p=2 \right)$

Fix $q \neq p$ a prime st $-pq \equiv 1 \bmod D$. We can write

$B = K \oplus Kj$ with $j^2 = -pq$, $\forall a \in K$, $jaj^{-1} = \bar{a}$.

Let $\partial = \partial_{K/\mathbb{Q}} = (\sqrt{D}) \subset \mathcal{O}_K$. q splits in $K/\mathbb{Q}$, $q = \mathfrak{q}\bar{\mathfrak{q}}$.

$$S = \{ \alpha + \beta j \in B \mid \alpha \in \partial^{-1}, \beta \in \partial^{-1}\mathfrak{q}^{-1}\mathfrak{n}, \alpha - \beta \in \mathcal{O}_\partial = \prod_{\ell | D} \mathcal{O}_{K,\ell} \}$$

$\rightsquigarrow$ S is an order in B st $\forall \ell$, $S_\ell \simeq R_\ell$ (admitted).

For $\mathfrak{b} \subset \mathcal{O}_K$ ideal, let $R^{(\mathfrak{b})} = \mathfrak{b} S \mathfrak{b}^{-1} \subset B$.

Thm (Eichler) $\{ R^{(\mathfrak{b})} \mid [\mathfrak{b}] \in \mathcal{C}\ell(\mathcal{O}_K) \}$ is a set of representatives for isom classes of orders in B, locally isom to R.

$\rightarrow \exists ! [\mathfrak{b}] \in \mathcal{C}\ell(\mathcal{O}_K)$ st $R^{(\mathfrak{b})} \simeq R$ $(= R_v)$.

Important remark: p tot split in H/K, $\sigma \in \mathrm{Gal}(H/K)$

$\sigma \leftrightarrow [\mathfrak{c}] \in \mathcal{C}\ell(\mathcal{O}_K)$, Then $R^{(\mathfrak{bc})} = R_{\sigma(v)}$.

$$\text{Let } (\underline{x} \cdot T_m \sigma(x))_p = \sum_{v | p} (\underline{x} \cdot T_m \sigma(x))_v = \sum_{[\mathfrak{b}] \in \mathcal{C}\ell(\mathcal{O}_K)} \sum_{\substack{b \in R^{(\mathfrak{b})}\mathfrak{a} \\ N(b) = mN(\mathfrak{a})}} \frac{1}{4} (1 + v_p(N(b^{-}))).$$

$\Rightarrow$ We don't need to identify $\mathfrak{b}$!

<u>Proposition</u> $(\underline{x} - T_m \underline{\sigma(x)})_p = \frac{1}{2}\mu^2 \sum_{\substack{0<m<\frac{m|D|}{N} \\ p|m}} v_p(pm)\,\delta(m)\, r_{\mathcal{A}}(m|D| - mN)\, R_{\mathcal{A}[qm]}\left(\frac{m}{p}\right).$

where $\mu = \frac{1}{2} H\mathcal{O}_K^\times$ $\quad \delta(m) = 2^{\#\{\ell \mid D, m\}}$ $\quad R_{\mathcal{B}}(k) = \sum_{\mathcal{C} \in Cl(\mathcal{O}_K)^2} r_{\mathcal{B}\mathcal{C}}(k).$

<u>Proof</u> Explicitly

$R^{(b)}{}_{\mathfrak{a}} = \left\{ \alpha + \beta j \in B \;\middle|\; \underbrace{\alpha \in \partial^{-1}\mathfrak{a},\ \beta \in \partial^{-1} q^{-1} m\, b \bar{b}^{-1} \bar{\mathfrak{a}}}_{(*)},\ \forall \ell \mid D\ \ \alpha - (-1)^{v_{p_\ell}(b)} \beta \in \mathcal{O}_{K,\ell} \right\}$

$\mathfrak{C} = (\alpha)\,\partial\,\mathfrak{a}^{-1}$ $\quad \mathfrak{C}' = (\beta)\,\partial\, q\, m^{-1}\, b^{-1}\, \bar{b}\, \bar{\mathfrak{a}}^{-1}$, $\quad \mathfrak{C} \in \mathcal{A}^{-1}$, $\quad \mathfrak{C}' \in [qm^{-1}]\,\mathcal{B}^{-2}\mathcal{A}$

$(\mathcal{B} = [b])$

$(*) \iff \mathfrak{C}, \mathfrak{C}' \subset \mathcal{O}_K$

$N(b) = m N\mathfrak{a} \iff N(\mathfrak{C}) + pN\,N(\mathfrak{C}') = m|D|$

$\mathcal{E} = \left\{ (b, [b]) \in B \times Cl(\mathcal{O}_K) \mid b \in R^{(b)},\ N(b) = mN\mathfrak{a} \right\}$

