Six Functors in Motivic Homotopy

Some history:

(1). Grothendieck introduce part of six functors for coherent sheaves. (1958)

(2) Then they build full six functor formalism later SGA 4 (1963-) SGA5 .... — for étale sheaves

(3) Morihiko Saito build s-f --f - for his mixed Hodge modules (1988-1990) — ...

1. Some motivation

Let $\mathcal{C}$ locally cartesian cat
(i.e has inner $\underline{\text{hom}}$, e.g. Topos, $\underline{Sh(X)}$))

$X \in \mathcal{C}$

étale morphism of topos

$$\begin{tikzcd} \mathcal{C}_{/X} \arrow[d, bend right=60, "p_\#"'] \arrow[d, bend left=60, "p_*"] \\ \mathcal{C} \arrow[u, "p^*"] \end{tikzcd} \qquad p_\# \dashv p^* \dashv p_*$$

$Y \to X$

$p^*(Y) := \begin{matrix} X \times Y \\ \downarrow \\ X \end{matrix}$

$p_\#\left(\begin{matrix} A \\ \downarrow \\ X \end{matrix}\right) := A$

$p_*\left(\begin{matrix} A \\ \downarrow q \\ X \end{matrix}\right) := \underline{\text{hom}}_X(X, A) \in \mathcal{C}$

"Space" section of q

$p_\#$ as "total space" "Σ", p_* as "Sections" "$\prod$"

if $\mathcal{C} = Sh(X, \tau)$

$Y \xrightarrow{f} X \in Site(X, \tau)$ (open subset, etale, smooth)

$$Sh(X)_{/Y} \simeq Sh(Y) \begin{array}{c} \xrightarrow{f_*} \\ \xleftarrow{f^*} \\ \xrightarrow{f_\#} \end{array} Sh(X)$$

We will have

(1) projection formula $f_\#(f^*A \times B) \cong A \times f_\# B$

(2) $\begin{tikzcd} T \arrow[r, "q"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "p"'] & Y \end{tikzcd}$ $\underline{f_\# p^*} \cong q^* \underline{g_\#}$ (smooth) base change

$f \in$ Site $\boxed{\text{e.g. for open immersion}}$

not hard to prove. only use topos theory (HTT, 6.3.5.11)

at that stage sheaf of Set/Space has 3.5 funtors $f^*, f_*, \underline{\hom} + \underline{f_\#}$ ($f \in$ Site)

to go further $\Rightarrow$ linear world

Set $\to$ Ab

Space $\to$ Spectra.

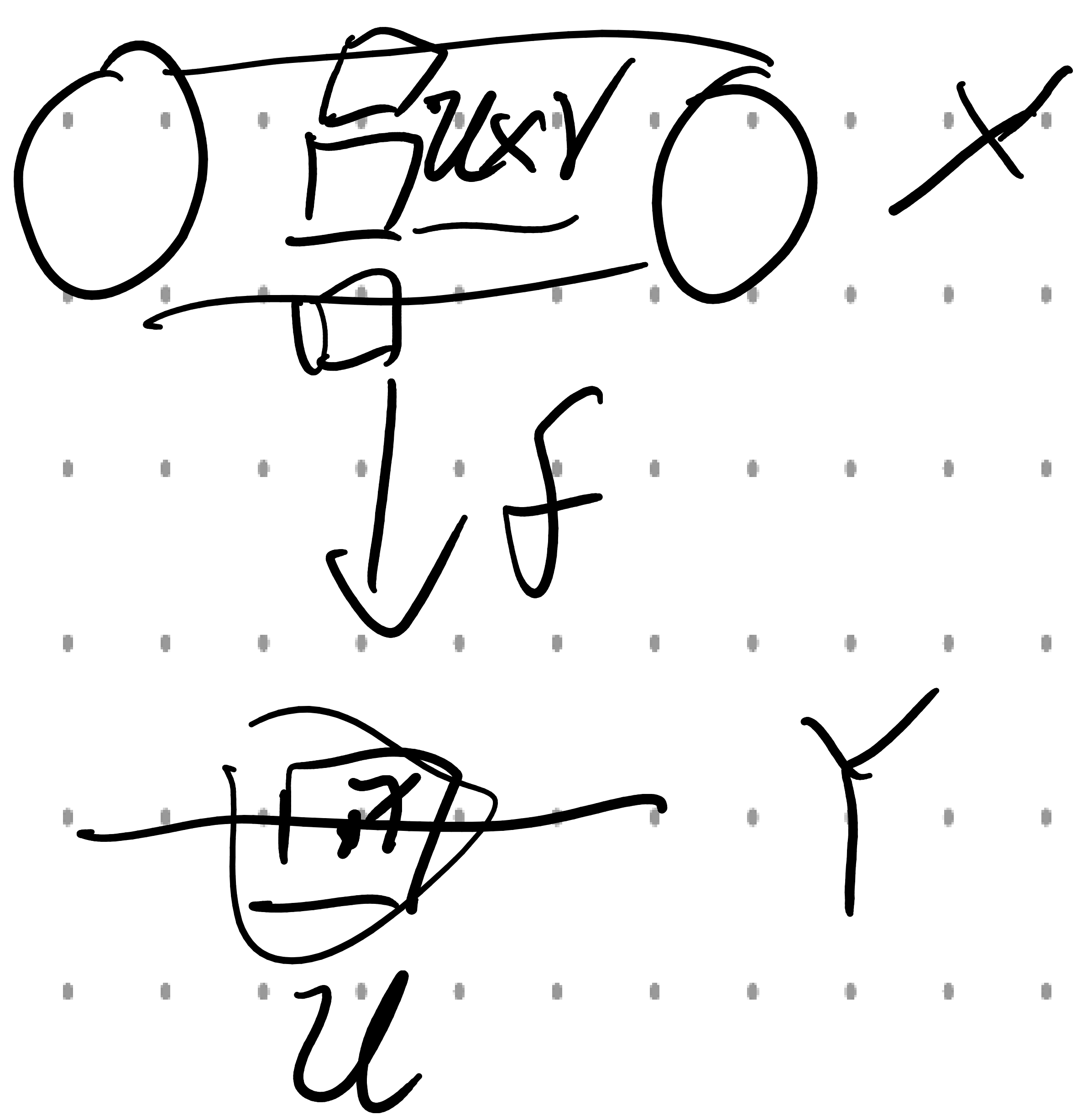
$\underline{\textcircled{+}\ \Sigma \simeq \prod \Pi}$

when "finite"

goal to extend "$f_\#$" to more general

use the f_*, for simplicity, mnfd case

$f: X \to Y$ proper and smooth



$F \in Sh(X)$ $\quad (Rf_*F(U) = \lim_{V \subset f^{-1}(x)} F(U \times V)$

$= \lim_{V \subset f^{-1}(x)} Hom(U \times V, F) \simeq \operatorname{colim}_{V \subset f^{-1}(x)} Hom(U, V^\vee \otimes F)$ $\quad \Sigma^{\infty}$

"Hom$(U, Hom(V, F))$"

$= "f_\#"(T^{-1}_{X/Y} \otimes F)$

Sheaves on mnfds

$f_! \simeq "f_\#"(T^{-1}_{X/Y})$

$\mathbb{A}^1$-homotopy

f proper $f_! = f_*$

f. open immersion $f_! = f_\#$

$f_! = p_* g_\#$ when p is compactifying

The proof will follow. A.A. Khan

VOEVODSKY'S CRITERION FOR CONSTRUCTIBLE CATEGORIES OF COEFFICIENTS

More details see also C-D

Triangulated categories of mixed motives

Road map

s smooth
i closed, p proper

$f_\# \dashv f^* \dashv f_*$

Topo'g

$s_\# f^* \to$ ✓

smooth base change

Voevodsky's conditions
(I) homotopy inv
(II) localization
(III) Thom stability

(II) for i

supp pro

s proper smooth

$s_\# i_*$, $s_\# p_*$

smooth-proper-exchange

$i_* f^*$, $p_* f^*$

proper-base-change

$s_\# \xrightarrow{\sim} s_*\langle T_f\rangle$

relative purity

projective

Atiyah Duality

$SH(X)$

well defined

Def. $f_! = p_* j_\#$

when $f = pj$

p proper, j open

(Nagata)