$\mathcal{F} = \left\{ (\mathfrak{C}, \mathfrak{C}', \mathcal{C}) \in \mathcal{I}(\mathcal{O}_K)^2 \times Cl(\mathcal{O}_K)^2 \mid \mathfrak{C} \in \mathcal{A}^{-1},\ \mathfrak{C}' \in [qm^{-1}]\,\mathcal{C}\mathcal{A},\ N(\mathfrak{C}) + pN\,N(\mathfrak{C}') = m|D| \right\}$

$f : \mathcal{E} \to \mathcal{F}$

$(b, [b]) \longmapsto (\mathfrak{C}, \mathfrak{C}', [b]^{-2})$ $\qquad (\alpha + \beta j, [b]) \in f^{-1}(\mathfrak{C}, \mathfrak{C}', \mathcal{C})$

$b = \alpha + \beta j$

$$\iff \left\{ \begin{array}{l} (\alpha) = \mathfrak{C}\,\partial^{-1}\mathfrak{a} \\ (\beta) = \ldots \\ \mathcal{C} = [b]^{-2} \end{array} \right) \ (**).$$

$\alpha - \beta(-1)^{v_{p_\ell}(b)} \in \mathcal{O}_{K,\ell}\ \ \forall \ell \mid D \quad (***)$

<u>Recall</u> $Cl(\mathcal{O}_K)[2]$ generated by the classes of $p_\ell \mid \partial$ with unique relation: $\prod p_\ell = (\sqrt{D})$.

Take (α, β) st $(**)$. either a) $\alpha, \beta \in \mathcal{O}_{K,\ell}$ condition $(***)$ empty

$\alpha \pm \beta \in \mathcal{O}_{K,\ell}$.

b) $v_{p_\ell}(\alpha) = v_{p_\ell}(\beta) = -1$, $\quad N(\alpha) - pq N(\beta) \in \mathbb{Q}\mathbb{Z} \Rightarrow \frac{\alpha}{\beta} \equiv \pm 1 \bmod p_\ell$ in $\mathcal{O}_{K,\ell}$.

$\rightsquigarrow \alpha \pm \beta \in \mathcal{O}_{K,\ell}$ for a unique sign

We are in a) iff $\ell \mid N(\mathfrak{C}')$.

$\rightsquigarrow$ $\boxed{\# f^{-1}(\mathfrak{C}, \mathfrak{C}', \mathcal{C}) = (2\mu)^2 \delta(m).}$

Set $n = p\,N(\mathfrak{L}')$. We have $\cancel{N}(v_p(N(\mathfrak{b}')) = v_p(N(\mathfrak{p})pq) = v_p(n)$.

$$\rightsquigarrow (\underline{x}\cdot\sigma(\underline{d}))_p = \frac{1}{2} \sum_{(\mathfrak{L},\mathfrak{L}',\varphi)\in\sigma_c} \frac{1}{2}(1+v_p(n))\,4\mu^2\,\delta(n) \qquad (n = N(\mathfrak{L}'))$$

$$= \mu^2 \sum_{\substack{0<n<\frac{m|D|}{N}\\ p\mid n}} \sum_{\mathcal{C}\in Cl(\mathcal{O}_K)^2} r_{\mathcal{A}\mathcal{C}[\mathfrak{q}n^{-1}]}\left(\frac{n}{p}\right) r_{\mathcal{A}^{-1}}(m|D|-nN).$$

$$= \mu^2 \sum R_{\mathcal{A}\mathcal{C}[\mathfrak{q}n^{-1}]}\left(\frac{n}{p}\right) r_{\mathcal{A}}(m|D|-nN).$$

We need to divide by 2 because of the relation in $Cl(\mathcal{O}_K)[2]$ $\square$

<u>Final Theorem</u> We have

$$\langle c, T_m\sigma(d)\rangle_p = -\mu\, r_{\mathcal{A}}(m)\, h_K\, v_p\left(\frac{m}{N}\right)\log p$$
$$+\begin{cases} 0 & \text{if } p \text{ split in } K \\ -\mu^2\log p \displaystyle\sum_{\substack{0<n<\frac{m|D|}{N}\\ p\mid n}} \underbrace{v_p(pn)}_{v_p(n)}\,\delta(n)\, r_{\mathcal{A}}(m|D|-Nn)\, R_{\mathcal{A}[\mathfrak{q}n]}\left(\frac{n}{p}\right) \end{cases}$$

p inert

$R_{\mathcal{A}[\mathfrak{q}pn]}\left(\frac{n}{p}\right)$ p ramified