$f_! \dashv f^!$

$f_\otimes \dashv f^\Delta$

$f^* g_!$

base change

$f^! \xrightarrow{\sim} \langle T_f\rangle f^*$

purity

f smooth

2. Recall construction of $S\mathcal{H}(S)$.

Sm/S f.p. smooth over S

finite present

$\rightsquigarrow$

$PSh(Sm/S)$ presheaf of Space $\longleftrightarrow$ freely add colim (homotopy)

Kan complex

$\rightsquigarrow$

$\mathcal{H}(S) = Sh_{Nis, \mathbb{A}^1}(Sm/S) \rightsquigarrow S\mathcal{H}(S)$ $\mathbb{P}^1$ stablize.

topos

(linearize)

"Ab" Sheaf, $\otimes$

$f_\#(X)$ $\qquad$ $\Sigma^\infty_{\mathbb{P}^1+}$

$S \xrightarrow{p} T$

smooth ...

$$\begin{tikzcd} \mathcal{H}(S) \arrow[r, bend left=40, "p_\#"] \arrow[r, bend right=40, "p_*"'] & \mathcal{H}(T) \arrow[l, "p^*"'] \end{tikzcd}$$

Voevodsky condition $SH(-)$: Sch $\to$ PrCat

Let $p : E \to S$ be v.b over S. $D(-)$:

(I) Homotopy invariance ($\mathbb{A}^1$ homotopy)

$id \xrightarrow{\sim} p_* p^*$ is invertible $\mathbb{A}^1 \times S \to S$

(II) Localization

$Z \overset{i}{\hookrightarrow} S \overset{j}{\hookleftarrow} U$

functor i_* with image spanned by kernel of j^*

$j_\# j^* \to id \to i_* i^*$

$i_* \dashv i^!$ $i^! \simeq \overset{(\mathrm{Ker})}{\mathrm{Fib}}(i^* \to i^* j_* j^*)$

$i_* i^! \to id \to j_* j^*$

$SH(S) \xrightarrow{(i^*, j^*)} SH(Z) \times SH(U)$ is conservative (iso $\Leftarrow$ iso)

$\Rightarrow$ Nisnevich seperation

(III) Thom stability $p: E \to S$ $s: S \to E$

Def Thom twist $\langle E \rangle$: $\mathcal{F} \mapsto \mathcal{F}\langle E \rangle := p_\# s_*(\mathcal{F})$

$\langle E \rangle$ is an equivalence

So we have

$\langle -E \rangle := \langle E \rangle^{-1}$

$1_S\langle E \rangle = \Sigma^\infty_{\mathbb{P}} Th(E)$ apply projection formula

$\mathcal{F}\langle E \rangle \simeq \mathcal{F} \otimes Th(E)$

(IV) f_* has right adjoint

3, Smooth-proper exchange (SPC), proper base change (PBC)

let $f: X \to Y$ proper. (X is noetherian)

(I) $PBC(f)$

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"'] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Ex_*^* $v^* f_* \xrightarrow{\sim} g_* u^*$.

(II) $SPC(f)$

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "p"'] & Y' \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

p, q smooth

$Ex_{\#,*}$: $q_\# g_* \xrightarrow{\sim} f_* p_\#$

Rmf. p, q is open $\Rightarrow$ $\square$ just commutative.

Support property

Idea proof. 1) for $f = i$ closed immersion. by V(II). (localization)

2) for f projective, by Atiyah Duality $\Rightarrow$ RP(f)

$RP(f) \xrightarrow{V.(IV)}$ PB(f), SPC(f)

3) Chow lemma. $f: X \to Y$ proper

$\exists$ $\pi: \tilde{X} \to X$ projective birational

s.t $f \circ \pi : \tilde{X} \to X \to Y$ is also projective.

+ cdh descent (noetherian induction)

We reduce to projective cases.

4. Relative purity

I. closed immersion: $RP(i)$

$i : X \hookrightarrow Y$, $\quad X \xrightarrow{p} S \xleftarrow{q} Y \in Sm/S$

$q_\# i_* \simeq p_\#(N_{X/Y})$

normal bundle

proof: $\underline{P_S(X,Y)} := q_\# i_* \qquad \underline{P_S(X, N_{X/Y})} = p_\# \pi_\# s_* \simeq p_\#(N_{X/Y})$

deformatonation

$(X, Y) \to (X \times A^1, D_{X/Y}) \leftarrow (X, N_{X/Y}) \qquad$ V.I, Nis

$P_S(X,Y) \xrightarrow{\simeq} P_S(X \times A^1, D_{X/Y}) \circ pr_1^* \xleftarrow{\simeq} P_S(X, N_{X/Y})$

Rmk $\Pi_S(X) := p_\#(1_X)$ $\quad \Pi_S(X,Y) = p_S(X,Y)(1_X)$

$\Pi_S(X,Y) \xrightarrow{\sim} \Pi(X)\langle N_{X/Y}\rangle =: \Pi_S(X; N_{X/Y})$

II Proper smooth case $f : X \to Y$

$SPC(f) \Rightarrow \varepsilon_f : f_\# \xrightarrow{\sim} f_*\langle T_f\rangle$ is an iso.

Proof

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr, bend right] & X\times_Y X \arrow[r, "pr_1"] \arrow[d, "pr"] & Y \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

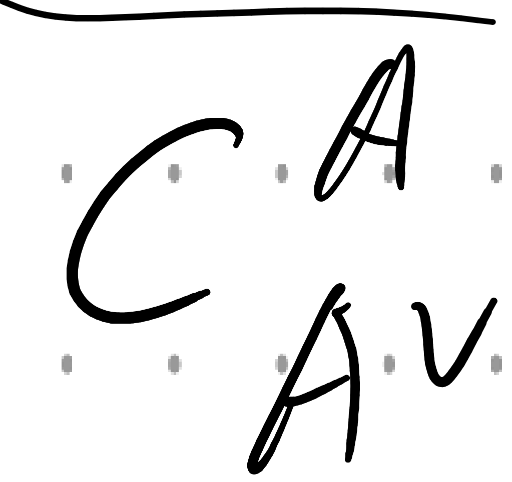
$\varepsilon_f : f_\# = f_\# pr_{1,*}\Delta_{f,*} \xrightarrow{Ex^{\#}_{,*}} f_* pr_{1\#}\Delta_{f,*}$ is due purity $f_*\langle T_S\rangle$

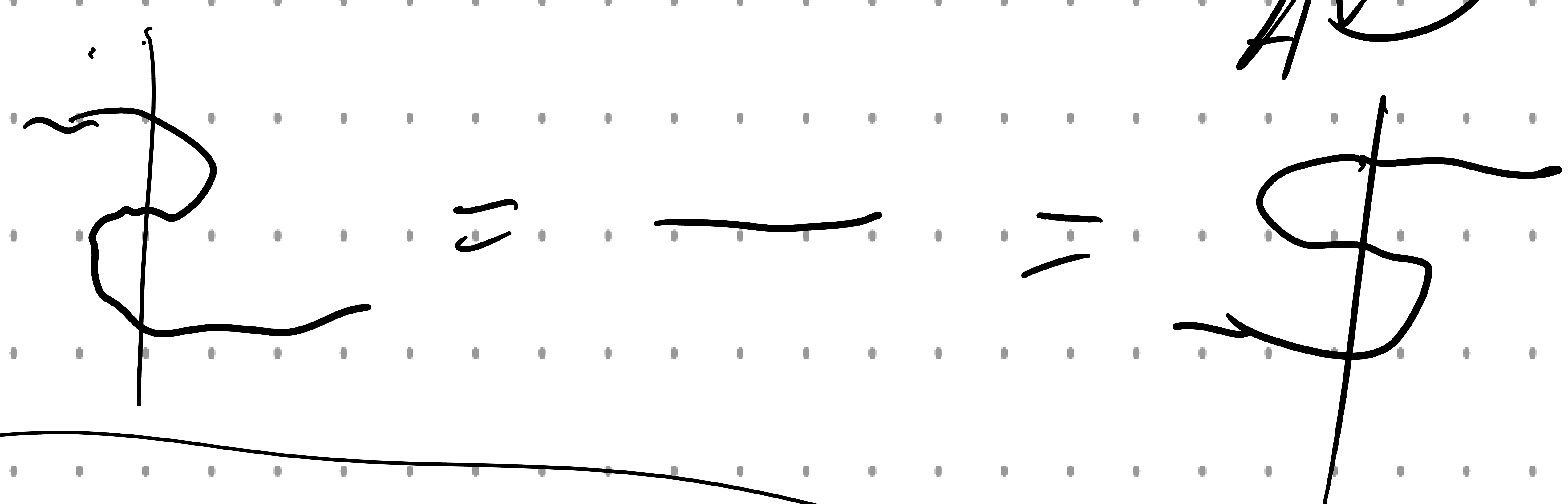
Rmk $RP(f) \Rightarrow SPC(f), PB(f)$

RP(f) $\Longleftrightarrow$ Atiyah Duality (Spectra)

$A = \Pi_Y(X) = f_\#(1_X)$ is rigid i.e. has strong dual $A^\vee = \Pi_Y(X; -T_f)$

$\eta : 1 \to A \otimes A^\vee$ $\qquad$ $\mu : A \otimes A^\vee \to 1$





$f = p \quad \mathbb{P}(E) \to S$ $\qquad p_*(1_X) = \underline{\mathrm{Hom}}(\Sigma_p^\infty X, 1_S)$

$\overset{\pi}{\underset{\text{v.b}}{}}$ $\quad \cancel{//}$

$p_\#(1_X) = \Sigma_{\mathbb{P}^+}^\infty X = \Pi_S(X)$

construction $\Sigma^\infty \mathbb{P}(E)$ $\qquad \boxed{1_X \mathbb{Z}[n](n)}$

$Th(-T_p) \cdot \langle E \rangle^{-1}$

$S_p : \Pi_S(X) = \underline{p_\# p^* 1_S} \to 1_S$

Pontryagin—Thom coll map.

$1\langle E \rangle^{-1} 1_X$

e^p : $1_S \to p_\# \langle -T_p \rangle 1_X = \Pi_X(X; -T_p)$

$\delta_p : \Pi_S(X) \otimes \Pi_S(X; T_p) \to \Pi_S(X)$

$\delta^p : \Pi_S(X; -T_p) \to ($ $\qquad)$

$\mu = S_p \circ \delta_p \qquad \eta = \delta^p \circ e^p$

5. $f_!, f^!$ $f: X \to Y$ (separated) finite type

Comp(f)

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "j'"'] & \tilde{X} \arrow[r, "p"] \arrow[d, "\pi"] & P \\ & \tilde{X}' \arrow[ur, "p'"'] & \end{tikzcd}$$

π : proper.

⇓ Supp: q

Bcomp(f)

contra

$$f_! = p_* \tilde{j}_\# = p'_* \pi_* j_\# \xrightarrow[\sim]{\text{Exc}_*} p'_* j'_\#$$

is well-defined.

$f_* \dashv f^\Delta \implies f_! \dashv f^!$

Rmk 1) $f_! \simeq f_\#$ f open. $f_! \xrightarrow{\sim} f_*$ f proper.

2)
$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner"] & Y' \arrow[d, "v"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$Ex^*_!$: $v^* f_! \xrightarrow{\sim} g_! u^*$

$Ex^!_*$: $u_* g^! \xrightarrow{\sim} f^! v_*$

3) $\mathcal{F} \otimes f_!(\mathcal{G}) \xrightarrow{\sim} f_!(f^*(\mathcal{F}) \otimes \mathcal{G})$

4) purity $f : X \to Y$ smooth

$pur_f : f^! \xrightarrow{\sim} \langle T_f \rangle f^*$.